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ON SOME APPROXIMATION PROPERTIES OF THE GAUSS-WEIERSTRASS OPERATORS

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Abstract. In this paper, we present some approximation properties of the Gauss-Weierstrass operators in exponential weighted spaces including norm convergence of them and Voronovskaya and quantitative Voronovskaya-type theorems.

1. Preliminaries

The Gauss-Weierstrass singular integral operator

$$\left(W_n f\right)(x) := \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} f\left(x+t\right) e^{-nt^2} dt, \tag{1}$$

where $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $n \to \infty$, was examined in [1], [3], [4], [8] for functions belonging to the space L_p and the classical Hölder spaces.

In this paper we examine the Gauss-Weierstrass operators W_n for functions f belonging to the exponential weighted spaces $L_q^p(\mathbb{R})$ and $L_q^{p,r}(\mathbb{R})$ which definitions are given bellow. We give some elementary properties, the orders of approximation and the Voronovskaya type theorem and quantitative Voronovskaya type theorem for these operators. Also simultaneous approximation property is obtained.

Let $q > 0$ be a fixed number and let

$$\nu_q(x) := e^{-qx^2}, \quad x \in \mathbb{R}. \tag{2}$$

For a fixed $1 \leq p \leq \infty$ and $q > 0$ we denote by L_q^p the set of all real-valued functions f defined on $\mathbb{R}$ for which the $p-$ th power of $\nu_q f$ is Lebesgue-integrable on $\mathbb{R}$ if $1 \leq p < \infty$, and $\nu_q f$ is uniformly continuous and bounded on $\mathbb{R}$ if $p = \infty$. Let the norm in L_q^p be given below by the formula

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$$\|f\|_{p,q} = \|f(.)\|_{p,q} := \begin{cases} \left(\int\limits_{-\infty}^{\infty} |\nu_q(x) f(x)|^p \, dx \right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \sup\limits_{x \in \mathbb{R}} \nu_q(x) |f(x)|, & \text{if } p = \infty. \end{cases} \tag{3}$$

Also, let $r \in \mathbb{N}_0$ and $L_q^{p,r} \equiv L_q^{p,r}(\mathbb{R})$ be the class of all $r-$times differentiable functions $f \in L_q^p$ having the derivatives $f^{(k)} \in L_q^p, 1 \le k \le r$. The norm in $L_q^{p,r}$ is given by (3). The spaces L_q^p and $L_q^{p,r}$are called exponential weighted spaces(see [2]).

For $f \in L_q^p$ we define the modulus of smoothness of the order two (see [5])

$$\omega_2\left(f, L_q^p; t\right) := \sup_{|h| \le t} \left\|\Delta_h^2 f(\cdot)\right\|_{p,q} \text{ for } t \ge 0, \tag{4}$$

where

$$\Delta_h^2 f(x) := f(x+h) - f(x-h) - 2f(x), \quad x, h \in \mathbb{R} \tag{5}$$

From (3)-(5) for $f \in L_q^p$ follows

$$\|f(\cdot + h)\|_{p,q} \le e^{qh^2} \|f(\cdot)\|_{p,q}, \quad h \in \mathbb{R}, \tag{6}$$

$$0 = \omega_2\left(f; L_p^q; 0\right) \le \omega_2\left(f, L_p^q; t_1\right) \le \omega_2\left(f, L_p^q; t_2\right) \text{ if } 0 \le t_1 < t_2. \tag{7}$$

Using the identity (see [6])

$$\Delta_{nh}^2 f(x) = \sum_{k=1}^{n} k \Delta_h^2 f(x - (n-k)h) + \sum_{k=1}^{n} (n-k) \Delta_h^2 f(x + kh),$$

$x, h \in \mathbb{R}$; $n = 2, 3, ...,$ and by (2) and (6) we can prove that

$$\omega_2\left(f, L_p^q; \lambda t\right) \le (1+\lambda)^2 e^{q(t\lambda)^2} \omega_2\left(f, L_p^q; t\right) \text{ for } \lambda, t \ge 0. \tag{8}$$

2. AUXILIARY RESULTS

In this part, we shall give some fundamental properties of the Gauss-Weierstrass integral operators W_n in the spaces $L_{p,2q}(\mathbb{R})$.

