Representations and $T^*$-extensions of $\delta$-Bihom-Jordan-Lie algebras

Abdelkader Ben Hassine$^{1,2}$, Liangyun Chen$^3$, Juan Li$^3$

$^1$ Department of Mathematics, Faculty of Science and Arts at Belqarn, University of Bisha, Kingdom of Saudi Arabia
$^2$ Faculty of Sciences, University of Sfax, BP 1171, 3000 Sfax, Tunisia
$^3$ School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

Abstract
The purpose of this article is to study representations of $\delta$-Bihom-Jordan-Lie algebras. In particular, adjoint representations, trivial representations, deformations, $T^*$-extensions of $\delta$-Bihom-Jordan-Lie algebras are studied in detail. Derivations and central extensions of $\delta$-Bihom-Jordan-Lie algebras are also discussed as an application.

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1. Introduction
The notion of Jordan-Lie algebras was introduced in [7], which is closely related to both Lie and Jordan superalgebras. Engel’s theorem of Jordan-Lie algebras was proved, and some properties of Cartan subalgebras of Jordan-Lie algebras were given in [8].

Recently, the definition of $\delta$-hom-Jordan-Lie algebras were introduced in [10], and their representations and $T^*$-extensions were studied in detail.

A Bihom-algebra is an algebra in such a way that the identities defining the structure are twisted by two homomorphisms $\alpha$, $\beta$. This class of algebras was introduced from a categorical approach in [4] as an extension of the class of Hom-algebras. The origin of Hom-structures can be found in the physics literature around 1900, appearing in the study of quasi deformations of Lie algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras in [5]. Since then, many authors have been interested in the study of Hom-algebras, mainly motivated by their applications in mathematical physics (see for instance the recent references [1,6]). The fundamental for getting the basic notions, motivations, and results on Bihom-algebras is the reference [4].

More applications of the Bihom-Lie algebras, Bihom-algebras, Bihom-Lie superalgebras and Bihom-Lie admissible superalgebras can be found in [3,9].

The notion of derivations, representations, and $T^*$-extensions of $\delta$-Bihom-Jordan Lie algebras are not so well developed.

*Corresponding Author.
Email addresses: benhassine.abdelkader@yahoo.fr (A. Ben Hassine), chenly640@nenu.edu.cn (L. Chen), lij355@nenu.edu.cn (J. Li)
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The paper is organized as follows. In Section 2 we give the definition of $\delta$-Bihom-Jordan-Lie algebras, and show that the direct sum of two $\delta$-Bihom-Jordan-Lie algebras is still a $\delta$-Bihom-Jordan-Lie algebra. A linear map between $\delta$-Bihom-Jordan-Lie algebras is a morphism if and only if its graph is a Bihom subalgebra. In Section 3 we study derivations of multiplicative $\delta$-Bihom-Jordan-Lie algebras. For any nonnegative integers $k$ and $l$, we define $\alpha^k\beta^l$-derivations of multiplicative $\delta$-Bihom-Jordan-Lie algebras. Considering the direct sum of the space of $\alpha^k\beta^l$-derivations, we prove that it is a Lie algebra. In particular, any $\alpha^k\beta^l$-derivation gives rise to a derivation extension of the multiplicative $\delta$-hom-Jordan-Lie algebra $(L, [\cdot, \cdot], \alpha, \beta)$ (Theorem 3.3). In Section 4 we give the definition of representations of multiplicative $\delta$-Bihom-Jordan-Lie algebras. We can obtain the semidirect product multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L \oplus M, [\cdot, \cdot], \alpha + \alpha_M, \beta + \beta_M)$ associated to any representation $\rho$ on $M$ of the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot], \alpha, \beta)$. In Section 5 we study trivial representations of multiplicative $\delta$-Bihom-Jordan-Lie algebras. We show that central extensions of a multiplicative $\delta$-Bihom-Jordan-Lie algebra are controlled by the second cohomology with coefficients in the trivial representation. In Section 6 we study the adjoint representation of a regular $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot], \alpha, \beta)$. For any integers $s, t$, we define the $\alpha^s\beta^t$-derivations. We show that a 1-cocycle associated to the $\alpha^s\beta^t$-derivation is exactly an $\alpha^{s+2}\beta^{t-1}$-derivation of the regular $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot], \alpha, \beta)$ in some conditions. We also give the definition of Bihom-Nijenhuis operators of regular $\delta$-Bihom-Jordan-Lie algebras. We show that the deformation generated by a Bihom-Nijenhuis operator is trivial. In Section 7 we study $T^*$-extensions of $\delta$-Bihom-Jordan-Lie algebras, show that $T^*$-extensions preserve many properties such as nilpotency, solvability and decomposition in some sense.

2. Definitions and properties of $\delta$-Bihom-Jordan-Lie algebras

**Definition 2.1.** ([7]) A $\delta$-Jordan Lie algebra is a couple $(L, [\cdot, \cdot], \delta)$ consisting of a vector space $L$ and a bilinear map (bracket) $[\cdot, \cdot]: L \times L \to L$ satisfying

\[ [x, y] = -\delta[y, x], \quad \delta = \pm 1, \]
\[ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad \forall x, y, z \in L. \]

**Definition 2.2.** ([10]) A $\delta$-hom-Jordan Lie algebra is a triple $(L, [\cdot, \cdot], \alpha)$ consisting of a vector space $L$, a bilinear map (bracket) $[\cdot, \cdot]: L \otimes L \to L$ and a linear map $\alpha: L \to L$ satisfying

\[ [x, y] = -\delta[y, x], \quad \delta = \pm 1, \]
\[ [\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0, \quad \forall x, y, z \in L. \]

Especially, for $\delta = 1$ one has a hom-Lie algebra and for $\delta = -1$ a hom-Jordan Lie algebra.

**Definition 2.3.** ([3]) A Bihom-Lie algebra is a 4-tuple $(L, [\cdot, \cdot], \alpha, \beta)$ consisting of vector space $L$, a bilinear map $[\cdot, \cdot]: L \times L \to L$ and two homomorphisms $\alpha, \beta: L \to L$ such that for all elements $x, y, z \in L$ we have

\[ \alpha \circ \beta = \beta \circ \alpha, \]
\[ [\beta(x), \alpha(y)] = -[\beta(y), \alpha(x)], \]
\[ [\beta^2(x), [\beta(y), \alpha(z)]] + [\beta^2(y), [\beta(z), \alpha(x)]] + [\beta^2(z), [\beta(x), \alpha(y)]] = 0 \]

(Bihom-Jacobi equation).

**Definition 2.4.** A $\delta$-Bihom-Jordan Lie algebra is a 4-tuple $(L, [\cdot, \cdot], \alpha, \beta)$ consisting of a vector space $L$, a bilinear map (bracket) $[\cdot, \cdot]: L \otimes L \to L$ and two linear maps
\( \alpha, \beta : L \rightarrow L \) satisfying
\[
\alpha \circ \beta = \beta \circ \alpha, \quad (2.1)
\]
\[
[\beta(x), \alpha(y)] = -\delta[\beta(y), \alpha(x)], \delta = \pm 1, \quad (2.2)
\]
\[
[\beta^2(x), [\beta(y), \alpha(z)]] + [\beta^2(y), [\beta(z), \alpha(x)]] + [\beta^2(z), [\beta(x), \alpha(y)]] = 0, \forall x, y, z \in L. \quad (2.3)
\]

Especially, for \( \delta = 1 \) one has a Bihom-Lie algebra and for \( \delta = -1 \) a Bihom-Jordan Lie algebra.

**Definition 2.5.**
1) A \( \delta \)-Bihom-Jordan Lie algebra \( (L, [\cdot, \cdot], \alpha, \beta) \) is multiplicative if \( \alpha \) and \( \beta \) are algebra morphisms, i.e., for any \( x, y \in L \), we have
\[
\alpha([x, y]_L) = [\alpha(x), \alpha(y)]_L \quad \text{and} \quad \beta([x, y]_L) = [\beta(x), \beta(y)]_L.
\]
2) A \( \delta \)-Bihom-Jordan Lie algebra \( (L, [\cdot, \cdot], \alpha, \beta) \) is regular if \( \alpha \) and \( \beta \) are algebra automorphisms.
3) A subvector space \( \eta \in L \) is a Bihom subalgebra of \( (L, [\cdot, \cdot], \alpha, \beta) \) if \( \alpha(\eta) \in \eta, \beta(\eta) \in \eta \) and
\[
[x, y]_L \in \eta, \quad \forall x, y \in \eta.
\]
4) A subvector space \( \eta \in L \) is a Bihom ideal of \( (L, [\cdot, \cdot], \alpha, \beta) \) if \( \alpha(\eta) \in \eta, \beta(\eta) \in \eta \) and
\[
[x, y]_L \in \eta, \quad \forall x \in \eta, y \in L.
\]

**Definition 2.6.** A \( \delta \)-Bihom associative algebra is a triple \( (L, \alpha, \beta) \) consisting of a vector space \( L \), a bilinear map on \( L \), and two linear commuting maps \( \alpha, \beta : L \rightarrow L \) satisfying
\[
\alpha(x)(yz) = \delta(x)y\beta(z), \quad \forall x, y, z \in L. \quad (2.4)
\]

**Proposition 2.7.** Let \( (L, \alpha, \beta) \) be a multiplicative \( \delta \)-Bihom associative algebra. Define a bilinear map (bracket) \( [\cdot, \cdot]_L : L \times L \rightarrow L \) satisfying
\[
[x, y]_L = xy - \delta\alpha^{-1}(\beta(y))\beta^{-1}(\alpha(x)), \forall x, y \in L. \quad (2.5)
\]
Then \( (L, [\cdot, \cdot]_L, \alpha, \beta) \) is a \( \delta \)-Bihom-Jordan-Lie algebra.

**Proof.** First we check that the bracket product \( [\cdot, \cdot] \) is compatible with the structure maps \( \alpha \) and \( \beta \). For any \( x, y \in L \), we have
\[
[\alpha(x), \alpha(y)] = \alpha(x)\alpha(y) - \delta(\alpha^{-1}\beta(\alpha(y)))(\alpha\beta^{-1}(\alpha(x)))
\]
\[
= \alpha(x)\alpha(y) - \delta\beta(y)(\alpha\beta^{-1}(\alpha(x)))
\]
\[
= \alpha([x, y]).
\]
Similarly, one can prove that \( \beta([x, y]) = [\beta(x), \beta(y)] \).

And
\[
[\beta(x), \alpha(y)] = \beta(x)\alpha(y) - \delta(\alpha^{-1}\beta(\alpha(y)))(\alpha\beta^{-1}(\beta(x)))
\]
\[
= \beta(x)\alpha(y) - \delta\beta(\alpha(y))(\alpha\beta^{-1}(\alpha(x)))
\]
\[
= -\delta[\beta(y), \alpha(x)].
\]

Now we prove the Bihom-Jacobi condition. For any elements \( x, y \in L \), we have
\[
[\beta^2(x), [\beta(y), \alpha(z)]] = [\beta^2(x), \beta(y)\alpha(z) - \delta\alpha^{-1}\beta(\alpha(z))\alpha\beta^{-1}(\beta(y))]
\]
\[
= [\beta^2(x), \beta(y)\alpha(z)] - \delta[\beta^2(x), \beta(z)\alpha(y)]
\]
\[
= \left( \beta^2(x)\beta(y)\alpha(z) - \delta(\alpha^{-1}(\beta^2(y))\beta(z)\alpha\beta(x)) \right)
\]
\[
- \delta \left( \beta^2(x)(\beta(z)\alpha(y)) - \delta(\alpha^{-1}(\beta^2(z))\beta(y)\alpha\beta(x)) \right).
\]
Similarly, we have
\[
\begin{align*}
\{\beta^2(y), [\beta(z), \alpha(x)]\} &= \left(\beta^2(y)(\beta(z)\alpha(x)) - \delta(\alpha^{-1}(\beta^2(z))\beta(x))\alpha(\beta(y))\right) \\
&\quad - \delta\left(\beta^2(y)(\beta(x)\alpha(z)) - \delta(\alpha^{-1}(\beta^2(x))\beta(z))\alpha(\beta(y))\right).
\end{align*}
\]
\[
\begin{align*}
\{\beta^2(z), [\beta(x), \alpha(y)]\} &= \left(\beta^2(z)(\beta(x)\alpha(y)) - \delta(\alpha^{-1}(\beta^2(x))\beta(y))\alpha(\beta(z))\right) \\
&\quad - \delta\left(\beta^2(z)(\beta(y)\alpha(x)) - \delta(\alpha^{-1}(\beta^2(y))\beta(x))\alpha(\beta(z))\right).
\end{align*}
\]
Note that
\[
\begin{align*}
\beta^2(x)(\beta(y)\alpha(z)) &= \delta(\alpha^{-1}(\beta^2(x))\beta(y))\alpha(\beta(z)), \\
\beta^2(y)(\beta(x)\alpha(z)) &= \delta(\alpha^{-1}(\beta^2(x))\beta(x))\alpha(\beta(z)), \\
\beta^2(x)(\beta(z)\alpha(y)) &= \delta(\alpha^{-1}(\beta^2(x))\beta(z))\alpha(\beta(y)), \\
\beta^2(y)(\beta(z)\alpha(x)) &= \delta(\alpha^{-1}(\beta^2(y))\beta(z))\alpha(\beta(x)), \\
\beta^2(z)(\beta(x)\alpha(y)) &= \delta(\alpha^{-1}(\beta^2(z))\beta(x))\alpha(\beta(y)), \\
\beta^2(z)(\beta(y)\alpha(x)) &= \delta(\alpha^{-1}(\beta^2(z))\beta(y))\alpha(\beta(x)).
\end{align*}
\] Then we obtain \([\beta^2(x), [\beta(y), \alpha(z)] + [\beta^2(y), [\beta(z), \alpha(x)] + [\beta^2(z), [\beta(x), \alpha(y)] = 0.\]

