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Research Article

Ideal based trace graph of matrices

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Abstract

Let R be a commutative ring and $M_n(R)$ be the set of all $n \times n$ matrices over R where $n \geq 2$. The trace graph of the matrix ring $M_n(R)$ with respect to an ideal I of R, denoted by $\Gamma_{I^t}(M_n(R))$, is the simple undirected graph with vertex set $M_n(R) \setminus M_n(I)$ and two distinct vertices A and B are adjacent if and only if $\mathrm{Tr}(AB) \in I$. Here $\mathrm{Tr}(A)$ represents the trace of the matrix A. In this paper, we exhibit some properties and structure of $\Gamma_{I^t}(M_n(R))$.

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1. Introduction

The concept of associating graphs to commutative rings was first introduced by Beck [3]. He introduced the concept of zero-divisor graph of a commutative ring R as an undirected graph whose vertices are the elements of R with two distinct vertices x and y joined by an edge if and only if $xy = 0$. Later on, Anderson and Livingston [2] modified the definition with vertex set, the set of all nonzero zero divisors of R and introduced the zero-divisor graph $\Gamma(R)$ corresponding to a commutative ring R. In [9], Redmond introduced the notion of the zero-divisor graph with respect to an ideal I of a commutative ring R, denoted by $\Gamma_I(R)$, as the graph with vertex set $\{x \in R \setminus I : xy \in I$ for some $y \in R \setminus I\}$, and two distinct vertices x and y are adjacent if and only if $xy \in I$. The concept of trace graph of a matrix ring over a commutative ring was introduced by Almahdi, Louartiti, and Tamekkante [1]. Several authors have extensively studied about zero-divisor graph with respectxzs to an ideal. For example one may refer [8]. Let R be a commutative ring and n be a positive integer. Let $M_n(R)$ denote the set of all $n \times n$ matrices over R, $M_n(R)^*$ denotes the set of all $n \times n$ non-zero matrices over R and let $\mathrm{Tr}(A)$ be the trace of the matrix $A \in M_n(R)$. The trace graph of the matrix ring $M_n(R)$, denoted by $\Gamma_t(M_n(R))$, is the simple undirected graph with vertex set $\{A \in M_n(R)^* :$ there exists $B \in M_n(R)^*$ such that $\mathrm{Tr}(AB) = 0\}$ and two distinct vertices A and B are adjacent if and only if $\mathrm{Tr}(AB) = 0$. Further study on the trace graph of matrices was done by authors [10].

In this paper, as a parallel approach of generalization of $\Gamma(R)$ to $\Gamma_I(R)$, we generalize the notion of the trace graph $\Gamma_t(M_n(R))$ of a matrix ring $M_n(R)$ to the trace graph $\Gamma_{I^t}(M_n(R))$ with respect to an ideal I of R. Actually $\Gamma_{I^t}(M_n(R))$ is the simple undirected

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graph with vertex set $M_n(R) \setminus M_n(I)$ and two distinct vertices A and B are adjacent if and only if $\mathrm{Tr}(AB) \in I$. Note that if $A \in M_n(I)$, then $\mathrm{Tr}(AB) \in I$ for every $B \in M_n(R)$. Due to this, matrices in $M_n(I)$ are not considered for the vertex set of $\Gamma_{I^t}(M_n(R))$. As usual, E_{ij} denotes the matrix whose ij^{th} entry is 1 and 0 elsewhere. For a set X, $|X|$ denotes the cardinality of X, $X \setminus Y$ denotes the set of elements that belong to X and not to set Y. For basic definitions on rings, one may refer [6] and for noncommutative rings see [5,7].

Let G be a graph. For distinct vertices x and y of G, let $d(x,y)$ be the length of the shortest path between x and y ($d(x,y) = \infty$ if there is no such path). The diameter of G is $\mathrm{diam}(G) = \sup\{d(x,y) : \ x \text{ and } y \text{ are distinct vertices of } G\}$. The girth of G, denoted by $\mathrm{gr}(G)$, is defined as the length of the shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G contains no cycles). For a graph G and a vertex $v \in V(G)$, the eccentricity $e(v)$ of v is the maximum distance to any vertex in the graph, i.e., $e(v) = \max\limits_{u \in V(G)} \{\mathrm{d}(v,u)\}$. The radius $rad(G)$ of a G is the minimum eccentricity among all vertices in G and a vertex of G is a central vertex if $e(v) = rad(G)$. G is self-centered if every vertex is in the center i.e., $e(v) = rad(G)$ for every vertex $v \in V(G)$. A subset Ω of $V(G)$ is called a clique if the induced subgraph of Ω is complete. The order of the largest clique in G is its clique number, which is denoted by $\omega(G)$. The chromatic number of a graph G, denoted by $\chi(G)$, is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. An independent set or stable set is a set of vertices in a graph G such that no two of them are adjacent. A maximum independent set is an independent set of largest possible size for the given graph G. This size is called the independence number of G and denoted by $\alpha(G)$.

If the edges of G are partitioned into subgraphs $H_1, \ldots, H_k, \ldots H_n$, then we write $G \cong H_1 \oplus \cdots \oplus H_n$, and if $H_i \cong H_j$ for all $1 \le i, j \le k$, then we write $G \cong kH \oplus H_{k+1} \oplus \cdots \oplus H_n$, where $H \cong H_i, (1 \le i \le k)$. For general reference of graph theoretical terms and results, we refer [11].

Remark 1.1. Let R be a commutative ring and n be a positive integer.

1. The graph $\Gamma_{I^t}(M_1(R))$ coincides with $\Gamma_I(R)$ (ideal based zero-divisor graph of the ring R).
2. If $I = (0)$, then $\Gamma_{I^t}(M_n(R)) = \Gamma_t(M_n(R))$ for all $n \ge 1$.
3. If $I = R$, then $\Gamma_{I^t}(M_n(R))$ is the null graph.