Lemma 1. *The equality*

$$\int_0^\infty t^r e^{-nt^2} dt = \frac{1}{2n^{\frac{r+1}{2}}} \Gamma\left(\frac{r+1}{2}\right)$$

holds for every $r \in \mathbb{N}_0$ *and* $n > 0$.

Lemma 2. *Let* $e_0(x) = 1$, $e_1(x) = x$ *and let* $\varphi_x(t) = t - x$ *for* $x, t \in \mathbb{R}$ *and* $k \in \mathbb{N}$. *Then,*

$$W_n(e_i; x) = e_i(x), \text{ for } x \in \mathbb{R}, n \in N, \ i = 0, 1 \tag{9}$$

$$W_n\left(\varphi_x^k(t); x\right) = \frac{\left((-1)^k + 1\right)\Gamma(\frac{k+1}{2})}{2\sqrt{\pi} n^{\frac{k}{2}}} \tag{10}$$

$$W_n\left(|\varphi_x(t)|^k \exp(q\,|\varphi_x(t)|^2; x\right) = \sqrt{\frac{n}{\pi}} \frac{\Gamma\left(\frac{k+1}{2}\right)}{(n-q)^{\frac{k+1}{2}}}\ , \ \ n > q+1 \tag{11}$$

Lemma 3. *Let* $f \in L_{p,q}(\mathbb{R})$, *with fixed* $1 \le p \le \infty, q > 0$. *Then for* $n > 2q + 1$, *we have*

$$\|W_n f\|_{p,2q} \le \sqrt{\frac{n}{n-2q}}\, \|f\|_{p,q}\ , \tag{12}$$

Lemma 3 shows that W_n are linear positive operators from $L_{p,q}(\mathbb{R})$ into $L_{p,2q}(\mathbb{R})$.

Proof. Arguing analogously to the proof of Lemma 2 in [7] we can obtain the above lemma. □

Lemma 4. *Let* $f \in L_{p,q}(\mathbb{R})$ *with fixed* $1 \le p \le \infty$ *and* $q > 0$ *and let* $n \in N$. *Let* $f \in L^r_{\infty,q}(\mathbb{R})$ *with a fixed* $r \in \mathbb{N}$. *Then* $W_n f \in L^r_{\infty,q}(\mathbb{R})$ *and for derivatives of* $W_n f$ *there holds*

$$\left\|(W_n f)^{(k)}\right\|_{\infty,2q} = \left\|W_n f^{(r)}\right\|_{\infty,2q} \le \sqrt{\frac{n}{n-2q}}\left\|f^{(k)}\right\|_{\infty,q}.$$

Proof. For details see [9]. □

3. APPROXIMATION RESULTS

Theorem 5. *Let* $f \in L_{p,q}(\mathbb{R})$ *with fixed* $1 \le p \le \infty$, $q > 0$ *and* $n > q + 1$. *Then we have*

$$\|W_n(f) - f\|_{p,2q} \le \omega_2\left(f, L_p^q; \frac{1}{\sqrt{n}}\right)\left[\frac{1}{2}\sqrt{\frac{n}{n-q}} + \frac{2n}{\sqrt{\pi}\,(n-q)} + \frac{n^{\frac{3}{2}}}{4\,(n-q)^{\frac{3}{2}}}\right].$$

Proof. From (1) and (5) we get

$$W_n(f; x) - f(x) = \sqrt{\frac{n}{\pi}} \int_0^\infty \Delta_t^2 f(x) e^{-nt^2} dt$$

for $x \in \mathbb{R}$ and $n > q + 1$. By (4) and (8), we get

$$\|W_n(f) - f\|_{p,2q} \quad \le \quad \sqrt{\frac{n}{\pi}} \int_0^\infty \left\|\Delta_t^2 f(x)\right\|_{p,q} e^{-nt^2} dt$$

$$\leq \sqrt{\frac{n}{\pi}}\omega_2\left(f, L_p^q; \frac{1}{\sqrt{n}}\right)\int\limits_0^\infty \left(1+\sqrt{n}t\right)^2 e^{-t^2(n-q)}dt.$$

Using Lemma 1, we obtain

$$\|W_n(f)-f\|_{p,2q} = \omega_2\left(f, L_p^q; \frac{1}{\sqrt{n}}\right)\left[\frac{1}{2}\sqrt{\frac{n}{n-q}} + \frac{2n}{\sqrt{\pi}\,(n-q)} + \frac{n^{\frac{3}{2}}}{4\,(n-q)^{\frac{3}{2}}}\right].$$

Thus the theorem is completed.