**Proposition 2.8.** Given two \(\delta\)-Bihom-Jordan-Lie algebras \((L, \cdot, L, \alpha_1, \beta_1)\) and \((L', \cdot, L', \alpha_2, \beta_2)\), there is a \(\delta\)-Bihom-Jordan-Lie algebra \((L \oplus L', \cdot, L \oplus L', \alpha_1 + \alpha_2, \beta_1 + \beta_2)\), where the bilinear map \([\cdot, \cdot]_{L \oplus L'} : L \oplus L' \times L \oplus L' \to L \oplus L'\) is given by
\[
[u_1 + v_1, u_2 + v_2]_{L \oplus L'} = [u_1, v_1]_L + [u_2, v_2]_{L'}, \forall u_1, u_2 \in L, v_1, v_2 \in L',
\]
and the two linear maps \(\alpha_1 + \alpha_2, \beta_1 + \beta_2 : L \oplus L' \to L \oplus L'\) defined by
\[
\begin{align*}
(\alpha_1 + \alpha_2)(u_1 + v_1) &= \alpha_1(u_1) + \alpha_2(v_1), \\
(\beta_1 + \beta_2)(u_1 + v_1) &= \beta_1(u_1) + \beta_2(v_1).
\end{align*}
\]

**Proof.** For any \(u_1, u_2, u_3 \in L\) and \(v_1, v_2, v_3 \in L'\) we have:
\[
\begin{align*}
[([\beta_1 + \beta_2](u_1 + v_1), (\alpha_1 + \alpha_2)(u_2 + v_2)]_{L \oplus L'} \\
&= [\beta_1(u_1), \alpha_1(u_2)]_L + [\beta_2(u_1), \alpha_2(u_2)]_L - \delta[\beta_1(u_2), \alpha_1(u_1)]_L - \delta[\beta_2(u_2), \alpha_2(u_1)]_L' \\
&= -\delta([\beta_1(u_2), \alpha_1(u_1)]_L + [\beta_2(u_2), \alpha_2(u_1)]_L' \\
&= -\delta([\beta_1 + \beta_2](u_2 + v_2), (\alpha_1 + \alpha_2)(u_1 + v_1)]_{L \oplus L'}.
\end{align*}
\]
\[
\begin{align*}
(\alpha_1 + \alpha_2) \circ (\beta_1 + \beta_2)(u_1 + v_1) \\
&= (\alpha_1 + \alpha_2)(\beta_1(u_1) + \beta_2(v_1)) = \alpha_1 \beta_1(u_1) + \alpha_2 \beta_2(v_1) \\
&= \beta_1 \circ \alpha_1(u_1) + \beta_2 \circ \alpha_2(v_1) \\
&= (\beta_1 + \beta_2) \circ (\alpha_1 + \alpha_2)(u_1 + v_1).
\end{align*}
\]
Then, we have \((\alpha_1 + \alpha_2) \circ (\beta_1 + \beta_2) = (\beta_1 + \beta_2) \circ (\alpha_1 + \alpha_2)\). By a direct computation, we have
\[
\begin{align*}
\circ_{(u_1 + v_1), (u_2 + v_2), (u_3 + v_3)} ([\beta_1 + \beta_2](u_1 + v_1), ([\beta_1 + \beta_2](u_2 + v_2), (\alpha_1 + \alpha_2)(u_3 + v_3)]_{L \oplus L'} & L \oplus L' \\
&= \circ_{(u_1 + v_1), (u_2 + v_2), (u_3 + v_3)} ([\beta_1^2(u_1) + \beta_2^2(v_1), [\beta_1(u_2), \alpha_1(u_3)]_L + [\beta_2(v_2), \alpha_2(v_3)]_L]_{L \oplus L'} \\
&= \circ_{u_1, u_2, u_3} [\beta_1^2(u_1), [\beta_1(u_2), \alpha_1(u_3)]_L]_L + \circ_{v_1, v_2, v_3} [\beta_2^2(v_1), [\beta_1(v_2), \alpha_1(v_3)]_L]_{L'} \\
&= 0,
\end{align*}
\]
where \(\circ_{x, y, z}\) denotes summation over the cyclic permutation on \(x, y, z\). \(\Box\)
Definition 2.9. Let \((L, [[ , ]]_L, \alpha_1, \beta_1)\) and \((L′, [[ , ]]_{L′}, \alpha_2, \beta_2)\) be two \(\delta\)-Bihom-Jordan-Lie algebras. A linear map \(\phi : L \to L′\) is said to be a morphism of \(\delta\)-Bihom-Jordan-Lie algebras if
\[
\phi[u, v]_L = [\phi(u), \phi(v)]_{L′}, \forall u, v \in L, \tag{2.6}
\]
\[
\phi \circ \alpha_1 = \beta_1 \circ \phi, \tag{2.7}
\]
\[
\phi \circ \alpha_2 = \beta_2 \circ \phi. \tag{2.8}
\]
Denote by \(\mathcal{G}_\phi \in L \oplus L′\) is the graph of a linear map \(\phi : L \to L′\).

Proposition 2.10. A map \(\phi : (L, [[ , ]]_L, \alpha_1, \beta_1) \to (L′, [[ , ]]_{L′}, \alpha_2, \beta_2)\) is a morphism of \(\delta\)-Bihom-Jordan-Lie algebras if and only if the graph \(\mathcal{G}_\phi \in L \oplus L′\) is a Bihom subalgebra of \((L \oplus L′, [[ , ]]_{L \oplus L′}, \alpha_1 + \alpha_2, \beta_1 + \beta_2)\).

Proof. Let \(\phi : (L, [[ , ]]_L, \alpha_1, \beta_1) \to (L′, [[ , ]]_{L′}, \alpha_2, \beta_2)\) be a morphism of \(\delta\)-Bihom-Jordan-Lie algebras, then for any \(u, v \in L\), we have
\[
[u + \phi(u), v + \phi(v)]_{L \oplus L′} = [u, v]_L + [\phi(u), \phi(v)]_{L′} = [u, v]_L + \phi[u, v]_L.
\]
Then the graph \(\mathcal{G}_\phi\) is closed under the bracket operation \([ , ]_{L \oplus L′}\). So, we obtain
\[
(\alpha_1 + \alpha_2)(u + \phi(u)) = \alpha_1(u) + \alpha_2 \circ \phi(u) = \alpha_1(u) + \phi \circ \alpha_2(u),
\]
and
\[
(\beta_1 + \beta_2)(u + \phi(u)) = \beta_1(u) + \beta_2 \circ \phi(u) = \beta_1(u) + \phi \circ \beta_2(u),
\]
which implies that \((\alpha_1 + \alpha_2)(\mathcal{G}_\phi) \subset \mathcal{G}_\phi\) and \((\beta_1 + \beta_2)(\mathcal{G}_\phi) \subset \mathcal{G}_\phi\). Then \(\mathcal{G}_\phi\) is a Bihom subalgebra of \((L \oplus L′, [[ , ]]_{L \oplus L′}, \alpha_1 + \alpha_2, \beta_1 + \beta_2)\).

Now, suppose that the graph \(\mathcal{G}_\phi \subset L \oplus L′\) is a Bihom subalgebra of \((L \oplus L′, [[ , ]]_{L \oplus L′}, \alpha_1 + \alpha_2, \beta_1 + \beta_2)\), then we have
\[
[u + \phi(u), v + \phi(v)]_{L \oplus L′} = [u, v]_L + [\phi(u), \phi(v)]_{L′} \in \mathcal{G}_\phi,
\]
which implies that
\[
[\phi(u), \phi(v)]_{L′} = \phi[u, v]_L.
\]
Furthermore, \((\alpha_1 + \alpha_2)(\mathcal{G}_\phi) \subset \mathcal{G}_\phi\) and \((\beta_1 + \beta_2)(\mathcal{G}_\phi) \subset \mathcal{G}_\phi\) implies
\[
(\alpha_1 + \alpha_2)(u + \phi(u)) = \alpha_1(u) + \alpha_2 \circ \phi(u) \in \mathcal{G}_\phi\) and \((\beta_1 + \beta_2)(u + \phi(u)) = \beta_1(u) + \beta_2 \circ \phi(u) \in \mathcal{G}_\phi\).

Which is equivalent to the condition \(\alpha_1 \circ \phi(u) = \phi \circ \beta_1(u)\), and \(\alpha_2 \circ \phi(u) = \phi \circ \beta_2(u)\) i.e.
\[
\alpha_1 \circ \phi = \phi \circ \beta_1
\]
and
\[
\alpha_2 \circ \phi = \phi \circ \beta_2.
\]
Therefore, \(\phi\) is a morphism of \(\delta\)-Bihom-Jordan-Lie algebras.

Example 2.11. Let \((L, [[ , ]]_L, \alpha, \beta) : L \to L\) two commuting linear maps such that \(\alpha([x, y]) = [\alpha(x), \alpha(y)]\) and \(\beta([x, y]) = [\beta(x), \beta(y)]\), for all \(x, y \in L\). Then \((L, [[ , ]]_L, \alpha, \beta)\), where \([x, y]_L = [\alpha(x), \beta(y)]\), is a \(\delta\)-Bihom-Jordan-Lie algebra. Moreover, suppose that \((L′, [[ , ]]_{L′})\) is another \(\delta\)-Jordan-Lie algebra and \(\alpha′, \beta′ : L′ \to L′\) be two algebra endomorphisms. If \(f : L \to L′\) is a \(\delta\)-Jordan-Lie algebra homomorphism that satisfies \(f \circ \alpha = \alpha′ \circ f\) and \(f \circ \beta = \beta′ \circ f\) then \(f : (L, [[ , ]]_L, \alpha, \beta) \to (L′, [[ , ]]_{L′}, \alpha′, \beta′)\) is also a homomorphism of \(\delta\)-Bihom-Jordan-Lie algebras.

Proof. It is easy to show that \((L, [[ , ]]_L, \alpha, \beta)\) satisfies \([\beta(x), \alpha(y)]_L = [\alpha(x), \beta(y)]\) is also a homomorphism of \(\delta\)-Bihom-Jordan-Lie algebras.
Then $(L, [\cdot, \cdot]_L, \alpha, \beta)$ is a $\delta$-Bihom-Jordan-Lie algebra.

The second assertion follows from

$$f([x, y]_L) = f([\alpha(x), \beta(y)]) = [f(\alpha(x)), f(\beta(y))] = [\alpha(f(x)), \beta(f(y))] = [f(x), f(y)]_L.$$ Then $f : (L, [\cdot, \cdot]_L, \alpha, \beta) \rightarrow (L', [\cdot, \cdot]_{L'}, \alpha', \beta')$ is also a homomorphism of $\delta$-Bihom-Jordan-Lie algebras.

**Example 2.12.** A three dimensional linear space $L$ has a basis

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $(L, [\cdot, \cdot])$ is a $\delta$-Jordan-Lie algebra with respect to the product:

$$\left[ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \end{pmatrix}, \begin{pmatrix} 0 & a' & b' \\ 0 & 0 & c' \end{pmatrix} \right] = \delta \begin{pmatrix} 0 & ac & ac' \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & a'c \\ 0 & 0 & 0 \end{pmatrix}.$$

If we define two algebra endomorphisms $\alpha$ and $\beta$ by

$$\alpha(e_1) = \delta e_1, \alpha(e_2) = e_3, \alpha(e_3) = e_2,$$

and

$$\beta(e_1) = \delta e_1, \beta(e_2) = e_3, \beta(e_3) = e_2.$$

Then $(L, \alpha \otimes \beta([\cdot, \cdot]_L) = [\alpha(\cdot), \beta(\cdot)], \alpha, \beta)$ is a $\delta$-Bihom-Jordan-Lie algebra.

**3. Derivations of $\delta$-Bihom-Jordan-Lie algebras**

In this section, we will study derivations of $\delta$-Bihom-Jordan-Lie algebras. Let $(L, [\cdot, \cdot]_L, \alpha, \beta)$ be a multiplicative $\delta$-Bihom-Jordan-Lie algebra. For any nonnegative integers $k, l$, denote by $\alpha^k$ the $k$-times composition of $\alpha$ and $\beta^l$ the $l$-times composition of $\beta$, i.e.

$$\alpha^k = \underbrace{\alpha \circ \cdots \circ \alpha}_{(k\text{-times})}, \quad \beta^l = \underbrace{\beta \circ \cdots \circ \beta}_{(l\text{-times})}.$$ Since the maps $\alpha, \beta$ commute, we denote by

$$\alpha^k \beta^l = \underbrace{\alpha \circ \cdots \circ \alpha}_{(k\text{-times})} \underbrace{\beta \circ \cdots \circ \beta}_{(l\text{-times})}.$$ In particular, $\alpha^0 \beta^0 = 1d, \alpha^1 \beta^1 = \alpha \beta, \alpha^{-k} \beta^{-l}$ is the inverse of $\alpha^k \beta^l$. If $(L, [\cdot, \cdot]_L, \alpha, \beta)$ is a regular $\delta$-Bihom-Jordan-Lie algebra, we denote by $\alpha^{-k}$ the $k$-times composition of $\alpha^{-1}$, the inverse of $\alpha$.