Throughout this paper, unless otherwise specified, R is a commutative ring with identity, $n \ge 2$ is an integer, and I is a non-trivial ideal of R. If $A = [a_{ij}] \in M_n(R) \setminus M_n(I)$ the corresponding matrix in $M_n(R/I)$ is $[a_{ij} + I]$. If $A = [a_{ij}] \in M_n(I)$, then the corresponding matrix in $M_n(R/I)$ is the zero matrix in $M_n(R/I)$. For convenience, we denote the matrix $[a_{ij} + I] \in M_n(R/I)$ as $\overline{A}$ corresponding to the matrix $A = (a_{ij})$. In Section 2, we prove that for $n \ge 2$, $\Gamma_{I^t}(M_n(R))$ is a connected graph of diameter 2 and of girth 3. In Section 3, we study the structure of $\Gamma_{I^t}(M_n(R))$ through the relationship between $\Gamma_{I^t}(M_n(R))$ and $\Gamma_t(M_n(R/I))$. In Section 4, we discuss the clique, chromatic, and independence numbers of $\Gamma_{I^t}(M_n(R))$.

2. Girth and diameter

In this section, we list some properties of the trace graph of matrix ring with respect to an ideal I of R that can be proved by similar arguments as in the case of the trace graph of matrix rings over commutative rings. For $A = [a_{ij}] \in M_n(R)$, we set $J_I(A) = \sum\limits_{1 \le i,j \le n} (R/I)(a_{ij} + I) \in (R/I)$; the sum of the ideals of R/I generated by all entries of $\overline{A} = [a_{ij} + I]$ over R/I. Note that $J_I(A)$ is an ideal of R/I.

Proposition 2.1. *For a non-zero ideal I of R and an integer $n \geq 2$, $\Gamma_{I^t}(M_n(R)))$ contains no isolated vertex.*

Proof. Let $A = [a_{ij}] \in M_n(R) \setminus M_n(I)$.

Case 1. If $A \neq I_n$ and $\mathrm{Tr}(A) \in I$, then A is adjacent to the identity matrix I_n.

Case 2. Assume that $\mathrm{Tr}(A) \notin I$.

Case 2.1. Suppose A has exactly one entry $a_{k\ell}$ such that $a_{k\ell} \notin I$. Choose $B = [b_{ij}]$ such that $b_{\ell k} \in I$ and $b_{ij} \notin I$ otherwise. Then $B \in M_n(R) \setminus M_n(I)$ and $\mathrm{Tr}(AB) = a_{11}b_{11} + \cdots + a_{1n}b_{n1} + a_{21}b_{12} + \cdots + a_{2n}b_{n2} + \cdots + a_{n1}b_{1n} + \cdots + a_{nn}b_{nn}$. Note that in each term of $\mathrm{Tr}(AB)$ either $a_{ij} \in I$ or $b_{ij} \in I$. Since I is an ideal of R, $a_{ij}b_{ij} \in I$ for every $1 \leq i, j \leq n$ and hence their sum belongs to I. Thus $\mathrm{Tr}(AB) \in I$.

Case 2.2. Suppose that A has at least two entries $a_{k\ell}, a_{k_1\ell_1}$ which are not elements of I. Then choose $B = [b_{ij}]$ such that $b_{\ell k} = -a_{k_1\ell_1}, b_{\ell_1 k_1} = a_{k\ell}, b_{ij} \in I$ elsewhere. Thus $B \in M_n(R) \setminus M_n(I)$ and $\mathrm{Tr}(AB) = a_{k\ell}b_{\ell k} + a_{k_1\ell_1}b_{\ell_1 k_1} +$ elements of I. Hence $\mathrm{Tr}(AB) \in I$.

Thus in all the cases for every $A \in M_n(R) \setminus M_n(I)$, there exists $B \in M_n(R) \setminus M_n(I)$ such that $\mathrm{Tr}(AB) \in I$. Hence, $\Gamma_{I^t}(M_n(R)))$ contains no isolated vertex. □

In the following, we prove that no vertex in $\Gamma_{I^t}(M_n(R)))$ is adjacent to all other vertices.

Proposition 2.2. *For a non-zero ideal I of R and an integer $n \geq 2$, no vertex of $\Gamma_{I^t}(M_n(R))$ is adjacent to every other vertex of $\Gamma_{I^t}(M_n(R))$.*

Proof. Given a matrix $A = [a_{ij}] \in M_n(R) \setminus M_n(I)$. There exists at least one entry $a_{k\ell}$ such that $a_{k\ell} \notin I$. Choose $B = [b_{ij}]$ such that $b_{\ell k} = 1$ and $b_{ij} = 0$ elsewhere. Thus $B \in V(\Gamma_{I^t}(M_n(R)))$ and $\mathrm{Tr}(AB) = a_{k\ell} \notin I$. If $A = B$, then $\mathrm{Tr}(AI_n) \notin I$. □

Now we obtain, the degree of vertices in $\Gamma_{I^t}(M_n(R))$.

Proposition 2.3. *Let R be a finite commutative ring and $n \geq 2$ be an integer.*

1. *For any vertex A of $\Gamma_{I^t}(M_n(R))$, we have:*
   a. *$deg(A) = \frac{|R|^{n^2}}{|J_I(A)|} - 1$ if $Tr(A^2) \notin I$, and*
   b. *$deg(A) = \frac{|R|^{n^2}}{|J_I(A)|} - 2$ if $Tr(A^2) \in I$.*
2. $\delta(\Gamma_t(M_n(R))) = |R|^{n^2-1}|I| - 2$.

Proof. **1.** Let $A \in M_n(R)$. Consider $f_A : M_n(R) \to R$ defined by $f_A(B) = \mathrm{Tr}(AB)$ and natural homomorphism $\varphi : R \to R/I$ by $\varphi(x) = x + I$. Clearly $\varphi \circ f_A : M_n(R) \to R/I$ is a surjective homomorphism with $(\varphi \circ f_A)(B) = \mathrm{Tr}(AB) + I$, $\mathrm{Im}(\varphi \circ f_A) = J_I(A)$ and $\ker(\varphi \circ f_A) = \{B \in M_n(R) | \ \mathrm{Tr}(AB) \in I\}$.

By the isomorphism theorem, $\frac{M_n(R)}{\ker(\varphi \circ f_A)} \cong J_I(A)$ and so $|\ker(\varphi \circ f_A)| = \frac{|M_n(R)|}{|J_I(A)|} = \frac{|R|^{n^2}}{|J_I(A)|}$. When $\mathrm{Tr}(A^2) \notin I$, $\ker(\varphi \circ f_A)$ contains exactly the vertices adjacent to A and the zero matrix. When $\mathrm{Tr}(A^2) \in I$, $\ker(\varphi \circ f_A)$ contains additionally A. Hence (a) and (b) hold.