Corollary 6. *Let* $f \in L_{p,q}(\mathbb{R})$ *with fixed* $1 \leq p \leq \infty$, $q > 0$ *and* $n > q+1$. *Then*

$$\lim_{n\to\infty} \|W_n(f)-f\|_{p,2q} = 0. \tag{13}$$

□

Applying Corollary 1, we shall prove the Voronovskaya-type theorem for W_n.

Theorem 7. *Let* $f \in L_q^{\infty,2}(\mathbb{R})$ *has second derivate at a point* $x \in \mathbb{R}$ *and with a fixed* $q > 0$. *Then we have*

$$\lim_{n\to\infty} n\left[W_n(f;x)-f(x)\right] = \frac{f^{''}(x)}{4}.$$

Proof. For $f \in L_q^{\infty,2}$ and $x \in \mathbb{R}$. Then we can use Taylor formula in the form

$$f(t) = f(x) + f^{'}(x)(t-x) + \frac{1}{2}f^{''}(x)(t-x)^2 + \mu(t;x)(t-x)^2 \text{ for } t \in \mathbb{R},$$

where $\mu(t) = \mu(t;x)$ is a function belonging to L_q^∞ and

$$\lim_{t\to x}\mu(t;x) = \mu(x) = 0.$$

Using the operator W_n, (9) and (10), we get

$$\begin{aligned} W_n(f(t);x) &= f(x) + f^{'}(x)W_n(t-x;x) \\ &\quad + \frac{1}{2}f^{''}(x)W_n((t-x)^2;x) + W_n\left(\mu(t)\varphi_x^2(t);x\right) \\ &= f(x) + \frac{1}{4n}f^{''}(x) + W_n\left(\mu(t)\varphi_x^2(t);x\right) \end{aligned} \tag{14}$$

and by the Hölder inequality and (10), we have

$$\begin{aligned} \left|W_n\left(\mu(t)\varphi_x^2(t);x\right)\right| &\leq (W_n(\mu^2(t);x)^{\frac{1}{2}}W_n(\varphi^4(t);x)^{\frac{1}{2}} \\ &= n^{-1}\left(\frac{3}{4}W_n(\mu^2(t);x)\right)^{\frac{1}{2}}. \end{aligned}$$

From properties of μ and (13) there result that

$$\lim_{n\to\infty} W_n(\mu^2(t);x) = \mu^2(x) = 0.$$

Thus we have

$$\lim_{n\to\infty} nW_n\left(\mu(t)\varphi_x^2(t);x\right)=0$$

frıom (14) we have desired result. □

Theorem 8. *Let* $f\in L_q^{\infty,2}(\mathbb{R})$ *with a fixed* $q>0.$ *Then*

$$\left\|4n\left[W_n(f)-f\right]-f^{''}\right\|_{\infty,2q}\leq\omega_1\left(f^{''};L_q^{\infty};\frac{1}{\sqrt{n}}\right)\left[\frac{1}{4}\left(\frac{n}{n-q}\right)^{\frac{3}{2}}+\frac{1}{2\sqrt{\pi}}\left(\frac{n}{n-q}\right)^2\right]. \tag{15}$$

Proof. For $f\in L_q^{\infty,2}$ and $x,t\in\mathbb{R}$ there holds the Taylor-type formula

$$f(t)=f(x)+f^{'}(x)(t-x)+\frac{1}{2}f^{''}(x)(t-x)^2+(t-x)^2I(t,x),$$

where

$$I(t,x):=\int_0^1(1-u)\left[f^{''}(x+u(t-x))-f^{''}(x)\right]du. \tag{16}$$

Using operator W_n, and (9)-(11), we get

$$W_n(f(t);x)=f(x)+\frac{1}{4n}f^{''}(x)+W_n\left(\varphi_x^2(t)I(t,x);x\right),$$

which implies that

$$4n\left[W_n(f;x)-f(x)\right]-f^{''}(x)=nW_n\left(\varphi_x^2(t)\left|I(t,x)\right|;x\right)$$