**Definition 3.1.** For any nonnegative integers $k, l$, a linear map $D : L \rightarrow L$ is called an $\alpha^k \beta^l$-derivation of the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$, if

$$[D, \alpha] = 0, \quad i.e. \quad D \circ \alpha = \alpha \circ D, \quad (3.1)$$

$$[D, \beta] = 0, \quad i.e. \quad D \circ \beta = \beta \circ D, \quad (3.2)$$

and

$$D[u, v]_L = \delta^k([D(u), \alpha^k \beta^l(v)]_L + [\alpha^k \beta^l(u), D(v)]_L), \forall u, v \in L. \quad (3.3)$$

For a regular $\delta$-Bihom-Jordan-Lie algebra, $\alpha^{-k} \beta^{-l}$-derivations can be defined similarly.

Note first that if $\alpha$ and $\beta$ are bijective, the skew-symmetry condition (2.3) implies

$$[u, v] = -\delta[\alpha^{-1} \beta(v), \alpha \beta^{-1}(u)]_L, \forall u, v \in L. \quad (3.4)$$

Denote by $\text{Der}_{\alpha^k \beta^l}(L)$ is the set of $\alpha^k \beta^l$-derivations of the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$. For any $u \in L$ satisfying $\alpha(u) = u$, and $\beta(u) = u$,
define $D_{k,l}(u) : L \to L$ by

$$D_{k,l}(u)(v) = -\delta[\alpha^k \beta^l(v), u]_L, \delta^k = 1, \ \forall v \in L.$$ 

By Equation (3.4),

$$D_{k,l}(u)(v) = -\delta[\alpha^k \beta^l(v), u]_L = \delta[\alpha^{-1} \beta(u), \alpha \beta^{-1}(\alpha^k \beta^l(v))]_L = \delta[u, \alpha^{k+1} \beta^{l-1}(v)]_L.$$ 

Then $D_{k,l}(u)$ is an $\alpha^{k+1} \beta^l$-derivation. We call an inner $\alpha^{k+1} \beta^l$-derivation. In fact, we have

$$D_{k,l}(u)(\alpha(v)) = -\delta[\alpha^{k+1} \beta^l(v), u]_L = -\alpha(\delta[\alpha^k \beta^l(v), u]_L) = \alpha \circ D_{k,l}(u)(v).$$

$$D_{k,l}(u)(\beta(v)) = -\delta[\alpha^k \beta^{l+1}(v), u]_L = -\beta(\delta[\alpha^k \beta^l(v), u]_L) = \beta \circ D_{k,l}(u)(v).$$

On the other hand, we have

$$D_{k,l}(u)([v, w]_L) = -\delta[\alpha^k \beta^l([v, w]_L), u]_L = -\delta[[\alpha \beta^l(v), \alpha \beta^l(w), \alpha(u)]_L, \beta^l(v)_L = \delta[\beta^l(v), \alpha \beta^l(w), \alpha(u)]_L - \delta[\alpha \beta^l(v), [\alpha \beta^l(w), \alpha(u)]_L] - \delta[\alpha \beta^l(v), \alpha \beta^l(w), \alpha(u)]_L = -\delta[\alpha \beta^l(v), \alpha \beta^l(w), \alpha(u)]_L - \delta[\alpha \beta^l(v), [\alpha \beta^l(w), \alpha(u)]_L].$$

Therefore, $D_{k,l}(u)$ is an $\alpha^{k+1} \beta^l$-derivation. Denote by $\text{Inn}_{\alpha^k \beta^l}(L)$ the set of inner $\alpha^k \beta^l$-derivations, i.e.

$$\text{Inn}_{\alpha^k \beta^l}(L) = \{-\delta[\alpha^{k-1} \beta^l(\cdot), u]_L | u \in L, \alpha(u) = u, \beta(u) = u, \delta^k = 1\}. \quad (3.5)$$

For any $D \in \text{Der}_{\alpha^k \beta^l}(L)$ and $D' \in \text{Der}_{\alpha^k \beta^l}(L)$, define their commutator $[D, D']$ as usual:

$$[D, D'] = D \circ D' - D' \circ D. \quad (3.6)$$

**Lemma 3.2.** For any $D \in \text{Der}_{\alpha^k \beta^l}(L)$ and $D' \in \text{Der}_{\alpha^k \beta^l}(L)$, we have

$$[D, D'] \in \text{Der}_{\alpha^{k+s} \beta^{l+t}}(L).$$

**Proof.** For any $u, v \in L$, we have

$$[D, D'](u, v)_L = D \circ D'([u, v]_L) - D' \circ D([u, v]_L) = \delta^k D([D(u), \alpha^k \beta^l(v)]_L - \delta^k D'([D(u), \alpha^k \beta^l(v)]_L + [\alpha^k \beta^l(v), D(v)]_L) = \delta^k D([D(u), \alpha^k \beta^l(v)]_L + [D \circ \alpha^k \beta^l(v), D(u)]_L + [\alpha^k \beta^l(v), D\circ \alpha^k \beta^l(v)]_L = \delta^k D([D(u), \alpha^k \beta^l(v)]_L + [D \circ \alpha^k \beta^l(v), D(u)]_L - [D' \circ D(u), \alpha^k \beta^l(v)]_L - [\alpha^k \beta^l(v), D\circ \alpha^k \beta^l(v)]_L).$$
Since any two of maps $D, D', \alpha, \beta$ commute, we have
\[
D \circ \alpha^s = \alpha^s \circ D, \quad D' \circ \alpha^k = \alpha^k \circ D',
\]
\[
D \circ \beta^t = \beta^t \circ D, \quad D' \circ \beta^l = \beta^l \circ D'.
\]
Therefore, we have
\[
[D, D']([u, v]_L) = \delta^{k+s}(D \circ D'(u) - D' \circ D(u), \alpha^{k+s}\beta^{l+t}(v))_L
\]
\[
+ [\alpha^{k+s}\beta^{l+t}(u), D \circ D'(u) - D' \circ D(v)]_L
\]
\[
= \delta^{k+s}([D, D'](u), \alpha^{k+s}\beta^{l+t}(v))_L + [\alpha^{k+s}\beta^{l+t}(u), [D, D'](v), ]_L.
\]
Furthermore, it is straightforward to see that
\[
[D, D'] \circ \alpha = D \circ D' \circ \alpha - D \circ D' \circ \alpha
\]
\[
= \alpha \circ D \circ D' - \alpha \circ D \circ D'
\]
\[
= \alpha \circ [D, D'],
\]
and
\[
[D, D'] \circ \beta = D \circ D' \circ \beta - D \circ D' \circ \beta
\]
\[
= \beta \circ D \circ D' - \beta \circ D \circ D'
\]
\[
= \beta \circ [D, D'].
\]
Therefore, $[D, D'] \in \text{Der}_{\alpha^{k+s}\beta^{l+t}}(L)$. \hfill \Box

For any integer $k, l$, denote by $\text{Der}(L) = \oplus_{k \geq 0, l \geq 0} \text{Der}_{\alpha^k\beta^l}(L)$. Obviously, $\text{Der}(L)$ is a Lie algebra, in which the Lie bracket is given by equation (3.6).

In the end, we consider the derivation extension of the regular $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$ and give an application of the $\alpha^0\beta^1$-derivation $\text{Der}_{\alpha^0\beta^1}(L)$.

For any linear map $D, \alpha, \beta : L \to L$, where $\alpha$ and $\beta$ are inverse, consider the vector space $L \oplus RD$. Define a skew-symmetric bilinear bracket operation $[\cdot, \cdot]_D$ on $L \oplus RD$ by
\[
[u, v]_D = [u, v]_L, [D, u]_D = -\delta^{[\alpha^{-1}\beta(u), \alpha\beta^{-1}D]}_D = D(u), \forall u, v \in L.
\]
Define two linear maps by $\alpha_D(u, D) = (\alpha(u), D), \quad \beta_D(u, D) = (\beta(u), D)$.

And the linear maps $\alpha, \beta$ involved in the definition of the bracket operation $[\cdot, \cdot]_D$ are required to be multiplicative, that is
\[
\alpha \circ [D, u]_D = [\alpha \circ D, \alpha(u)]_D, \quad \beta \circ [D, u]_D = [\beta \circ D, \beta(u)]_D.
\]
Then, we have
\[
[u, D]_D = -\delta^{[\alpha^{-1}\beta D, \alpha\beta^{-1}u]}_D
\]
\[
= -\delta^{\alpha^{-1}\beta[D, \alpha^2\beta^{-2}(u)]}_D
\]
\[
= -\delta^{\alpha^{-1}\beta D(\alpha^2\beta^{-2}(u))}_D
\]
\[
= -\delta^{\alpha\beta^{-1}D(u)}_D.
\]

**Theorem 3.3.** With the above notations, $(L \oplus RD, [\cdot, \cdot]_D, \alpha_D, \beta_D)$ is a multiplicative $\delta$-Bihom-Jordan-Lie algebra if and only if $D$ is an $\alpha^0\beta^1$-derivation of the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$.

**Proof.** For any $u, v \in L, m, n \in R$, we have
\[
\alpha_D \circ \beta_D(u, mD) = \alpha_D(\beta(u), mD) = (\alpha \circ \beta(u), mD),
\]
and
\[
\beta_D \circ \alpha_D(u, mD) = \beta_D(\alpha(u), mD) = (\beta \circ \alpha(u), mD).
\]
Hence, we have

$$\alpha_D \circ \beta_D = \beta_D \circ \alpha_D \iff \alpha \circ \beta = \beta \circ \alpha.$$  

On the other hand,

$$\alpha_D[(u, mD), (v, nD)]_D = \alpha_D([u, v]_L + [u, nD]_D + [mD, v]_D)$$
$$\quad = \alpha_D([u, v]_L - \delta nD \circ \alpha \beta^{-1}(u) + mD(v))$$
$$\quad = \alpha([u, v]_L) - \delta n \circ D \circ \alpha \beta^{-1}(u) + m \circ D(v),$$

$$[\alpha_D(u, mD), \alpha_D(v, nD)]_D = [(\alpha(u), mD), (\alpha(v), nD)]_D$$
$$\quad = [\alpha(u), \alpha(v)]_L + [\alpha(u), nD]_D + [mD, \alpha(v)]_D$$
$$\quad = [\alpha(u), \alpha(v)]_L - \delta nD \circ \alpha \beta^{-1}(\alpha(u)) + mD(\alpha(v)).$$

Since $$\alpha([u, v]_L) = [\alpha(u), \alpha(v)]_L,$$

$$\alpha_D[(u, mD), (v, nD)]_D = [\alpha_D(u, mD), \alpha_D(v, nD)]_D$$
if and only if

$$D \circ \alpha = \alpha \circ D, \quad D \circ \beta = \beta \circ D.$$  

Similarly

$$\beta_D[(u, mD), (v, nD)]_D = [\beta_D(u, mD), \beta_D(v, nD)]_D$$
if and only if

$$D \circ \alpha = \alpha \circ D, \quad D \circ \beta = \beta \circ D.$$  

Next, we have

$$[\beta_D(v, nD), \alpha_D(u, mD)]_D = [(\beta(v), nD), (\alpha(u), mD)]_D$$
$$\quad = [\beta(v), \alpha(u)]_L + [\beta(v), mD]_D + [nD, \alpha(u)]_D$$
$$\quad = [\beta(v), \alpha(u)]_L - \delta m \alpha \beta^{-1} \circ D \circ (\beta(v)) + nD(\alpha(u))$$
$$\quad = -\delta([\beta(u), \alpha(v)]_L + m \alpha \beta^{-1} \circ D \circ (\beta(v)) - \delta nD(\alpha(u))),$$

$$[\beta_D(u, mD), \alpha_D(v, nD)]_D = [(\beta(u), mD), (\alpha(v), nD)]_D$$
$$\quad = [\beta(u), \alpha(v)]_L + [\beta(u), nD]_D + [mD, \alpha(v)]_D$$
$$\quad = [\beta(u), \alpha(v)]_L - \delta n \alpha \beta^{-1} \circ D \circ (\beta(u)) + mD(\alpha(v)),$$

thus

$$[\beta_D(v, nD), \alpha_D(u, mD)]_D = -\delta[\beta_D(u, mD), \alpha_D(v, nD)]_D$$
if and only if

$$D \circ \alpha = \alpha \circ D, \quad D \circ \beta = \beta \circ D.$$
On the other hand, we have

\[
[b_2^D(u, mD), [b_2^D(v, nD), \alpha(w, lD)]_D]_D + [b_2^D(v, nD), [b_2^D(w, lD), \alpha_D(u, mD)]_D]_D \\
+ [b_2^D(w, lD), [b_2^D(v, nD), \alpha_D(u, mD)]_D]_D \\
= [(b_2^D(u, mD), [(b_2^D(v, nD), \alpha(w, lD)]_D) + [(b_2^D(v, nD), [(b_2^D(w, lD), (\alpha(u, mD)]_D)_D)]] + [(b_2^D(w, lD), [(b_2^D(v, nD), (\alpha(u, mD)]_D)_D)]]]
\]

\[
= [(b_2^D(u, mD), [(b_2^D(v, nD), \alpha(w)]_D - \delta \alpha \circ D(v) + nD \circ \alpha(w)]_D) + [(b_2^D(v, nD), [(b_2^D(w, lD), (\alpha(u, mD)]_D)_D)]] + [(b_2^D(w, lD), [(b_2^D(v, nD), (\alpha(u, mD)]_D)_D)]]
\]

\[
= [(b_2^D(u, [(b_2^D(v, \alpha(w)]_D) - \delta b_2^D(u, l\alpha \circ D(v)]_D) + [(b_2^D(v, nD), \alpha \circ \alpha(w)]_D) + [(b_2^D(v, [(b_2^D(w, \alpha(u)]_D) - \delta nD, \alpha \circ D(u)]_D + [nD, \alpha \circ \alpha(u)]_D) + [(b_2^D(w, [(b_2^D(w, \alpha(u)]_D) - \delta \alpha \circ D(u)]_D) + [(b_2^D(w, mD \circ \alpha(u)]_D) + [(b_2^D(w, [(b_2^D(u, \alpha(u)]_D) - \delta mD, \alpha \circ D(v)]_D + [mD, \alpha \circ \alpha(v)]_D) + [(b_2^D(u, [(b_2^D(v, \alpha(w)]_D) - \delta mD, \alpha \circ D(v)]_D + [mD, \alpha \circ \alpha(w)]_D) + [(b_2^D(v, [(b_2^D(u, \alpha(u)]_D) - \delta nD, \alpha \circ D(u)]_D) + [(b_2^D(w, mD \circ \alpha(v)]_D) + [(b_2^D(w, [(b_2^D(u, \alpha(w)]_D) - \delta mD, \alpha \circ D(v)]_D + [mD, \alpha \circ \alpha(v)]_D)
\]