2. Consider the matrix $A = [a_{ij}] \in M_n(R) \setminus M_n(I)$ with $a_{ii} \in I$ for every $1 \leq i \leq n$, $a_{ij} \notin I$ implies $a_{ji} \in I$ for every $i \neq j$ and a_{ij} is a unit for some i and j. Clearly $J_I(A) = R/I$ and $\mathrm{Tr}(A^2) \in I$. Thus by 1(b), we have $\deg(A) = |R|^{n^2-1}|I| - 2$ and so $\delta \leq |R|^{n^2-1}|I| - 2$.

Since $|J_I(A)| \leq |R/I|$ for every ideal $J_I(A)$ of R/I, $\frac{|R|^{n^2}}{|J_I(A)|} \geq |R|^{n^2-1}|I|$. From this $|R|^{n^2-1}|I| - 2 \leq \frac{|R|^{n^2}}{|J_I(A)|} - 2 \leq deg(A)$ for every $A \in M_n(R)$. Thus, $\delta = |R|^{n^2-1}|I| - 2$. □

From Proposition 2.3, for a finite commutative ring R, $\Gamma_{I^t}(M_n(R))$ can never be an Eulerian graph. For, consider the matrices E_{11} and E_{1n} where $n \neq 1$. $\mathrm{Tr}(E_{11}^2) = 1 \notin I$ and $\mathrm{Tr}(E_{1n}^2) = 0 \in I$. Hence, by the Proposition 2.3, either of E_{11} and E_{1n} must have odd degree.

Proposition 2.4. *Let R be a commutative ring, $n \geq 2$ be an integer and I be a non trivial ideal of R. Then $\Gamma_{I^t}(M_n(R))$ is connected with $diam(\Gamma_{I^t}(M_n(R))) = 2$ and $gr(\Gamma_{I^t}(M_n(R)))$ $= 3$.*

Proof. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two distinct elements of $M_n(R) \backslash M_n(I)$. If $\mathrm{Tr}(AB) \in I$, then $d(A, B) = 1$. Assume that $\mathrm{Tr}(AB) \notin I$. By Proposition 2.2, $\mathrm{diam}(\Gamma_{I^t}(M_n(R))) > 1$. Now let us consider two cases:

Case 1. Suppose $a_{ij}b_{k\ell} - a_{k\ell}b_{ij} \in I$ for each $(i,j),(k,l) \in \{1,\ldots,n\}^2$.
Let (i_0, j_0) and (i_1, j_1) be two distinct elements of $\{1,\ldots,n\}^2$ such that $a_{i_0j_0} \notin I$. Consider the matrix $C = [c_{ij}]$ with $c_{j_0i_0} = -a_{i_1j_1}, c_{j_1i_1} = a_{i_0j_0}$, and $c_{k\ell} \in I$ elsewhere. Then $C \in M_n(R) \setminus M_n(I)$ and

$$\begin{aligned}\mathrm{Tr}(AC) &= a_{i_0j_0}c_{j_0i_0} + a_{i_1j_1}c_{j_1i_1} + \text{ elements of } I\\ &= -a_{i_0j_0}a_{i_1j_1} + a_{i_1j_1}a_{i_0j_0} + \text{ elements of } I \in I\end{aligned}$$

and

$$\begin{aligned}\mathrm{Tr}(BC) &= b_{i_0j_0}c_{j_0i_0} + b_{i_1j_1}c_{j_1i_1} + \text{ elements of } I\\ &= -b_{i_0j_0}a_{i_1j_1} + b_{i_1j_1}a_{i_0j_0} + \text{ elements of } I \in I.\end{aligned}$$

Case 2. Suppose there exist $(i_0, j_0), (i_1, j_1) \in \{1,\ldots,n\}^2$ such that $a_{i_0j_0}b_{i_1j_1} - a_{i_1j_1}b_{i_0j_0} \notin I$.
Let $(i_2, j_2) \in \{1,\ldots,n\}^2 \setminus \{(i_0, j_0), (i_1, j_1)\}$ and consider the matrix $C = [c_{ij}]$ where

$$\begin{aligned}c_{j_0i_0} &= a_{i_1j_1}b_{i_2j_2} - a_{i_2j_2}b_{i_1j_1},\\ c_{j_1i_1} &= a_{i_2j_2}b_{i_0j_0} - a_{i_0j_0}b_{i_2j_2},\\ c_{j_2i_2} &= a_{i_0j_0}b_{i_1j_1} - a_{i_1j_1}b_{i_0j_0} \text{ and}\\ c_{k\ell} &\in I \text{ elsewhere.}\end{aligned}$$

Then $C \in M_n(R) \setminus M_n(I)$ and

$$\begin{aligned}\mathrm{Tr}(AC) &= a_{i_0j_0}c_{j_0i_0} + a_{i_1j_1}c_{j_1i_1} + a_{i_2j_2}c_{j_2i_2}\\ &= a_{i_0j_0}a_{i_1j_1}b_{i_2j_2} - a_{i_0j_0}a_{i_2j_2}b_{i_1j_1} + a_{i_1j_1}a_{i_2j_2}b_{i_0j_0}\\ &\quad - a_{i_1j_1}a_{i_0j_0}b_{i_2j_2} + a_{i_2j_2}a_{i_0j_0}b_{i_1j_1} - a_{i_2j_2}a_{i_1j_1}b_{i_0j_0} + \text{ elements of } I \in I,\\ \text{and}\quad \mathrm{Tr}(BC) &= b_{i_0j_0}c_{j_0i_0} + b_{i_1j_1}c_{j_1i_1} + b_{i_2j_2}c_{j_2i_2}\\ &= b_{i_0j_0}a_{i_1j_1}b_{i_2j_2} - b_{i_0j_0}a_{i_2j_2}b_{i_1j_1} + b_{i_1j_1}a_{i_2j_2}b_{i_0j_0}\\ &\quad - b_{i_1j_1}a_{i_0j_0}b_{i_2j_2} + b_{i_2j_2}a_{i_0j_0}b_{i_1j_1} - b_{i_2j_2}a_{i_1j_1}b_{i_0j_0} + \text{ elements of } I \in I.\end{aligned}$$