for $x\in\mathbb{R}$. Now, applying (4), (7) and (8), we get

$$\begin{aligned}\left|I(t,x)\right| &\leq \int_0^1(1-u)\omega_1\left(f^{''};L_q^{\infty};u\left|t-x\right|\right)e^{qx^2}du\\ &\leq \frac{1}{2}\omega_1\left(f^{''};L_q^{\infty};\left|t-x\right|\right)e^{qx^2}\\ &\leq \frac{1}{2}\omega_1\left(f^{''};L_q^{\infty};\frac{1}{\sqrt{n}}\right)(1+\sqrt{n}\left|t-x\right|)e^{qx^2+q\left|t-x\right|^2}\end{aligned}$$

and next by (2) and (11), we can write for $x\in\mathbb{R}$ and $n>q+1$,

$$\begin{aligned}n\nu_q(x)W_n(\varphi_x^2(t)I(t,x);x) &\leq \frac{n}{2}\omega_1\left(f^{''};L_q^{\infty};\frac{1}{\sqrt{n}}\right)\\ &\quad\times\left\{W_n((t-x)^2e^{q\left|t-x\right|^2};x)+\sqrt{n}W_n(\left|t-x\right|^3e^{q\left|t-x\right|^2};x\right\}\\ &= \omega_1\left(f^{''};L_q^{\infty};\frac{1}{\sqrt{n}}\right)\left[\frac{1}{4}\left(\frac{n}{n-q}\right)^{\frac{3}{2}}+\frac{1}{2\sqrt{\pi}}\left(\frac{n}{n-q}\right)^2\right].\end{aligned}$$

Now the estimate (15) is obtained by (16), the last inequality and (3). □

Theorem 9. *Let $f \in L_q^{\infty,r}$,with fixed $q > 0$ and $r \in \mathbb{N}$. Then*

$$\left\| W_n^{(r)}(f) - f^{(r)} \right\|_{\infty,2q} \leq \omega_2\left(f^{^{(k)}}; L_q^{\infty}; \frac{1}{\sqrt{n}}\right) \times \left(\sqrt{\frac{n}{n-2q}} + \frac{n}{(n-2q)\sqrt{\pi}} + \frac{1}{4}\left(\frac{n}{n-2q}\right)^{\frac{3}{2}}\right) \tag{17}$$

for $n > 2q + 1$.

Proof. If $f \in L_q^{\infty,r}$, then for r-th derivative of $W_n(f)$ we have by Lemma 4, (9) and (10):

$$\begin{aligned} W_n^{(r)}(f;x) - f^{(r)}(x) &= \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} \left[f^{(r)}(x+t) - f^{(r)}(x)\right] e^{-nt^2} dt \\ &= \sqrt{\frac{n}{\pi}} \int_{0}^{\infty} \left[\Delta_t^2 f^{(r)}(x-t)\right] e^{-nt^2} dt. \end{aligned}$$

from this and by (4),(8) and Lemma 1 we deduce that

$$\begin{aligned} \left\| W^{(r)}r(f;x) - f^{(r)}(x) \right\|_{\infty,2q} &\leq \sqrt{\frac{n}{\pi}} \int_0^{\infty} \omega_2\left(f^{(r)}; L_q^{\infty}; t\right) e^{-(n-q)t^2} dt \\ &\leq \omega_2\left(f^{(r)}; L_q^{\infty}; \frac{1}{\sqrt{n}}\right) \sqrt{\frac{n}{\pi}} \int_0^{\infty} (1+\sqrt{n}t)^2 e^{-t^2(n-2q)} dt \\ &= \omega_2\left(f^{(r)}; L_q^{\infty}; \frac{1}{\sqrt{n}}\right) \\ &\quad \times \sqrt{\frac{n}{\pi}} \left[\int_0^{\infty} e^{-t^2(n-2q)} dt + 2\sqrt{n} \int_0^{\infty} t e^{-t^2(n-2q)} dt + n \int_0^{\infty} t^2 e^{-t^2(n-2q)} dt\right] \\ &= \omega_2\left(f^{(r)}; L_q^{\infty}; \frac{1}{\sqrt{n}}\right) \\ &\quad \times \left(\sqrt{\frac{n}{n-2q}} + \frac{n}{(n-2q)\sqrt{\pi}} + \frac{1}{4}\left(\frac{n}{n-2q}\right)^{\frac{3}{2}}\right), \end{aligned}$$

for $n > 2q + 1$, which yields the estimate (17). □

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