If \( D \) is an \( \alpha^0 \beta^1 \)-derivation of the multiplicative \( \delta \)-Bihom-Jordan-Lie algebra \( (L, [\cdot, \cdot]_L, \alpha, \beta) \), then

\[
[mD, [b_2^D(v, \alpha(w)]_D] = mD[b_2^D(v, \alpha(w)]_D = \delta mD \circ \beta(v, \alpha^0 \beta^1(\alpha(w))]_D + [\alpha^0 \beta^1(v), mD \circ \alpha(w)]_D
\]

\[
= -\delta [\alpha^0 \beta^1(v), mD \circ \alpha(w)]_D + [\alpha^0 \beta^2(v), mD \circ \alpha(w)]_D
\]

Similarly

\[
[nD, [b_2^D(v, \alpha(u)]_D] = nD[b_2^D(v, \alpha(u)]_D = -\delta [\beta^2(v), nD \circ \alpha(u)]_D + [\beta^2(v), mD \circ \alpha(u)]_D
\]

And

\[
[lD, [b_2^D(v, \alpha(u)]_D] = -\delta [\beta^2(v), lD \circ \alpha(u)]_D + [\beta^2(v), lD \circ \alpha(u)]_D.
\]

Therefore, the \( \delta \)-Bihom-Jacobi identity is satisfied if and only if \( D \) is an \( \alpha^0 \beta^1 \)-derivation of \( (L, [\cdot, \cdot]_L, \alpha, \beta) \). Thus \( (L + RD, [\cdot, \cdot]_D, \alpha_D, \beta_D) \) is a multiplicative \( \delta \)-Bihom-Jordan-Lie algebra if and only if \( D \) is an \( \alpha^0 \beta^1 \)-derivation of \( (L, [\cdot, \cdot]_L, \alpha, \beta) \).

4. Representations of \( \delta \)-Bihom-Jordan-Lie algebras

In this section we study representations of \( \delta \)-Bihom-Jordan-Lie algebras and give the corresponding coboundary operators. We can also construct the semidirect product of \( \delta \)-Bihom-Jordan-Lie algebras. Let \( A \in \text{End}(V) \) be an arbitrary linear transformation from \( V \) to \( V \).
Definition 4.1. Let \((L, \cdot, \cdot)_L, \alpha, \beta\) be a multiplicative \(\delta\)-Bihom-Jordan-Lie algebra. A representation of \(L\) is a 4-tuple \((M, \rho, \alpha_M, \beta_M)\), where \(M\) is a linear space, \(\alpha_M, \beta_M : M \to M\) are two commuting linear maps and \(\rho : L \to \text{End}(M)\) is a linear map such that, for all \(u, v \in L\), we have

\[
\rho(\alpha(u)) \circ \alpha_M = \alpha_M \circ \rho(u),
\]
\[
\rho(\beta(u)) \circ \beta_M = \beta_M \circ \rho(u),
\]
\[
\rho([\beta(u), v]_L) \circ \beta_M = \rho(\alpha(\beta(u))) \circ \rho(v) - \delta \rho(\beta(v)) \circ \rho(\alpha(u)).
\]

Let \((L, \cdot, \cdot)_L, \alpha, \beta\) be a regular \(\delta\)-Bihom-Jordan-Lie algebra. The set of \(k\)-cochains on \(L\) with values in \(M\), which we denote by \(C^k(L; M)\), is the set of \(k\)-linear maps from \(L \times \cdots \times L\) (\(k\)-times) to \(M\):

\[
C^k(L; M) \triangleq \{ f : L \times \cdots \times L(k \text{-times}) \to M \text{ is a linear map} \}.
\]

A \(k\)-Bihom-cochain on \(L\) with values in \(M\) is defined to be a \(k\)-cochain \(f \in C^k(L; M)\) such that it is compatible with \(\alpha, \beta\) and \(\alpha_M, \beta_M\) in the sense that \(\alpha_M \circ f = f \circ \alpha, \beta_M \circ f = f \circ \beta\), i.e.

\[
\alpha_M(f(u_1, \ldots, u_k)) = f(\alpha(u_1), \ldots, \alpha(u_k)),
\]
\[
\beta_M(f(u_1, \ldots, u_k)) = f(\beta(u_1), \ldots, \beta(u_k)).
\]

Denote by \(C^k_{(\alpha, \alpha_M)}(L, M)\) the set of \(k\)-Bihom-cochains:

\[
C^k_{(\alpha, \alpha_M)}(L, M) \triangleq \{ f \in C^k(L, M) | \alpha_M \circ f = f \circ \alpha \}.
\]

Define the linear map \(d^k_\rho : C^k_{(\alpha, \alpha_M)}(L, M) \to C^{k+1}_{(\beta, \beta_M)}(L, M)(k = 1, 2)\) as follows: we set

\[
d^1_\rho f(u_1, u_2) = \rho(\alpha(u_1))f(u_2) - \delta \rho(\alpha(u_1))f(u_1) - \delta f([\alpha^{-1}\beta(u_1), u_2]_L),
\]
\[
d^2_\rho f(u_1, u_2, u_3) = \rho(\alpha \beta(u_1))f(u_2, u_3) - \delta \rho(\alpha \beta(u_2))f(u_1, u_3) + \rho(\alpha \beta(u_3))f(u_1, u_2)
\]
\[
- f([\alpha^{-1}\beta(u_1), u_2]_L, \beta(u_3)) + \delta f([\alpha^{-1}\beta(u_1), u_3]_L, \beta(u_2))
\]
\[
- f([\alpha^{-1}\beta(u_2), u_3]_L, \beta(u_1)).
\]

Lemma 4.2. With the above notations, for any \(f \in C^k_{(\alpha, \alpha_M)}(L, M)\), we have

\[(d^k_\rho \circ f) \circ \alpha = \alpha_M \circ d^k_\rho f,\]
\[(d^k_\rho \circ f) \circ \beta = \beta_M \circ d^k_\rho f.\]

Thus we obtain a well-defined map

\[
d^k_\rho : C^k_{(\alpha, \alpha_M)}(L, M) \to C^{k+1}_{(\beta, \beta_M)}(L, M)
\]

with \(k = 1, 2\).

Proposition 4.3. With the above notations, we have \(d^2_\rho \circ d^1_\rho = 0\).
Similarly, we have
\[ d^2_B \circ d^1_L f(u_1, u_2, u_3) = \rho(\alpha(\beta(u_1)))d^1_L f(u_2, u_3) - \delta(\alpha(\beta(u_2)))d^1_L f(u_1, u_3) + \rho(\alpha(\beta(u_3)))d^1_L f(u_1, u_2) \]
\[ -d^1_L f([\alpha^{-1}\beta(u_1), u_2], \beta(u_3)) \]
\[ -\delta d^1_L f([\alpha^{-1}\beta(u_1), u_3], \beta(u_2)) \]
\[ -d^1_L f([\alpha^{-1}\beta(u_3), u_2], \beta(u_1)) \]
\[ = \rho(\alpha(\beta(u_1)))\rho(\alpha(\beta(u_2)))f(u_3) - \delta\rho(\alpha(\beta(u_2)))f(u_2) - \delta f([\alpha^{-1}\beta(u_2), u_3], L) \]
\[ -\delta\rho(\alpha(\beta(u_2)))\rho(\alpha(\beta(u_1)))f(u_3) - \delta\rho(\alpha(\beta(u_3)))f(u_1) - \delta f([\alpha^{-1}\beta(u_1), u_3], L) \]
\[ +\rho(\alpha(\beta(u_3)))\rho(\alpha(\beta(u_1)))f(u_2) - \delta\rho(\alpha(\beta(u_2)))f(u_1) - \delta f([\alpha^{-1}\beta(u_1), u_2], L) \]
\[ -\rho(\alpha([\alpha^{-1}\beta(u_1), u_2], L))f(\beta(u_3)) + \delta\rho(\alpha(\beta(u_3)))f(u_1) \]
\[ +\rho(\alpha([\alpha^{-1}\beta(u_1), u_3], L))f(\beta(u_2)) + \rho(\alpha(\beta(u_2)))f(u_1) \]
\[ +f([\alpha^{-1}\beta(u_1), u_3], L, \beta(u_2)]L) \]
\[ +\delta\rho(\alpha([\alpha^{-1}\beta(u_2), u_3], L))f(\beta(u_1)) - \rho(\alpha(\beta(u_1)))f(u_2) \]
\[ -f([\alpha^{-1}\beta([\alpha^{-1}\beta(u_2), u_3], \beta(u_1)]L) \]
\[ = 0. \]

Then \( d^2_B \circ d^1_L f(u_1, u_2, u_3) = 0. \)

Associated to the representation \( \rho \), we obtain the complex \( (C^k_{(\alpha,\alpha_M)}(L, M), d^\rho) \) Denote the set of closed \( k \)-Bihom-cochains by \( Z^k_{\alpha,\beta}(L; \rho) \) and the set of exact \( k \)-Bihom-cochains by \( B^k_{\alpha,\beta}(L, \rho) \), \( k = 1, 2 \).

Denote the corresponding cohomology by
\[ H^k_{\alpha,\beta}(L, \rho) = Z^k_{\alpha,\beta}(L; \rho)/B^k_{\alpha,\beta}(L, \rho), \]
where
\[ Z^k_{\alpha,\beta}(L; \rho) = \{ f \in C^k_{(\alpha,\alpha_M)}(L, M) \mid d^k_B f = 0 \}, \]
\[ B^k_{\alpha,\beta}(L, \rho) = \{ d^k_B g \mid g \in C^{k-1}_{(\beta,\beta_M)}(L, M) \}. \]

In the case of Lie algebras, we can form semidirect products when given representations. Similarly, we have

**Proposition 4.4.** Let \( (L, [\cdot, \cdot]_L, \alpha, \beta) \) be a multiplicative \( \delta \)-Bihom-Jordan-Lie algebra and \( (M, \rho, \alpha_M, \beta_M) \) a representation of \( L \). Assume that the maps \( \alpha_M \) and \( \beta_M \) be bijective. Then \( L \ltimes M = (L \oplus M, [\cdot, \cdot]_L, \alpha \oplus \alpha_M, \beta \oplus \beta_M) \) is a \( \delta \)-Bihom-Jordan-Lie algebra, where \( \alpha \oplus \alpha_M, \beta \oplus \beta_M : L \oplus M \to L \oplus M \) are defined by \( (\alpha \oplus \alpha_M)(u + x) = \alpha(u) + \alpha_M(x) \) and...
\((\beta \oplus \beta_M)(u + x) = \beta(u) + \beta_M(x)\), for all \(u, v \in L\) and \(x, y \in M\), the bracket \([\cdot, \cdot]_\rho\) is defined by

\[
[u + v, x + y]_\rho = [u, v]_L + \delta \rho(u)(y) - \rho(\alpha^{-1}(v))(\alpha_M \beta_M^{-1}(x)).
\]

We call \(L \rtimes M\) the semidirect product of the multiplicative \(\delta\)-Bihom-Jordan-Lie algebra \((L, [\cdot, \cdot]_L, \alpha, \beta)\) and \(M\).