In both cases, $A \neq C$ and $B \neq C$ (otherwise $\mathrm{Tr}(AB) \in I$) and hence $d(A, B) = 2$. Consequently, $\Gamma_{I^t}(M_n(R))$ is connected and $\mathrm{diam}(\Gamma_{I^t}(M_n(R))) = 2$.
Consider nonzero distinct matrices $A = [a_{ij}]$ with $a_{11} = 1$ and $a_{ij} \in I$ elsewhere, $B = [b_{ij}]$ with $b_{nn} = 1$ and $b_{ij} \in I$ elsewhere and $C = [c_{ij}]$ with $c_{1n} = 1$ and $c_{ij} \in I$ elsewhere. By the choice of A, B, C, we have $\mathrm{Tr}(AB)$, $\mathrm{Tr}(BC)$, $\mathrm{Tr}(AC) \in I$. Thus $A - B - C - A$ is a cycle, and so $\mathrm{gr}(\Gamma_{I^t}(M_n(R))) = 3$. □

Remark 2.5. (i). By Propositions 2.2 and 2.4, the eccentricity of every vertex in $\Gamma_{I^t}(M_n(R))$ is 2 and hence the radius of $\Gamma_{I^t}(M_n(R))$ is 2. i.e., the graph $\Gamma_{I^t}(M_n(R))$ is self-centered.

(ii). By Proposition 2.4 $\Gamma_{I^t}(M_n(R))$ contains an odd cycle, and so $\Gamma_{I^t}(M_n(R))$ can never be a bipartite graph.

3. Relationship between $\Gamma_{I^t}(M_n(R))$ and $\Gamma_t(M_n(R/I))$

In this section, we study the graph $\Gamma_{I^t}(M_n(R))$ through $\Gamma_t(M_n(R/I))$. The following theorem is useful in the further discussion of this paper.

Theorem 3.1. *Let R be a ring and I be an ideal of R. Then $M_n(R)/M_n(I) \cong M_n(R/I)$.*

Proof. The map $\varphi : M_n(R)/M_n(I) \to M_n(R/I)$ by $[a_{ij}] + M_n(I) = [a_{ij} + I]$ defines an isomorphism between $M_n(R)/M_n(I)$ and $M_n(R/I)$. □

Note 3.2. From the isomorphism defined in Theorem 3.1, given an ideal I of R and a matrix $A \in M_n(R)$, we can view the trace of the coset $A + M_n(I)$ in $M_n(R)/M_n(I)$ as the trace of $\overline{A}$ in $M_n(R/I)$. Thus, the trace graph of $M_n(R)/M_n(I)$ is the trace graph of $M_n(R/I)$.

Theorem 3.3. *Let I be an ideal of a commutative ring R, $n \geq 2$ be a positive integer and $A = [a_{ij}], B = [b_{ij}] \in M_n(R) \setminus M_n(I)$ Then the following are true:*

1. *If $\overline{A}$ is adjacent to $\overline{B}$ in $\Gamma_t(M_n(R/I))$, then A and B are adjacent in $\Gamma_{I^t}(M_n(R))$.*
2. *If A is adjacent to B in $\Gamma_{I^t}(M_n(R))$ and $\overline{A} \neq \overline{B}$, then $\overline{A}$ is adjacent to $\overline{B}$ in $\Gamma_t(M_n(R/I))$.*
3. *If A is adjacent to B in $\Gamma_{I^t}(M_n(R))$ and $\overline{A} = \overline{B}$, then $Tr(A^2)$, $Tr(B^2) \in I$.*
4. *If $Tr(A^2) \in I$ and $\overline{A} = \overline{B}$, then A is adjacent to B in $\Gamma_{I^t}(M_n(R))$ and $Tr(B^2) \in I$.*
5. *If A and B are (distinct) adjacent vertices in $\Gamma_{I^t}(M_n(R))$, then all (distinct) elements of $\overline{A}$ are adjacent to all elements of $\overline{B}$ in $\Gamma_{I^t}(M_n(R))$. In particular, if $Tr(A^2) \in I$, then all the distinct elements of $\overline{A}$ are adjacent in $\Gamma_{I^t}(M_n(R))$.*

Proof. **1.** In view of the fact mentioned in Note 3.2, it is enough to prove that $A+M_n(I)$ is adjacent to $B+M_n(I)$ in $\Gamma_t(M_n(R)/M_n(I))$ implies A is adjacent to B in $\Gamma_{I^t}(M_n(R))$. When $A + M_n(I)$ is adjacent to $B + M_n(I)$ in $\Gamma_t(M_n(R)/M_n(I))$, we have $\mathrm{Tr}(AB + M_n(I)) = M_n(I)$ and so $\mathrm{Tr}(AB) \in I$. Thus A is adjacent to B in $\Gamma_{I^t}(M_n(R))$.

2. If A is adjacent to B in $\Gamma_{I^t}(M_n(R))$, then $\mathrm{Tr}(AB) \in I$. This gives that $\mathrm{Tr}(AB)+I = I$ and hence $\mathrm{Tr}(AB + M_n(I)) = M_n(I)$. Thus, $\mathrm{Tr}((A + M_n(I))(B + M_n(I))) = M_n(I)$, and so $A + M_n(I)$ is adjacent to $B + M_n(I)$ in $\Gamma_t(M_n(R)/M_n(I))$.

3. If A is adjacent to B in $\Gamma_{I^t}(M_n(R))$, by (2) above $\mathrm{Tr}((A + M_n(I))(B + M_n(I))) = M_n(I)$. Since $A + M_n(I) = B + M_n(I)$, $\mathrm{Tr}((A + M_n(I))(A + M_n(I))) = M_n(I)$. i.e., $\mathrm{Tr}(A^2+M_n(I)) = M_n(I)$ giving $\mathrm{Tr}(A^2)+I = I$. Thus $\mathrm{Tr}(A^2) \in I$ and similarly $\mathrm{Tr}(B^2) \in I$.