**Proof.** First we show that \([\cdot, \cdot]_\rho\) satisfies antisymmetry,

\[
([\beta \oplus \beta_M](v + y), (\alpha \oplus \alpha_M)(u + x)]_\rho
\]

\[
= [\beta(v) + \beta_M(y), \alpha(u) + \alpha_M(x)]_\rho
\]

\[
= [\beta(v), \alpha(u)]_L + \delta \rho(\beta(v))(\alpha_M(x)) - \rho(\alpha^{-1}(\beta(v)))(\alpha_M \beta_M^{-1}(\beta_M(v)))
\]

\[
= [\beta(v), \alpha(u)]_L + \delta \rho(\beta(v))(\alpha_M(x)) - \rho(\beta(u))(\alpha_M(v))
\]

\[
= -\delta([\beta(u), \alpha(v)]_L + \delta \rho(\beta(u))(\alpha_M(y)) - \rho(\beta(v))(\alpha_M(u))
\]

\[
= -\delta([\beta(\rho(u)), \alpha(v)]_L + \delta \rho(\beta(u))(\alpha_M(y)) - \rho(\beta(v))(\alpha_M(u)))
\]

Next we show that \((\alpha \oplus \alpha_M)\) and \((\beta \oplus \beta_M)\) are algebra morphisms. On the one hand, we have

\[
(\alpha \oplus \alpha_M)(u + x, v + y)_\rho
\]

\[
= (\alpha(u), \alpha(v)]_L + \rho(\alpha(u))(\alpha_M(y)) - \rho(\alpha^{-1}(\beta(v)))(\alpha_M \beta_M^{-1}(\beta_M(x)))
\]

\[
= [\alpha(u), \alpha(v)]_L + \rho(\alpha(u))(\alpha_M(y)) - \rho(\alpha^{-1}(\beta(v)))(\alpha_M \beta_M^{-1}(\beta_M(x)))
\]

\[
= [\alpha(u), \alpha(v)]_L + \rho(\alpha(u))(\alpha_M(y)) - \rho(\beta(v))(\alpha_M(u))
\]

\[
= [\alpha(\beta_M(u) + \beta_M(v)), (\alpha \oplus \alpha_M)(v + y)]_\rho.
\]

Similarly, we obtain

\[
(\beta \oplus \beta_M)[u + x, v + y]_\rho = ([\beta \oplus \beta_M](u + x), (\beta \oplus \beta_M)(v + y)]_\rho.
\]

Furthermore

\[
([\beta \oplus \beta_M]^2(u + x), [\beta \oplus \beta_M][v + y], (\alpha \oplus \alpha_M)(w + z)]_\rho
\]

\[
= [\beta^2(v) + \beta_M^2(x), \beta(v) + \beta_M(x), \alpha(w) + \alpha_M(z)]_\rho
\]

\[
= [\beta^2(v) + \beta_M^2(x), \beta(v), \alpha(w)]_L + \delta \rho(\beta(v))(\alpha_M(z)) - \rho(\alpha^{-1}(\beta(v)))(\alpha_M \beta_M^{-1}(\beta_M(y)))_\rho
\]

\[
= [\beta^2(v) + \beta_M^2(x), \beta(v), \alpha(w)]_L + \delta \rho(\beta(v))(\alpha_M(z)) - \rho(\beta(u))(\alpha_M(y))
\]

\[
= [\beta^2(u), [\beta(v), \alpha(w)]_L + \delta \rho(\beta(v))(\alpha_M(z)) - \rho(\beta(u))(\alpha_M(y))
\]

\[
-\rho(\alpha^{-1}(\beta(v), \alpha(w)]_L(\alpha_M \beta_M^{-1}(\beta_M(x))))
\]

\[
= [\beta^2(u), [\beta(v), \alpha(w)]_L + \delta \rho(\beta(v))(\alpha_M(z)) - \rho(\beta(u))(\alpha_M(y))
\]

\[
-\rho(\alpha^{-1}(\beta(v), \beta(u)]_L(\alpha_M \beta_M(y))
\]

Similarly,

\[
([\beta \oplus \beta_M]^2(v + y), [\beta \oplus \beta_M](u + z), (\alpha \oplus \alpha_M)(u + x)]_\rho
\]

\[
= [\beta^2(v), [\beta(w), \alpha(u)]_L + \rho(\beta(v))(\alpha_M(z)) - \rho(\beta(u))(\alpha_M(y))
\]

\[
-\rho(\alpha^{-1}(\beta^2(w), \beta(u)]_L(\alpha_M \beta_M(y))
\]

And

\[
([\beta \oplus \beta_M]^2(u + x), [\beta \oplus \beta_M](u + z), (\alpha \oplus \alpha_M)(v + y)]_\rho
\]

\[
= [\beta^2(v), [\beta(w), \alpha(u)]_L + \rho(\beta(v))(\alpha_M(z)) - \rho(\beta(u))(\alpha_M(y))
\]

\[
-\rho(\alpha^{-1}(\beta^2(w), \beta(v)]_L(\alpha_M \beta_M(z))
\]

By (4.3), the \(\delta\)-Bihom-Jacobi identity is satisfied. Thus, \((L \oplus M, [\cdot, \cdot]_\rho, \alpha \oplus \alpha_M, \beta \oplus \beta_M)\) is a multiplicative \(\delta\)-Bihom-Jordan-Lie algebra. \(
\square
\)
5. The trivial representation of $\delta$-Bihom-Jordan-Lie algebras

In this section, we study the trivial representation of multiplicative $\delta$-hom-Jordan-Lie algebras. As an application, we show that the central extension of a multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L,[\cdot,\cdot]_L,\alpha,\beta)$ is controlled by the second cohomology of $L$ with coefficients in the trivial representation.

Now let $M = \mathbb{R}$. Then we have $\text{End}(M) = \mathbb{R}$. Any $\alpha_M, \beta_M \in \text{End}(M)$ is exactly two real numbers, which we denote by $r_1, r_2$ respectively. Let $\rho : L \to \text{End}(M) = \mathbb{R}$ be the zero map. Obviously, $\rho$ is a representation of the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L,[\cdot,\cdot]_L,\alpha,\beta)$ with respect to any $r_1, r_2 \in \mathbb{R}$. We will always assume that $r_1 = r_2 = 1$. We call this representation the trivial representation of the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L,[\cdot,\cdot]_L,\alpha,\beta)$.

Associated to the trivial representation, the set of $k$-cochains on $L$, which we denote by $C^k(V) = \wedge^k L^*$, is the set of skew-symmetric $k$-linear maps from $V \times \cdots \times V$ to $\mathbb{R}$. The set of $k$-Bihom-cochains is given by

$$C^k_{\alpha,\beta}(L) = \{ f \in C^k(L) | f \circ \alpha = f, f \circ \beta = f \}.$$

The corresponding coboundary operator $d_T : C^k_{\alpha,\beta}(L) \to C^{k+1}_{\alpha,\beta}(L)$ is given by

$$d_T f(u_1, u_2) = -\delta f(\alpha^{-1}\beta(u_1), u_2),$$

with the following bracket

$$R_{\alpha,\beta} (g) = \theta \circ \alpha \circ \beta = \theta \circ \beta \circ \alpha.$$

In the following we consider central extensions of the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(L,[\cdot,\cdot]_L,\alpha,\beta)$. Obviously, $(\mathbb{R},0,1,1)$ is an abelian multiplicative $\delta$-Bihom-Jordan-Lie algebra with the trivial bracket and the identity morphism. Let $\theta \in C^2_{\alpha,\beta}(L)$, we have

$$\theta \circ \alpha = \theta, \quad \theta \circ \beta = \theta \quad \text{and} \quad \theta(u,v) = -\delta \theta(v,u), \quad \forall u,v \in L.$$

We consider the direct sum $g = L \oplus \mathbb{R}$ with the following bracket

$$[u + s,v + t]_{\theta} = [u,v]_L + \theta(\alpha^{-1}\beta^{-1}(u),v), \quad \forall u,v \in L, s,t \in \mathbb{R}.$$

Define $\alpha_\theta, \beta_\theta : g \to g$ by $\alpha_\theta(u+s) = \alpha(u) + s$, and $\beta_\theta(u+s) = \beta(u) + s$.

**Theorem 5.1.** With the above notations, the 4-tuple $(g,[\cdot,\cdot]_g,\alpha_\theta,\beta_\theta)$ is a multiplicative $\delta$-Bihom-Jordan-Lie algebra if and only if $\theta \in C^2_{\alpha,\beta}(L)$ is a 2-cocycle associated to the trivial representation, i.e.

$$d_T \theta = 0.$$

We call the multiplicative $\delta$-Bihom-Jordan-Lie algebra $(g,[\cdot,\cdot]_g,\alpha_\theta,\beta_\theta)$ the central extension of $(L,[\cdot,\cdot]_L,\alpha,\beta)$ by the abelian $\delta$-Bihom-Jordan-Lie algebra $(\mathbb{R},0,1,1)$.

**Proof.** Obviously, since $\alpha \circ \beta = \beta \circ \alpha$, we have $\alpha_\theta \circ \beta_\theta = \beta_\theta \circ \alpha_\theta$. Then we show that $\alpha_\theta$ is an algebra morphism with respect to the bracket $[\cdot,\cdot]_\theta$. On one hand, we have

$$\alpha_\theta([u+s,v+t])_\theta = \alpha_\theta([u,v]_L + \theta(\alpha^{-1}\beta^{-1}(u),v)) = \alpha([u,v]_L + \theta(\alpha^{-1}\beta^{-1}(u),v)).$$

On the other hand, we have

$$[\alpha_\theta(u+s),\alpha_\theta(v+t)]_\theta = [\alpha(u) + s,\alpha(v) + t]_\theta = [\alpha(u),\alpha(v)]_L + \theta(\alpha^{-1}(\alpha(u)),\alpha(v)).$$

Since $\alpha$ is an algebra morphism and $\theta(\alpha^{-1}(\alpha(u)),\alpha(v)) = \theta(\alpha^{-1}(\alpha(u)),\alpha(v)) = \theta(\alpha^{-1}(u),v)$, we have $\alpha_\theta$ is an algebra morphism.

Similarly, we have $\beta_\theta$ is also an algebra morphism.
Furthermore, we have
\[
[\beta_{g}(u + s), \alpha_{g}(v + t)]_{\theta} = [\beta(u) + s, \alpha(v) + t]_{\theta} \\
= [\beta(u), \alpha(v)]_{L} + \theta(\alpha \beta^{-1}(\beta(u)), \alpha(v)) \\
= [\beta(u), \alpha(v)]_{L} + \theta(\alpha(u), \alpha(v)) \\
= [\beta(u), \alpha(v)]_{L} + \theta(u, v)
\]
and
\[
[\beta_{g}(v + t), \alpha_{g}(u + s)]_{\theta} = [\beta(v) + t, \alpha(u) + s]_{\theta} \\
= [\beta(v), \alpha(u)]_{L} + \theta(\alpha \beta^{-1}(\beta(v)), \alpha(u)) \\
= [\beta(v), \alpha(u)]_{L} + \theta(\alpha(v), \alpha(u)) \\
= [\beta(v), \alpha(u)]_{L} + \theta(v, u)
\]
Then \([\beta_{g}(u + s), \alpha_{g}(v + t)]_{\theta} = -\delta[\beta_{g}(v + t), \alpha_{g}(u + s)]_{\theta}\).
By direct computations, we have
\[
[\beta_{g}^{2}(u + s), [\beta_{g}(v + t), \alpha_{g}(w + r)]_{\theta}]_{\theta} + [\beta_{g}^{2}(v + t), [\beta_{g}(u + r), \alpha_{g}(u + s)]_{\theta}]_{\theta} \\
+ [\beta_{g}^{2}(w + r), [\beta_{g}(u + s), \alpha_{g}(v + t)]_{\theta}]_{\theta} \\
= [\beta^{2}(u) + s, [\beta(v) + t, \alpha(w) + r]_{\theta}]_{\theta} + [\beta^{2}(v) + t, [\beta(w) + r, \alpha(u) + s]_{\theta}]_{\theta} \\
+ [\beta^{2}(w) + r, [\beta(u) + s, \alpha(v) + t]_{\theta}]_{\theta} \\
= [\beta^{2}(u), [\beta(v)\alpha(w)]_{L} + \theta(\alpha \beta^{-1}(\beta(v)), \alpha(w))]_{\theta} \\
+ [\beta^{2}(v), [\beta(w)\alpha(u)]_{L} + \theta(\alpha \beta^{-1}(\beta(w)), \alpha(u))]_{\theta} \\
+ [\beta^{2}(w), [\beta(u)\alpha(v)]_{L} + \theta(\alpha \beta^{-1}(\beta(u)), \alpha(v))]_{\theta} \\
= [\beta^{2}(u), [\beta(v)\alpha(w)]_{L} + \theta(\alpha \beta(w), [\beta(v), \alpha(w)]_{L}) \\
+ [\beta^{2}(v), [\beta(w)\alpha(u)]_{L} + \theta(\alpha \beta(v), [\beta(w), \alpha(u)]_{L}) \\
+ [\beta^{2}(w), [\beta(u)\alpha(v)]_{L} + \theta(\alpha \beta(w), [\beta(u), \alpha(v)]_{L})]
\]
Thus by the Bihom-Jacobi identity of \(L\), \([\cdot, \cdot]_{\theta}\) satisfies the \(\delta\)-Bihom-Jacobi identity if and only if
\[
\theta(\alpha \beta(w)), [\beta(v), \alpha(w)]_{L} + \theta(\alpha \beta(v)), [\beta(w), \alpha(u)]_{L} + \theta(\alpha \beta(w)), [\beta(u), \alpha(v)]_{L} = 0.
\]
Namely,
\[
\theta(\beta(u), [\alpha^{-1} \beta(v), w]_{L}) + \theta(\beta(v), [\alpha^{-1} \beta(w), u]_{L}) + \theta(\beta(w), [\alpha^{-1} \beta(u), v]_{L}) = 0.
\]
On the other hand,
\[
d_{T} \theta(u, v, w) \\
= \delta(\delta \theta(\alpha \beta^{-1}(\beta(u)), \alpha(v)) - \theta(\alpha \beta^{-1}(\beta(u)), \alpha(v))) + \theta(\alpha \beta^{-1}(\beta(u)), \alpha(v)) \\
= -\theta(\alpha \beta^{-1}(\beta(u)), \alpha(v)) + \theta(\alpha \beta^{-1}(\beta(u)), \alpha(v)) \\
= 0.
\]
Then the 4-tuple \((g, [\cdot, \cdot]_{\theta}, \alpha, \beta)\) is a multiplicative \(\delta\)-Bihom-Jordan-Lie algebra if and only if \(\theta \in C_{\alpha, \beta}^{2}(L)\) satisfies \(d_{T} \theta = 0\). \(\square\)
Proposition 5.2. For $\theta_1, \theta_2 \in Z^2(V)$, if $\delta(\theta_1 - \theta_2)$ is exact, the corresponding two central extensions $(\mathfrak{g}, [\cdot, \cdot]_{\theta_1}, \alpha_\theta, \beta_\theta)$ and $(\mathfrak{g}, [\cdot, \cdot]_{\theta_2}, \alpha_\theta, \beta_\theta)$ are isomorphic.