4. If $\mathrm{Tr}(A^2) \in I$, then $\mathrm{Tr}(A^2) + I = I$ and so $\mathrm{Tr}(A^2 + M_n(I)) = M_n(I)$. Thus $\mathrm{Tr}((A + M_n(I))(B+M_n(I))) = M_n(I)$ giving $\mathrm{Tr}(AB)+I = I$. Thus $\mathrm{Tr}(AB) \in I$. i.e., A is adjacent to B in $\Gamma_{I^t}(M_n(R))$. By (3), $\mathrm{Tr}(B^2) \in I$.

5. It is enough to prove that if A and B are (distinct) adjacent vertices in $\Gamma_{I^t}(M_n(R))$, then all (distinct) elements of $A + M_n(I)$ are adjacent to all elements of $B + M_n(I)$ in $\Gamma_{I^t}(M_n(R))$. In particular, if $Tr(A^2) \in I$, then all the distinct elements of $A + M_n(I)$ are adjacent in $\Gamma_{I^t}(M_n(R))$.

By (1) and (2), if A and B are adjacent vertices in $\Gamma_{I^t}(M_n(R))$, then all (distinct) elements of $A + M_n(I)$ and $B + M_n(I)$ are adjacent in $\Gamma_{I^t}(M_n(R))$. As a particular case, taking $B = A$, we get if $Tr(A^2) \in I$, then all the distinct elements of $A + M_n(I)$ are adjacent in $\Gamma_{I^t}(M_n(R))$. □

Corollary 3.4. *Let I be an ideal of a commutative ring R and $n \geq 2$ be a positive integer. Then $\Gamma_{I^t}(M_n(R))$ contains $|M_n(I)|$ disjoint subgraphs each isomorphic to $\Gamma_t(M_n(R/I))$.*

Proof. Let $\{A_i\}_{i\in\Lambda}$ be distinct coset representatives of elements in the quotient ring $M_n(R)/M_n(I)$. Then the vertex set of $\Gamma_t(M_n(R)/M_n(I))$ is partitioned into $\{A_i+M_n(I)\}_{i\in\Lambda}$.

Note that $A_i + M_n(I) \neq A_j + M_n(I)$ for $i \neq j$. Fix $X \in M_n(I)$. Consider the subgraph H_X with vertex set $\{A_i + X : i \in \Lambda\} \subseteq V(\Gamma_{I^t}(M_n(R)))$ and two vertices $A_i + X$ and $A_j + X$ are adjacent in H_X if $A_i + M_n(I)$ and $A_j + M_n(I)$ are adjacent in $\Gamma_t\left(M_n(R)/M_n(I)\right)$. Clearly, H_X is isomorphic to $\Gamma_t\left(M_n(R)/M_n(I)\right)$.

Assume that $A_i + X$ and $A_j + X$ are adjacent in H_X. By the definition of H_X, $A_i + M_n(I)$ is adjacent to $A_j + M_n(I)$ in $\Gamma_t\left(M_n(R)/M_n(I)\right)$. By Theorem 3.3(1), A_i and A_j are adjacent in $\Gamma_{I^t}(M_n(R))$. By Theorem 3.3(4), $A_i + X$ and $A_j + X$ are adjacent in $\Gamma_{I^t}(M_n(R))$. Hence H_X is a subgraph of $\Gamma_{I^t}(M_n(R))$.

Also, for any $Y(\neq X) \in M_n(I)$, $V(H_X) \cap V(H_Y) = \phi$. Thus, $\Gamma_{I^t}(M_n(R))$ contains $|M_n(I)|$ disjoint subgraphs each isomorphic to $\Gamma_t\left(M_n(R)/M_n(I)\right)$ and so contains $|M_n(I)|$ disjoint subgraphs isomorphic to $\Gamma_t\left(M_n(R/I)\right)$. ☐

Remark 3.5. The following are true:

1. $\Gamma_t\left(M_n(R)/M_n(I)\right)$ is a graph with $|M_n(R/I)| - 1$ vertices.
2. $\Gamma_{I^t}(M_n(R))$ is a graph with $|M_n(R)| - |M_n(I)|$ vertices.
3. Let R be a finite commutative ring. Note that Corollary 3.4 exhibits a partition of $\Gamma_{I^t}(M_n(R))$ into vertex disjoint subgraphs. Thus

$$|M_n(I)|\,|V\left(\Gamma_t\left(M_n(R/I)\right)\right)| = |V(\Gamma_{I^t}(M_n(R)))|\,.$$

The following theorem puts forth a partition of $\Gamma_{I^t}(M_n(R))$ into edge disjoint subgraphs. In view of Proposition 3.3(4), if $\mathrm{Tr}(A^2) \in I$ and $\overline{A} = \overline{B}$, then A is adjacent to B in $\Gamma_{I^t}(M_n(R))$ and $\mathrm{Tr}(B^2) \in I$. This means that if $\mathrm{Tr}(A^2) \in I$ for a matrix A, then the same is true for all matrices in the coset of A.

Theorem 3.6. *Let R be a commutative ring with identity, I be a non trivial ideal of R, $n \geq 2$ be an integer and*

$$\lambda = |\{\overline{A} \in V(\Gamma_t(M_n(R/I))) : \mathit{Tr}(A^2) \in I \text{ and } A \text{ is a coset representative of } \overline{A}\}|.$$

Then $\Gamma_{I^t}(M_n(R)) \cong |M_n(I)|^2 \Gamma_t(M_n(R/I)) \oplus \lambda K_{|M_n(I)|}$.

Proof. Consider the partition of edges of $\Gamma_t(M_n(R/I))$ given below:

$E_1 = \{e = (\overline{A}, \overline{B}) : \mathrm{Tr}(A^2), \mathrm{Tr}(B^2) \notin I\}$
$E_2 = \{e = (\overline{A}, \overline{B}) : \mathrm{Tr}(A^2), \mathrm{Tr}(B^2) \in I\}$
$E_3 = \{e = (\overline{A}, \overline{B}) : \text{either } \mathrm{Tr}(A^2) \text{ or } \mathrm{Tr}(B^2) \in I\}$.

Let $e = (\overline{A}, \overline{B}) \in E(\Gamma_t(M_n(R/I)))$. By Theorem 3.3(1) and (4), the subgraph induced by the set $V_e = \{A + N_1, B + N_2 : N_1, N_2 \in M_n(I)\}$ in $\Gamma_{I^t}(M_n(R))$ is

$$\langle V_e \rangle = \begin{cases} K_{|M_n(I)|,|M_n(I)|} & \text{if } \mathrm{Tr}(A^2), \mathrm{Tr}(B^2) \notin I; \\ K_{|M_n(I)|,|M_n(I)|} \oplus 2K_{|M_n(I)|} & \text{if } \mathrm{Tr}(A^2), \mathrm{Tr}(B^2) \in I; \\ K_{|M_n(I)|,|M_n(I)|} \oplus K_{|M_n(I)|} & \text{if either } \mathrm{Tr}(A^2) \text{ or } \mathrm{Tr}(B^2) \in I. \end{cases}$$

By [4, p.192], we have $K_{|M_n(I)|,|M_n(I)|} \cong M_1^{(e)} \oplus \cdots \oplus M_{|M_n(I)|}^{(e)}$, where each of $M_i^{(e)}$ is a perfect matching of $K_{|M_n(I)|,|M_n(I)|}$. Thus,

$$\langle V_e \rangle = \begin{cases} M_1^{(e)} \oplus \cdots \oplus M_{|M_n(I)|}^{(e)} & \text{if } \mathrm{Tr}(A^2), \mathrm{Tr}(B^2) \notin I; \\ M_1^{(e)} \oplus \cdots \oplus M_{|M_n(I)|}^{(e)} \oplus 2K_{|M_n(I)|} & \text{if } \mathrm{Tr}(A^2), \mathrm{Tr}(B^2) \in I; \\ M_1^{(e)} \oplus \cdots \oplus M_{|M_n(I)|}^{(e)} \oplus K_{|M_n(I)|} & \text{if either } \mathrm{Tr}(A^2) \text{ or } \mathrm{Tr}(B^2) \in I. \end{cases}$$

Note that $H_i = \bigoplus_{e \in E(\Gamma_t M_n(R/I))} M_i^{(e)}$ is a subgraph of $\Gamma_{I^t}(M_n(R))$ and H_i can be divided into $|M_n(I)|$ edge disjoint subgraphs each isomorphic to $\Gamma_t(M_n(R/I))$, i.e., $H_i \cong |M_n(I)|\Gamma_t(M_n(R/I))$.

Clearly $H = H_1 \oplus \cdots \oplus H_{|M_n(I)|}$ is a subgraph with vertex set $M_n(R) \setminus M_n(I)$ and $H \cong |M_n(I)|^2 \Gamma_t(M_n(R/I))$. Thus $\Gamma_{I^t}(M_n(R)) \cong |M_n(I)|^2 \Gamma_t(M_n(R/I)) \oplus \lambda K_{|M_n(I)|}$ where

$\lambda = |\{\overline{A} \in V(\Gamma_t(M_n(R/I)) : \mathrm{Tr}(A^2) \in I \text{ and } A \text{ is a coset representative of } \overline{A}\}|$. ☐

4. Chromatic, clique and independence numbers of $\Gamma_{I^t}(M_n(R))$

In this section, we obtain bounds for the clique, chromatic and independence numbers of $\Gamma_{I^t}(M_n(R))$ and obtain a condition for the chromatic and clique numbers of $\Gamma_{I^t}(M_n(R))$ to be equal.

Theorem 4.1. *Let $n \geq 2$ be an integer, R be a commutative ring and I be a non trivial ideal of R. Then the following hold:*

1. $\omega(\Gamma_t(M_n(R/I))) \leq \omega(\Gamma_{I^t}(M_n(R))) \leq |M_n(I)|\omega(\Gamma_t(M_n(R/I)))$. *Moreover, the equality $\omega(\Gamma_{I^t}(M_n(R))) = |M_n(I)|\omega(\Gamma_t(M_n(R/I)))$ holds if there exists a clique of maximum order in $\Gamma_t(M_n(R/I))$ such that $Tr(A^2) \in I$ for every vertex $\overline{A}$ in the clique.*
2. $\chi(\Gamma_t(M_n(R/I))) \leq \chi(\Gamma_{I^t}(M_n(R))) \leq |M_n(I)|\chi(\Gamma_t(M_n(R/I)))$.

***Proof.* 1.** The first inequality follows from the fact that $\Gamma_t(M_n(R/I))$ is a subgraph of $\Gamma_{I^t}(M_n(R))$. Let $\omega(\Gamma_t(M_n(R/I))) = k$. To conclude the proof, it is enough to prove that $\omega(\Gamma_{I^t}(M_n(R))) \leq k|M_n(I)|$. Since $\Gamma_t(M_n(R/I)) \cong \Gamma_t(M_n(R)/M_n(I))$, we have $\omega(\Gamma_t(M_n(R)/M_n(I))) = k$.

Suppose there exists a clique of order $k|M_n(I)| + 1$ in $\Gamma_{I^t}(M_n(R))$. Let $\{B_1, \ldots, B_{k|M_n(I)|+1}\}$ be a clique in $\Gamma_{I^t}(M_n(R))$. Consider the set

$$X = \{B_1 + M_n(I), \ldots, B_{k|M_n(I)|+1} + M_n(I)\} \subseteq V(\Gamma_t(M_n(R)/M_n(I))).$$

Since B_i is adjacent to B_j in $\Gamma_{I^t}(M_n(R))$, for $i \neq j$, either $B_i + M_n(I) = B_j + M_n(I)$ or $B_i + M_n(I)$ is adjacent to $B_j + M_n(I)$ in $\Gamma_t(M_n(R)/M_n(I))$. Since $|B_i + M_n(I)| = |M_n(I)|$ we have at least $k+1$ distinct elements in X such that the $k+1$ elements are adjacent to each other in $\Gamma_t(M_n(R)/M_n(I))$. Thus $\omega(\Gamma_t(M_n(R)/M_n(I))) \geq k+1$, which is a contradiction. Hence $\omega(\Gamma_{I^t}(M_n(R))) \leq k|M_n(I)|$. The moreover case is clear from the preceding arguments and Theorem 3.3(1) and (5).