Proof. Assume that $\theta_1 - \theta_2 = \delta d_T f$, $f \in C^1_{\alpha, \beta}(L)$. Thus we have
$$\theta_1(\alpha\beta^{-1}(u), v) - \theta_2(u, v) = \delta d_T f(\alpha\beta^{-1}(u), v) = -f([\alpha^{-1}\beta \circ \alpha^{-1}(u), v]) = -f([u, v]).$$

Define $\varphi_\theta : \mathfrak{g} \to \mathfrak{g}$ by
$$\varphi_\theta(u + s) = u + s + f(u).$$

Obviously, $\varphi_\theta$ is an isomorphism of vector spaces. The fact that $\varphi_\theta$ is a morphism of the $\delta$-Bihom-Jordan-Lie algebra follows from the fact $\theta \circ \alpha = \theta, \theta \circ \beta = \theta$. More precisely, we have
$$\varphi_\theta \circ \alpha_\theta(u + s) = \varphi_\theta(\alpha(u) + s) = \alpha(u) + s + f(\alpha(u)) = \alpha(u) + s + f(u).$$

On the other hand, we have
$$\alpha_\theta \circ \varphi_\theta(u + s) = \alpha_\theta(u + s + f(u)) = \alpha(u) + s + f(u).$$

Thus, we obtain that $\varphi_\theta \circ \alpha_\theta = \alpha_\theta \circ \varphi_\theta$. Similarly
$$\varphi_\theta \circ \beta_\theta = \beta_\theta \circ \varphi_\theta.$$

We also have
$$\varphi_\theta[u + s, v + t]_{\theta_1} = \varphi_\theta([u, v]_L + \theta_1(\alpha\beta^{-1}(u), v))$$
$$= \alpha_\theta([u, v]_L + \theta_1(\alpha\beta^{-1}(u), v) + f([u, v]_L) = ([u, v]_L, \theta_2(\alpha\beta^{-1}(u), v)$$
$$= [\varphi_\theta(u + s), \varphi_\theta(v + t)]_{\theta_2}.$$

Therefore, $\varphi_\theta$ is also an isomorphism of multiplicative $\delta$-Bihom-Jordan-Lie algebras. \qed

6. The adjoint representation of $\delta$-Bihom-Jordan-Lie algebras

Let $(L, [\cdot, \cdot]_L, \alpha, \beta)$ be a regular $\delta$-Bihom-Jordan-Lie algebra. We consider that $L$ represents on itself via the bracket with respect to the morphisms $\alpha, \beta$. A very interesting phenomenon is that the adjoint representation of a $\delta$-Bihom-Jordan-Lie algebra is not unique as one will see in sequel.

Definition 6.1. For any integer $s,t$, the $\alpha^s\beta^t$-adjoint representation of the regular $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$, which we denote by $\text{ad}_{s,t}$, is defined by
$$\text{ad}_{s,t}(u)(v) = \delta[\alpha^s\beta^t(u), v]_L, \forall u, v \in L.$$

Lemma 6.2. With the above notations, we have
$$\text{ad}_{s,t}(\alpha(u)) \circ \alpha = \alpha \circ \text{ad}_{s,t}(u);$$
$$\text{ad}_{s,t}(\beta(u)) \circ \beta = \beta \circ \text{ad}_{s,t}(u);$$
$$\text{ad}_{s,t}([\beta(u), v]_L) \circ \beta = \text{ad}_{s,t}([\beta(\beta(u)), v]_L).$$

Thus the definition of $\alpha^s\beta^t$-adjoint representation is well defined.

Proof. For any $u, v, w \in L$, first we show that $\text{ad}_{s,t}(\alpha(u)) \circ \alpha = \alpha \circ \text{ad}_{s,t}(u)$
$$\text{ad}_{s,t}(\alpha(u))(\alpha(v)) = \delta[\alpha^{s+1}\beta(u), \alpha(v)]_L$$
$$= \alpha(\delta[\alpha^s\beta^t(u), v]_L) = \alpha \circ \text{ad}_{s,t}(u)(v).$$

Similarly, we have
$$\text{ad}_{s,t}(\beta(u)) \circ \beta = \beta \circ \text{ad}_{s,t}(u).$$
Note that the skew-symmetry condition implies

\[
\text{ad}_s(u)(v) = \delta [\alpha^s \beta^t(u), v]_L \\
= \delta [\beta(\alpha^s \beta^t(u)), \alpha(\alpha^{-1}(v))]_L \\
= -\delta^2 [\alpha^{-1} \beta(v), \alpha^{s+1} \beta^t(u)]_L \\
= -[\alpha^{-1} \beta(v), \alpha^{s+1} \beta^t(u)]_L, \forall u, v \in L.
\]

On one hand, we have

\[
\text{ad}_{s,t}([\beta(u), v]_L) \circ \beta(w) = \text{ad}_{s,t}([\beta(u), v]_L)(\beta(w)) \\
= -[\alpha^{-1} \beta(\beta(w)), \alpha^{s+1} \beta^t([\beta(u), v]_L)]_L \\
= -[\alpha^{-1} \beta^2(w), \alpha^{s+1} \beta^t(u), \alpha^{s+1} \beta^t(v)]_L.
\]

On the other hand, we have

\[
\text{ad}_{s,t}(\alpha(u)) \circ \text{ad}_{s,t}(\beta(v)) = \text{ad}_{s,t}(\alpha(u))(\text{ad}_{s,t}(\beta(v))) \\
= \text{ad}_{s,t}(\alpha(u))(-[\alpha^{-1} \beta(w), \alpha^{s+1} \beta^t(u)]_L) \\
- \delta \text{ad}_{s,t}(\alpha(v))(-[\alpha^{-1} \beta(w), \alpha^{s+1} \beta^t(u)]_L) \\
= [\alpha^{-1} \beta([\alpha^{-1} \beta(w), \alpha^{s+1} \beta^t(u)]_L), \alpha^{s+1} \beta^t(-1)(\alpha(u))]_L \\
- \delta [\alpha^{-1} \beta([\alpha^{-1} \beta(w), \alpha^{s+1} \beta^t(u)]_L), \alpha^{s+1} \beta^t(-1)(\beta(v))]_L \\
= [\beta([\alpha^{-1} \beta(w), \alpha^{s+1} \beta^t(u)]_L), \alpha^{s+1} \beta^t(-1)(\beta(v))]_L \\
- \delta [\beta([\alpha^{-1} \beta(w), \alpha^{s+1} \beta^t(u)]_L), \alpha^{s+1} \beta^t(-1)(\beta(v))]_L \\
= [\alpha^{s+1} \beta^t(u), [\alpha^{-1} \beta(w), \alpha^{s+1} \beta^t(u)]_L]_L \\
+ [\alpha^{s+1} \beta^t(u), [\alpha^{-1} \beta(w), \alpha^{s+1} \beta^t(u)]_L)_L \\
= [\alpha^{s+1} \beta^t(u), \alpha^{s+1} \beta^t(u)]_L \\
+ [\alpha^{s+1} \beta^t(u), [\alpha^{-1} \beta(w), \alpha^{s+1} \beta^t(u)]_L)_L \\
= [\beta^2(\alpha^{s+1} \beta^t(u), [\beta(\alpha^{s+1} \beta^t(u), \alpha(\alpha^{-1}(w))]_L \\
+ [\beta^2(\alpha^{s+1} \beta^t(u), [\beta(\alpha^{s+1} \beta^t(u), \alpha(\alpha^{-1}(w))]_L) \\
= -[\beta^2(\alpha^{-1}(w), [\beta(\alpha^{s+1} \beta^t(u), \alpha(\alpha^{-1}(w))]_L \\
= -[\alpha^{-1} \beta^2(w), [\alpha^{s+1} \beta^t(u), \alpha^{s+1} \beta^t(u)]_L)_L.
\]

Thus, the definition of \(\alpha^s \beta^t\)-adjoint representation is well defined. The proof is completed. \(\square\)

The set of \(k\)-Bihom-cochains on \(L\) with coefficients in \(L\), which we denote by \(C^k_{\alpha, \beta}(L; L)\), is given by

\[
C^k_{\alpha, \beta}(L; L) = \{ f \in C^k(L; L) | \alpha \circ f = f \circ \alpha, \beta \circ f = f \circ \beta \}.
\]

In particular, the set of 0-Bihom-cochains is given by:

\[
C^0_{\alpha, \beta}(L; L) = \{ u \in L | \alpha(u) = u, \beta(u) = u \}.
\]

Associated to the \(\alpha^s \beta^t\)-adjoint representation, the corresponding operator

\[
d_{s,t} : C^k_{\alpha, \beta}(L; L) \rightarrow C^{k+1}_{\alpha, \beta}(L; L)(k = 1, 2)
\]

is given by

\[
d_{s,t} f(u_1, u_2) = \delta [\alpha^{1+s} \beta^t(u_1), \beta(u_2)] - \delta^2 [\alpha^{1+s} \beta^t(u_2), f(u_1)] - \delta f([\alpha^{-1} \beta(u_1), u_2]); \quad (6.1)
\]
\[ d_{s,t} f(u_1, u_2, u_3) = \delta[\alpha^{1+s}\beta^{t+1}(u_1), f(u_2, u_3)] - [\alpha^{1+s}\beta^{t+1}(u_2), f(u_1, u_3)] + \delta[\alpha^{1+s}\beta^{t+1}(u_3), f(u_1, u_2)] - f([\alpha^{-1}\beta(u_1), u_2], \beta(u_3)) + \delta f([\alpha^{-1}\beta(u_1), u_3], \beta(u_2)) - f([\alpha^{-1}\beta(u_2), u_3], \beta(u_1)). \]

For the \( \alpha^s\beta^t \)-adjoint representation \( \text{ad}_{s,t} \), we obtain the \( \alpha^s\beta^t \)-adjoint complex \( (C^\ast_{\alpha,\beta}(L; L), d_{s,t}) \).

We have known that a 1-cocycle associated to the adjoint representation is a derivation for Lie algebras and Hom-Lie algebras. Similarly, we have

**Proposition 6.3.** Associated to the \( \alpha^s\beta^t \)-adjoint representations \( \text{ad}_{s,t} \) of the regular \( \delta \)-Bihom-Jordan-Lie algebra \( (L, [\cdot, \cdot], \alpha, \beta) \), it satisfies \( \delta^{s+1} = 1 \), \( D \in C^1_{\alpha,\beta}(L; L) \) is a 1-cocycle if and only if \( D \) is an \( \alpha^{s+2}\beta^{t-1} \)-derivation, i.e. \( D \in \text{Der}_{\alpha^{s+2}\beta^{t-1}}(L) \).

**Proof.** The conclusion follows directly from the definition of the operator \( d_{s,t} \). \( D \) is closed if and only if

\[ d_{s,t}(D)(u, v) = \delta[\alpha^{s+1}\beta^t(u), D(v)]_L - [\alpha^{s+1}\beta^t(v), D(u)]_L - \delta D[\alpha^{-1}\beta(u), v]_L = 0. \]

\( D \) is an \( \alpha^{s+2}\beta^{t-1} \)-derivation if and only if

\[ D[\alpha^{-1}\beta(u), v]_L = -\delta[\alpha^{s+2}\beta^{t-1}\alpha^{-1}\beta(v), D(u)]_L + [\alpha^{s+2}\beta^{t-1}\alpha^{-1}\beta(u), D(v)]_L = \delta^{s+1}([D(u), \alpha^{s+1}\beta^t(v)]_L + [\alpha^{s+1}\beta^t(u), D(v)]_L). \]

Then, \( D \in C^1_{\alpha,\beta}(L; L) \) is a 1-cocycle if and only if \( D \) is an \( \alpha^{s+2}\beta^{t-1} \)-derivation, i.e. \( D \in \text{Der}_{\alpha^{s+2}\beta^{t-1}}(L) \). \( \square \)

Let \( \psi \in C^2_{\alpha,\beta}(L; L) \) be a bilinear operator commuting with \( \alpha \) and \( \beta \), also \( \psi(u, v) = -\delta \psi(v, u) \). Consider a t-parameterized family of bilinear operations

\[ [u, v]_t = [u, v]_L + t \psi(u, v). \quad (6.2) \]

Since \( \psi \) commutes with \( \alpha, \beta \), then \( \alpha, \beta \) are morphisms with respect to the bracket \( [\cdot, \cdot]_t \) for every \( t \). If all the brackets \( [\cdot, \cdot]_t \) endow \( (L, [\cdot, \cdot], \alpha, \beta) \) with regular \( \delta \)-Bihom-Jordan-Lie algebra structures, we say that \( \psi \) generates a deformation of the regular \( \delta \)-Bihom-Jordan-Lie algebra \( (L, [\cdot, \cdot], \alpha, \beta) \). The anti-symmetry of \( [\cdot, \cdot]_t \) means that

\[ [\beta(v), \alpha(u)]_t = [\beta(v), \alpha(u)]_L + t \psi(\beta(v), \alpha(u)) \]

and \( [\beta(u), \alpha(v)]_t = [\beta(u), \alpha(v)]_L + t \psi(\beta(u), \alpha(v)) \).

Then \( [\beta(v), \alpha(u)]_t = -\delta [\beta(u), \alpha(v)]_t \) if and only if

\[ \psi(\beta(v), \alpha(u)) = -\delta \psi(\beta(u), \alpha(v)). \quad (6.3) \]
By computing the Bihom-Jacobi identity of \([\cdot, \cdot]_L\)
\[
[\beta^2(u), [\beta(v), \alpha(w)]]_L + [\beta^2(v), [\beta(w), \alpha(u)]]_L + [\beta^2(w), [\beta(u), \alpha(v)]]_L
\]
\[
= [\beta^2(u), [\beta(v), \alpha(w)]_L + t\psi([\beta(v), \alpha(w))]_L
+ [\beta^2(v), [\beta(w), \alpha(u)]_L + t\psi([\beta(w), \alpha(u))]_L
+ [\beta^2(w), [\beta(u), \alpha(v)]_L + t\psi([\beta(u), \alpha(v)]_L
\]
\[
= [\beta^2(u), [\beta(v), \alpha(w)]_L + [\beta^2(v), t\psi([\beta(v), \alpha(w)]_L
+ [\beta^2(v), [\beta(w), \alpha(u)]_L + [\beta^2(v), t\psi([\beta(w), \alpha(u)]_L
+ [\beta^2(u), [\beta(w), \alpha(u)]_L + [\beta^2(w), t\psi([\beta(w), \alpha(u)]_L
\]
\[
= [\beta^2(v), [\beta(w), \alpha(u)]_L + [\beta^2(w), t\psi([\beta(w), \alpha(u)]_L
+ [\beta^2(w), [\beta(u), \alpha(v)]_L + t\psi([\beta(w), \beta(v), \alpha(w))]_L
\]
\[
This is equivalent to the conditions
\[
\psi([\beta^2(u), [\beta(v), \alpha(w)]_L + [\beta^2(v), [\beta(w), \alpha(u)]_L + [\beta^2(w), [\beta(u), \alpha(v)]_L
= 0. \quad (6.4)
\]
\[
\psi([\beta^2(u), [\beta(v), \alpha(w)]_L + [\beta^2(v), [\beta(w), \alpha(u)]_L + [\beta^2(w), [\beta(u), \alpha(v)]_L
= 0. \quad (6.5)
\]

Obviously, (6.4) and (6.3) means that \(\psi\) must itself define a \(\delta\)-Bihom-Jordan-Lie algebra structure on \(L\). Furthermore, (6.5) means that \(\psi\) is closed with respect to the \(\alpha^{-1}\beta\)-adjoint representation \(\text{ad}_{-1,1}\), i.e. \(d_{-1,1}\psi = 0\).
\[
d_{-1,1}\psi(u, v, w)
= \delta[\beta^2(u), [\beta(v), \alpha(w)]_L - [\beta^2(v), [\beta(w), \alpha(u)]_L + [\beta^2(w), [\beta(u), \alpha(v)]_L
\]
\[
= -\psi([\alpha^{-1}\beta(u), v]_L, \beta(w)) + \psi([\alpha^{-1}\beta(u), [\beta(v), \alpha(w)]_L, [\beta(w)]_L
\]
\[
= \delta[\beta^2(v), [\beta(w), \alpha(u)]_L + [\delta[\beta^2(v), [\beta(w), \alpha(u)]_L
\]
\[
= 0.
\]

A deformation is said to be trivial if there is a linear operator \(N \in C^1_{\alpha,\beta}(L; L)\) such that for \(T_t = \text{id} + tN\), there holds
\[
T_t[u, v]_L = [T_t(u), T_t(v)]_L. \quad (6.6)
\]

**Definition 6.4.** A linear operator \(N \in C^1_{\alpha,\beta}(L, L)\) is called a Bihom-Nijenhuis operator if we have
\[
[Nu, Nv]_L = [u, v]_N, \quad (6.7)
\]
where the bracket \([\cdot, \cdot]_N\) is defined by
\[
[u, v]_N \triangleq [Nu, Nv]_L + [u, Nv]_L - [u, v]_L. \quad (6.8)
\]

**Theorem 6.5.** Let \(N \in C^1_{\alpha,\beta}(L, L)\) be a Bihom-Nijenhuis operator. Then a deformation of the regular \(\delta\)-Bihom-Jordan-Lie algebra \((L, [\cdot, \cdot]_L, \alpha, \beta)\) can be obtained by putting
\[
\psi(u, v) = \delta d_{-1,1} N(u, v) = [u, v]_N.
\]

Furthermore, this deformation is trivial.
Thus we need to check the Bihom-Jacobi identity for $\psi$. Using the explicit expression of $\psi$, and we denote $\mathcal{O}_{u,v,w}$ the summation over the cyclic permutation on $u,v,w$. We have

\[
\mathcal{O}_{u,v,w} \psi(\beta^2(u), \psi(\beta(v), \alpha(w))) = \mathcal{O}_{u,v,w} \psi(\beta^2(u), [N\beta(v), \alpha(w)] + [\beta(v), N\alpha(u)] - N[\beta(v), \alpha(w)])
\]

Furthermore, also by the fact that $u$, (6.7) and (6.8), we have

\[
\mathcal{O}_{u,v,w} \psi(\beta^2(u), [N\beta(v), \alpha(w)]) + \psi(\beta^2(u), [\beta(v), N\alpha(u)]) - \psi(\beta^2(u), N[\beta(v), \alpha(w)])
\]

\[
= \mathcal{O}_{u,v,w} \left[ N\beta^2(u), [N\beta(v), \alpha(w)] + [N^2\beta^2(v), [\beta(w), N\beta(u)] + [\beta^2(w), N[\beta(u), \alpha(v)]] \right]
\]

\[
+ \mathcal{O}_{u,v,w} N[\beta^2(v), N[\beta(w), \alpha(u)]] - [N\beta^2(v), N[\beta(w), \alpha(u)]] = N\beta^2(v), [\beta(w), \alpha(u)]) + N^2[\beta^2(v), [\beta(w), \alpha(u)]]
\]

By the Bihom-Jacobi identity of $L$, we have

\[
\mathcal{O}_{u,v,w} N[\beta^2(v), N[\beta(w), \alpha(u)]] - [N\beta^2(v), N[\beta(w), \alpha(u)]]
\]

Then, we have

\[
\mathcal{O}_{u,v,w} \psi(\beta^2(u), \psi(\beta(v), \alpha(w))) = -N[N\beta^2(v), [\beta(w), \alpha(u)]] - N[\beta^2(v), [N\beta(v), \alpha(w)] + [\beta^2(w), [\beta(u), N\beta(v)]]
\]

\[
- N[\beta^2(Nv), [\beta(w), \alpha(u)]] + [\beta^2(u), [N\beta(v), \alpha(w)] + [\beta^2(w), [\beta(u), N\beta(v)]] = 0.
\]

Thus $\psi$ generates a deformation of the $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$. Let $T_I = \text{id} + tN$, then we have

\[
T_I[u,v]_L = (\text{id} + tN)([u,v]_L + t\psi(u,v)) = (\text{id} + tN)([u,v]_L + t[u,v]_N)
\]

\[
= [u,v]_L + t([u,v]_N + N[u,v]_L] + t^2N[u,v]_N.
\]

On the other hand, we have

\[
[T_I(u), T_I(v)]_L = [u + tNu,v + tNv]_L
\]

\[
= [u,v]_L + t([N[u,v]_L + [u,Nv]_L] + t^2[Nu,Nv]_L.
\]

By the equations (6.7) and (6.8), we have

\[
T_I[u,v]_L = [T_I(u), T_I(v)]_L,
\]

which implies that the deformation is trivial. \qed

7. $T^*$-extensions of $\delta$-Bihom-Jordan-Lie algebras

The last part deals with $T^*$-extension. We provide in this section, for $\delta$-Bihom-Jordan-Lie algebras, characterizations of $T^*$-extensions and observations about $T^*$-extensions of nilpotent and solvable $\delta$-Bihom-Jordan-Lie algebras. This method was introduced by Martin Bordemann in [2].
Definition 7.1. Let $(L, [\cdot, \cdot]_L, \alpha, \beta)$ be a $\delta$-Bihom-Jordan-Lie algebra. A bilinear form $f$ on $L$ is said to be nondegenerate if

$$L^\perp = \{ x \in L | f(x, y) = 0, \forall y \in L \} = 0;$$

$\alpha\beta$-invariant if

$$f([\beta(x), \alpha(y)], \alpha(z)) = f(\alpha(x), [\beta(y), \alpha(z)]), \forall x, y, z \in L;$$

symmetric if

$$f(x, y) = f(y, x).$$

A subspace $I$ of $L$ is called isotropic if $I \subseteq I^\perp$.

Definition 7.2. Let $(L, [\cdot, \cdot]_L, \alpha, \beta)$ be a $\delta$-Bihom-Jordan-Lie algebra over a field $\mathbb{K}$. If $L$ admits a nondegenerate invariant symmetric bilinear form $f$, then we call $(L, f, \alpha, \beta)$ a quadratic $\delta$-Bihom-Jordan-Lie algebra. In particular, a quadratic vector space $V$ is a vector space admitting a nondegenerate symmetric bilinear form.

Let $(L, [\cdot, \cdot]_L, \alpha_1, \beta_1)$ be another $\delta$-Bihom-Jordan-Lie algebra. Two quadratic $\delta$-Bihom-Jordan-Lie algebras $(L, f, \alpha, \beta)$ and $(L', f', \alpha_1, \beta_1)$ are said to be isometric if there exists a $\delta$-Bihom-Jordan-Lie algebra isomorphism $\phi : L \rightarrow L'$ such that

$$f(x, y) = f'(\phi(x), \phi(y)), \forall x, y \in L.$$ 

Lemma 7.3. Let $ad$ be the adjoint representation of a $\delta$-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$. Let us consider $L^*$ the dual space of $L$, $\tilde{\alpha}, \tilde{\beta} : L^* \rightarrow L^*$ two homomorphisms defined by

$$\tilde{\alpha}(f) = f \circ \alpha, \quad \tilde{\beta}(f) = f \circ \beta, \forall f \in L^*.$$

Then the linear map $\pi : L \rightarrow \text{End}(L^*)$ defined by, $\pi(x)(f)(y) = -\delta f \circ \text{ad}(x)(y), \forall x, y \in L$, is a representation of $L$ on $(L^*, \tilde{\alpha}, \tilde{\beta})$ if and only if

$$\alpha \circ \text{ad} \circ x = \text{ad} x \circ \alpha; \quad (7.1)$$

$$\beta \circ \text{ad} \circ x = \text{ad} x \circ \beta; \quad (7.2)$$

$$\text{ad}(\alpha(x)) \circ \text{ad}(\beta(y)) - \delta \text{ad} y \circ \text{ad}(\alpha\beta(x)) = \beta \circ \text{ad}[\beta(x), y]_L. \quad (7.3)$$

We call the representation $\pi$ the coadjoint representation of $L$.