2. The first inequality is clear since $\Gamma_t(M_n(R/I))$ is a subgraph of $\Gamma_{I^t}(M_n(R))$. Let $\chi(\Gamma_t(M_n(R/I))) = k$ and $C_1, \ldots, C_k$ be the color classes of $\Gamma_t(M_n(R/I))$. Consider $\overline{A} \in \Gamma_t(M_n(R/I))$ belongs to the color class C_1 and the set $X_A = \{[a_{ij}] \in \Gamma_{I^t}(M_n(R)) : [a_{ij} + I] = \overline{A}\}$. Note that $|X_A| = |M_n(I)|$. Assign $|M_n(I)|$ distinct colors $C_{11}, \ldots, C_{1|M_n(I)|}$ to the vertices of X_A. Assign the same colors $C_{11}, \ldots, C_{1|M_n(I)|}$ for the vertices arising out of other vertices $\overline{B} \in \Gamma_t(M_n(R/I))$ belonging to the color class C_1. Since $\overline{A}$ is not adjacent to $\overline{B}$ no vertex of X_A is adjacent to X_B. Similarly for $2 \leq i \leq k$ assigning colors, $C_{i1}, \ldots, C_{i|M_n(I)|}$ to the vertices of $\Gamma_{I^t}(M_n(R))$ arising out of the vertices of the color class C_i we have $k|M_n(I)|$ colors and the coloring is proper. Thus $\chi(\Gamma_{I^t}(M_n(R))) \leq k|M_n(I)|$. □

The following theorem is a generalization of the moreover case of Theorem 4.1(1).

Theorem 4.2. *Let $n \geq 2$ be an integer, R be a commutative ring and I be a non trivial ideal in R. Let S be a clique of maximum order in $\Gamma_t(M_n(R/I))$ and S have the largest number of elements $\overline{A}$ with $Tr(A^2) \in I$. Let $X = \{\overline{A} \in S : Tr(A^2) \notin I\}$. Then $\omega(\Gamma_{I^t}(M_n(R))) = |X| + |M_n(I)|(\omega(\Gamma_t(M_n(R/I))) - |X|)$.*

Proof. Let $|X| = |\{\overline{A} \in S : \mathrm{Tr}(A^2) \notin I\}| = k_1$, $|\{\overline{A} \in S : \mathrm{Tr}(A^2) \in I\}| = k_2$ and $\omega(\Gamma_t(M_n(R/I))) = |S| = k$. Then $k_1 + k_2 = k$. In view of Note 3.2, $\Gamma_t(M_n(R/I)) \cong \Gamma_t(M_n(R)/M_n(I))$ and so $\omega(\Gamma_t(M_n(R)/M_n(I))) = k$.

Further by our assumption on S, any maximal clique of $\Gamma_t(M_n(R/I))$ and hence of $\Gamma_t(M_n(R)/M_n(I))$ can have at most k_2 number of vertices $\overline{A}$ with $\mathrm{Tr}(A^2) \in I$.

Hence one can take the clique corresponding to S of $\Gamma_t(M_n(R/I))$ as a clique $< \{A_1 + M_n(I), \ldots, A_k + M_n(I)\} >$ of $\Gamma_t(M_n(R)/M_n(I))$ with $\mathrm{Tr}(A_i^2) \in I$ for $1 \leq i \leq k_2$ and $\mathrm{Tr}(A_i^2) \notin I$ for $k_2 + 1 \leq i \leq k$. Clearly the set $\{A_{ij} : A_{ij} \in A_i + M_n(I), 1 \leq i \leq$

k_2 and $1 \leq j \leq |M_n(I)|\} \cup \{A_{i2} = A_i : k_2+1 \leq i \leq k\}$ is a clique of size $|M_n(I)|k_2 + k_1$ in $\omega(\Gamma_{I^t}(M_n(R)))$.

Hence $\omega(\Gamma_{I^t}(M_n(R))) \geq k_1 + |M_n(I)|k_2$.

To prove our result, it is enough to prove that $\omega(\Gamma_{I^t}(M_n(R))) \leq k_1 + |M_n(I)|k_2$. Suppose $\Gamma_{I^t}(M_n(R))$ has a clique S' of order $k_1 + |M_n(I)|k_2 + 1$. Without loss of generality, we may assume that S' is a maximal clique of order $\geq k_1 + |M_n(I)|k_2 + 1$ in $\Gamma_{I^t}(M_n(R))$. Let $A \in S'$ with $\mathrm{Tr}(A^2) \notin I$.

By Theorem 3.3 (3), no vertex in the set $\{A+B : B \in M_n(I)^*\}$ is adjacent to A. Hence $\{A+B : B \in M_n(I)^*\}$ has no intersection with S'. Also note that due to the maximality of the clique S', if $A \in S'$ with $\mathrm{Tr}(A^2) \in I$ then by Theorem 3.3 (2) and (4), the set $\{A+B : B \in M_n(I)\} \subset S'$.

If S' contains at least $k_2|M_n(I)|+1$ vertices with $\mathrm{Tr}(A^2) \in I$, then by Theorem 3.3 (2), the clique S'_I of $\Gamma_t(M_n(R)/M_n(I))$ with respect to S' contains at least k_2+1 vertices with $\mathrm{Tr}(A^2) \in I$ which is a contradiction to our assumption that among the cliques of $\Gamma_t(M_n(R/I))$, S has the largest number of elements $\overline{A}$ with $\mathrm{Tr}(A^2) \in I$.

Hence the number of vertices in S' with $\mathrm{Tr}(A^2) \in I$ is less than or equal to $k_2|M_n(I)|$. i.e., S' contains at least k_1+1 vertices with $\mathrm{Tr}(A^2) \notin I$. Now, the clique S'_I of $\Gamma_t(M_n(R)/M_n(I))$ corresponding to S' contains at least $k_2 + k_1 + 1$ vertices which is a contradiction to $\omega(\Gamma_t(M_n(R/I))) = k$. Thus $\omega(\Gamma_{I^t}(M_n(R))) \leq k_1 + |M_n(I)|k_2$ and hence $\omega(\Gamma_{I^t}(M_n(R))) = k_1 + |M_n(I)|k_2$. ☐

Theorem 4.3. *Let $n \geq 2$ be an integer, R be a commutative ring and I be a non trivial ideal of R. Let $\Gamma_t(M_n(R/I))$ contain a clique of maximum order such that $Tr(A^2) \in I$ for every $\overline{A}$ in the clique. If $\chi(\Gamma_t(M_n(R/I))) = \omega(\Gamma_t(M_n(R/I)))$, then $\chi(\Gamma_{I^t}(M_n(R))) = \omega(\Gamma_{I^t}(M_n(R)))$.*

Proof. Firstly, let us assume that $\chi(\Gamma_t(M_n(R/I))) = \omega(\Gamma_t(M_n(R/I))) = k$. From this we have $\chi(\Gamma_t(M_n(R)/M_n(I))) = \omega(\Gamma_t(M_n(R)/M_n(I))) = k$.

Let $\{A_1 + M_n(I), \ldots, A_k + M_n(I)\}$ be a clique of order k in $\Gamma_t(M_n(R)/M_n(I))$ such that $\mathrm{Tr}(A_i^2) \in I,\ 1 \leq i \leq k$. Let $\{c_1, \ldots, c_k\}$ be a set of minimum colors required for a proper coloring of the graph $\Gamma_t(M_n(R)/M_n(I))$. Without loss of generality assume that $A_i + M_n(I)$ is colored by the color c_i. Since $A_i^2 \in I$, the set $X = \{A \in \Gamma_{I^t}(M_n(R)) : A \in A_i + M_n(I)$ for some $i \in \{1, \ldots, k\}\}$ forms a clique of order $|M_n(I)|k$ in $\Gamma_{I^t}(M_n(R))$.

By Theorem 4.1(1), this clique is maximum and $\omega(\Gamma_{I^t}(M_n(R))) = k|M_n(I)|$. Assign $k|M_n(I)|$ distinct colors $c'_1, \ldots, c'_{k|M_n(I)|}$ to the vertices in the set X.

For a vertex $B \in V(\Gamma_{I^t}(M_n(R))) \setminus X$, there exists $M \in M_n(I)$ such that $B = B_\ell + M \in B_\ell + M_n(I)$ for some $B_\ell + M_n(I) \notin \{A_1 + M_n(I), \ldots, A_k + M_n(I)\}$. Let c_j be the color of $B_\ell + M_n(I)$ in $\Gamma_t(M_n(R)/M_n(I))$. Note that $A_j + M_n(I)$ belongs to the color class c_j in $\Gamma_t(M_n(R)/M_n(I))$.

Assign the color of $A_j + M$ in $\Gamma_{I^t}(M_n(R))$ to $B = B_\ell + M$ in $\Gamma_{I^t}(M_n(R))$. Let $C \in V(\Gamma_{I^t}(M_n(R))) \setminus X$ be adjacent to $B \in V(\Gamma_{I^t}(M_n(R))) \setminus X$. Then $B_\ell + M_n(I)$ is adjacent to $C + M_n(I)$ in $\Gamma_t(M_n(R)/M_n(I))$ and hence they belong to different color classes in $\Gamma_t(M_n(R)/M_n(I))$ and so B and C belong to different color classes in $\Gamma_{I^t}(M_n(R))$.

Thus we have given a proper coloring for the graph $\Gamma_{I^t}(M_n(R))$ with $k|M_n(I)|$ colors and so $\chi(\Gamma_{I^t}(M_n(R))) \leq k|M_n(I)|$. Since $k|M_n(I)| = \omega(\Gamma_{I^t}(M_n(R))) \leq \chi(\Gamma_{I^t}(M_n(R)))$, $\chi(\Gamma_{I^t}(M_n(R))) = k|M_n(I)|$. ☐

Theorem 4.4. *Let $n \geq 2$ be an integer, R be a commutative ring and I be a non trivial ideal of R. Then*

1. $\alpha(\Gamma_t(M_n(R/I))) \leq \alpha(\Gamma_{I^t}(M_n(R))) \leq |M_n(I)|\alpha(\Gamma_t(M_n(R/I)))$;
2. *In particular, if there exists an independent set of maximum order in $\Gamma_t(M_n(R/I))$ such that $Tr(A^2) \notin I$ for every vertex A in the independent set, then $\alpha(\Gamma_{I^t}(M_n(R)))$ $= |M_n(I)|\alpha(\Gamma_t(M_n(R/I)))$.*

Proof. **1.** Let $\alpha(\Gamma_t(M_n(R/I))) = k$ and X be the corresponding maximum independent set of $\Gamma_t(M_n(R/I))$. Consider the set $X_1 = \{A : \overline{A} \in X\} \subseteq V(\Gamma_{I^t}(M_n(R)))$. By the Theorem 3.3(2), we have that X_1 is an independent set of order k in $\Gamma_{I^t}(M_n(R))$. Hence $\alpha(\Gamma_t(M_n(R/I))) \leq \alpha(\Gamma_{I^t}(M_n(R)))$. By Note 3.2, we have $\alpha(\Gamma_t(M_n(R)/M_n(I))) = k$.

Suppose that there exists an independent set of order $k|M_n(I)| + 1$ in $\Gamma_{I^t}(M_n(R))$. Let $\{B_1, \ldots, B_{k|M_n(I)|+1}\}$ be an independent set in $\Gamma_{I^t}(M_n(R))$. Consider the set $X = \{B_1 + M_n(I), \ldots, B_{k|M_n(I)|+1} + M_n(I)\} \subseteq V(\Gamma_t(M_n(R)/M_n(I)))$.

Note that for $i \neq j$, $B_i + M_n(I) = B_j + M_n(I)$ or $B_i + M_n(I)$ is not adjacent to $B_j + M_n(I)$ in $\Gamma_t(M_n(R)/M_n(I))$. Since $|B_i + M_n(I)| = |M_n(I)|$ we have at least $k + 1$ distinct elements in X such that the $k + 1$ elements are not adjacent to each other in $\Gamma_t(M_n(R)/M_n(I))$,i.e., $\alpha(\Gamma_t(M_n(R)/M_n(I))) \geq k + 1$ which is a contradiction. Hence $\alpha(\Gamma_{I^t}(M_n(R))) \leq k|M_n(I)|$.

2. From Theorem 3.3(2) and (3), we have $\alpha(\Gamma_{I^t}(M_n(R))) \geq |M_n(I)|\alpha(\Gamma_t(M_n(R/I)))$. By the previous part, we have $\alpha(\Gamma_{I^t}(M_n(R))) = |M_n(I)|\alpha(\Gamma_t(M_n(R/I)))$. □

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