Proof. Firstly, we have

$$(\pi(\alpha(x)) \circ \tilde{\alpha})(f) = -\delta \tilde{\alpha}(f) \circ \text{ad} \alpha(x) = -\delta f \circ \alpha \circ \text{ad} \alpha(x),$$

and

$$\tilde{\alpha}(\pi(x))(f) = -\delta \tilde{\alpha}(f \circ \text{ad} x) = -\delta f \circ \text{ad} x \circ \alpha.$$

Similarly,

$$(\pi(\beta(x)) \circ \tilde{\beta})(f) = -\delta \tilde{\beta}(f) \circ \text{ad} \beta(x) = -\delta f \circ \beta \circ \text{ad} \beta(x),$$

and

$$\tilde{\beta}(\pi(x))(f) = -\delta \tilde{\beta}(f \circ \text{ad} x) = -\delta f \circ \text{ad} x \circ \beta.$$

Therefore,

$$(\pi([\beta(x), y]) \circ \tilde{\beta})(f) = -\delta f \circ \beta \circ \text{ad}[\beta(x), y];$$

$$=(\pi(\alpha\beta(x)) \circ \pi(y) - \delta \pi(\beta(y)) \circ \pi(\alpha(x)))(f)$$

$$= -\delta \pi(\alpha\beta(x))(f \circ \text{ad} y) + \pi(\beta(y))(f \circ \text{ad} \alpha(x))$$

$$= f \circ \text{ad} y \circ \text{ad} \beta(x) - \delta f \circ \text{ad} \alpha(x) \circ \text{ad} \beta(y)$$

$$= -\delta f \circ (\text{ad} \alpha(x) \circ \text{ad} \beta(y) - \delta \text{ad} y \circ \text{ad} \alpha \beta(x)).$$
Then we have
\[
\pi(\alpha(x)) \circ \tilde{\alpha} = \tilde{\alpha}(\pi(x));
\]
\[
\pi(\beta(x)) \circ \tilde{\beta} = \tilde{\beta}(\pi(x));
\]
\[
\pi([\beta(x), y]) \circ \tilde{\beta} = \pi(\alpha(\beta(x))) \circ \pi(y) - \pi(\beta(y)) \circ \pi(\alpha(x)).
\]

Then \(\pi\) is a representation of \(L\) on \((L^*, \tilde{\alpha}, \tilde{\beta})\).

\[\Box\]

**Lemma 7.4.** Under the above notations, let \((L, [\cdot, \cdot]_L, \alpha, \beta)\) be a \(\delta\)-Bihom-Jordan-Lie algebra, and \(\omega : L \times L \to L^*\) be a bilinear map. Assume that the coadjoint representation exists. The space \(L \oplus L^*\), provided with the following bracket and a linear map defined respectively by
\[
[x + f, y + g]_{L \oplus L^*} = [x, y]_L + \omega(x, y) + \delta \pi \circ \pi(z) - \pi(\alpha^{-1}\beta(\pi(y))) \circ \tilde{\beta}^{-1}(f),
\]

\[
\alpha'(x + f) = \alpha(x) + f \circ \alpha,
\]

\[
\beta'(x + f) = \beta(x) + f \circ \beta.
\]

Then \((L \oplus L^*, [\cdot, \cdot]_{L \oplus L^*}, \alpha', \beta')\) is a \(\delta\)-Bihom-Jordan-Lie algebra if and only if \(\omega\) is a 2-cocycle: \(L \times L \to L^*\), i.e. \(\omega \in Z^2(L, L^*)\).

**Proof.** For any elements \(x + f, y + g, z + h \in L \oplus L^*\). We have
\[
[\beta'(x + f), \alpha'(y + g)] = [\beta(x) + f \circ \beta, \alpha(y) + g \circ \alpha]
\]
\[
= [\beta(x), \alpha(y)]_L + w(\beta(x), \alpha(y)) + \delta \pi \circ \pi(z) - \pi(\alpha^{-1}\beta(\alpha(y))) \circ \tilde{\beta}^{-1}(f \circ \beta)
\]
\[
= [\beta(x), \alpha(y)]_L + w(\beta(x), \alpha(y)) + \delta \pi(\beta(y)) \circ \tilde{\beta}^{-1}(f \circ \beta)
\]

Similarly, we have
\[
[\beta'(y + g), \alpha'(x + f)] = [\beta(y), \alpha(x)]_L + w(\beta(x), \alpha(y)) + \delta \pi(\beta(y)) \circ \tilde{\beta}^{-1}(f \circ \beta) - \pi(\beta(y))(f \circ \alpha).
\]

Then, we have \([\beta'(x + f), \alpha'(y + g)] = -\delta[\beta'(y + g), \alpha'(x + f)]\) if and only if
\[
w(\beta(x), \alpha(y)) = -\delta w(\beta(y), \alpha(x)).
\]

Therefore,
\[
[\beta'(x + f), \beta'(y + g), \alpha'(z + h)]
\]
\[
= [\beta^2(x) + f \circ \beta^2, [\beta(y), \alpha(z)]_L + w(\beta(y), \alpha(z)) + \delta \pi(\beta(y))(h \circ \alpha) - \pi(\alpha^{-1}\beta(\alpha(z))) \circ \tilde{\beta}^{-1}(g \circ \beta)]
\]
\[
= [\beta^2(x) + f \circ \beta^2, [\beta(y), \alpha(z)]_L + w(\beta(y), \alpha(z)) + \delta \pi(\beta(y))(h \circ \alpha) - \pi(\beta(z))(g \circ \alpha)]
\]
\[
= [\beta^2(x), [\beta(y), \alpha(z)]_L + w(\beta(y), \alpha(z)) + \delta \pi(\beta^2(x)w(\beta(y), \alpha(z)) + \pi(\beta^2(x))\pi(\beta(y))(h \circ \alpha) - \delta \pi(\beta^2(x))\pi(\beta(z))(g \circ \alpha) - \pi(\alpha^{-1}\beta(\beta(y), \alpha(z)))(f \circ \beta \circ \alpha))
\]

And
\[
[\beta'(y + g), \alpha'(x + f)]
\]
\[
= [\beta^2(y), [\beta(z), \alpha(x)]_L + w(\beta^2(y), [\beta(z), \alpha(x)]_L + \delta \pi(\beta^2(y))w(\beta(z), \alpha(x)) + \pi(\beta^2(y))\pi(\beta(z))(f \circ \alpha) - \delta \pi(\beta^2(y))\pi(\beta(z))(h \circ \alpha) - \pi(\alpha^{-1}\beta(\beta(z), \alpha(x)))(g \circ \beta \circ \alpha),
\]

\[
\begin{aligned}
\omega &= [\beta^2(z), [\beta(x), \alpha(y)]_L] + w(\beta^2(z), [\beta(x), \alpha(y)]_L) \\
& \quad + \delta \pi(\beta^2(z)) w(\beta(x), \alpha(y)) + \pi(\beta^2(z)) \pi(\beta(x))(g \circ \alpha) \\
& \quad - \delta \pi(\beta^2(z)) \pi(\beta(y))(f \circ \alpha) - \pi(\alpha^{-1} \beta [\beta(x), \alpha(y)])(h \circ \beta \circ \alpha).
\end{aligned}
\]

Since \(\pi\) is the coadjoint representation of \(L\), we have
\[
\begin{aligned}
\pi(\alpha^{-1} \beta [\beta(x), \alpha(y)]_L) h \circ \beta \circ \alpha &= \pi(\beta^2(y)) \pi(\beta(z))(f \circ \alpha) - \delta \pi(\beta^2(z)) \pi(\beta(y))(f \circ \alpha), \\
\end{aligned}
\]

and
\[
\begin{aligned}
\pi(\alpha^{-1} \beta [\beta(y), \alpha(z)]_L) g \circ \beta \circ \alpha &= \pi(\beta^2(z)) \pi(\beta(x))(g \circ \alpha) - \delta \pi(\beta^2(x)) \pi(\beta(z))(g \circ \alpha).
\end{aligned}
\]

Consequently, \([\beta^2(x + f), [\beta(x), \alpha(z)] + \beta^2(y + g), [\beta(x), \alpha(x)] + \beta^2(z + h), [\beta(x + f), \alpha(y + g)]\) if and only if
\[
\begin{aligned}
0 &= [\beta^2(x), [\beta(y), \alpha(x)]_L] + \delta \pi(\beta^2(x)) w(\beta(x), \alpha(y)) \\
& \quad + w(\beta^2(y), [\beta(z), \alpha(x)]_L) - \delta \pi(\beta^2(y)) w(\beta(z), \alpha(x)) \\
& \quad + \delta w(\beta^2(z), [\beta(x), \alpha(y)]_L) + \delta \pi(\beta^2(z)) w(\beta(x), \alpha(y)) \\
& \quad - \delta w([\beta(y), \alpha(z)]_L, \beta^2(x)) + w(\beta^2(y), [\beta(x), \alpha(y)]_L) - \delta w([\beta(x), \alpha(y)]_L, \beta^2(z)) \\
& \quad = \delta d_{-1,1} \omega(x, y, z).
\end{aligned}
\]

That is \(\omega \in Z^2_{\alpha, \beta}(L, L^*)\). Then confirmation holds if and only if \(\omega \in Z^2(L, L^*)\). Consequently, we prove the lemma.

Clearly, \(L^*\) is an abelian Bihom-ideal of \((L \oplus L^*, [\cdot, \cdot], \alpha', \beta')\) and \(L\) is isomorphic to the factor \(\delta\)-Bihom-Jordan-Lie algebra \((L \oplus L^*)/L^*\). Moreover, consider the following symmetric bilinear form \(q_L\) on \(L \oplus L^*\) for all \(x + f, y + g \in L \oplus L^*\),
\[
q_L(x + f, y + g) = f(y) + g(x).
\]

Then we have the following lemma.

**Lemma 7.5.** Let \(L, L^*, \omega\) and \(q_L\) be as above. Then the 4-tuple \((L \oplus L^*, q_L, \alpha', \beta')\) is a quadratic \(\delta\)-Bihom-Jordan-Lie algebra if and only if \(\omega\) is Jordancyclic in the following sense:
\[
\omega(\beta(x), \alpha(y))(\alpha(z)) = \omega(\beta(y), \alpha(z))(\alpha(x)) \text{ for all } x, y, z \in L.
\]

**Proof.** If \(x + f\) is orthogonal to all elements of \(L \oplus L^*\), then \(f(y) = 0\) and \(g(x) = 0\), which implies that \(x = 0\) and \(f = 0\). So the symmetric bilinear form \(q_L\) is nondegenerate.
Now suppose that \( x + f, y + g, z + h \in L \oplus L^* \), then
\[
q_L([\beta'(x + f), \alpha'(y + g)]_{L \oplus L^*}, \alpha'(z + h)) = q_L([\beta(x) + f \circ \beta, \alpha(y) + g \circ \alpha]_{L \oplus L^*}, \alpha(z) + h \circ \alpha)
\]
\[
= q_L([\beta(x), \alpha(y)]_L + \omega(\beta(x), \alpha(y)) + \delta \pi(\beta(x)) g \circ \alpha - \pi(\alpha^{-1} \beta \alpha(y)) \tilde{\alpha} \tilde{\beta}^{-1}(f \circ \beta), \alpha(z) + h \circ \alpha)
\]
\[
= \omega(\beta(x, \alpha(y))(\alpha(z)) - \delta g \circ \alpha([\beta(x), \alpha(z)]_L) + f \circ \alpha([\beta(y), \alpha(z)]_L)
\]
\[
+ h \circ \alpha((\beta(x, \alpha(y)]_L)
\]
\[
= \omega(\beta(x, \alpha(y))(\alpha(z)) - \delta g \circ \alpha([\beta(x), \alpha(z)]_L) + f \circ \alpha([\beta(y), \alpha(z)]_L)
\]
\[
- \delta h \circ \alpha([\beta(y), \alpha(x)]_L).
\]

On the other hand,
\[
q_L(\alpha'(x + f), [\beta'(y + g), \alpha'(z + h)]_{L \oplus L^*}) = q_L(\alpha(x) + f \circ \alpha, [\beta(y), \alpha(z)]_L + \omega(\beta(y), \alpha(z)) + \delta \pi(\beta(y)) h \circ \alpha
\]
\[- \pi(\alpha^{-1} \beta \alpha(y)) \tilde{\alpha} \tilde{\beta}^{-1}(g \circ \beta))
\]
\[
= q_L(\alpha(x) + f \circ \alpha, [\beta(y), \alpha(z)]_L + \omega(\beta(y), \alpha(z)) + \delta \pi(\beta(y)) h \circ \alpha - \pi(\beta(y)) (g \circ \alpha))
\]
\[
= f \circ \alpha([\beta(y), \alpha(z)]_L) + \omega(\beta(y), \alpha(z))(\alpha(x)) + \delta \pi(\beta(y)) h \circ \alpha(\alpha(x))
\]
\[- \pi(\beta(y)) (g \circ \alpha(\alpha(x)))
\]
\[
= \omega(\beta(y, \alpha(z))(\alpha(x)) + g \circ \alpha([\beta(z), \alpha(x)]_L) + f \circ \alpha([\beta(y), \alpha(z)]_L)
\]
\[- \delta h \circ \alpha([\beta(y), \alpha(x)]_L).
\]

Hence the lemma follows. \(\square\)

Now, for a Jordan cyclic 2-cocycle \( \omega \) we shall call the quadratic \( \delta \)-Bihom-Jordan-Lie algebra \((L \oplus L^*, q_L, \alpha', \beta')\) the \( T^*\)-extension of \( L \) (by \( \omega \)) and denote the \( \delta \)-Bihom-Jordan-Lie algebra \((L \oplus L^*, [\cdot, \cdot], \alpha', \beta')\) by \( T^*_\omega L \).

**Definition 7.6.** Let \( L \) be a \( \delta \)-Bihom-Jordan-Lie algebra over a field \( \mathbb{K} \). We inductively define a derived series
\[
(L^{(n)})_{n \geq 0} : L^{(0)} = L, \quad L^{(n+1)} = [L^{(n)}, L],
\]
and a central descending series
\[
(L^n)_{n \geq 0} : L^0 = L, \quad L^{n+1} = [L^n, L].
\]

\( L \) is called solvable and nilpotent (of length \( k \)) if and only if there is a (smallest) integer \( k \) such that \( L^{(k)} = 0 \) and \( L^{k+1} = 0 \), respectively.

In the following theorem we discuss some properties of \( T^*_\omega L \).

**Theorem 7.7.** Let \((L, [\cdot, \cdot]_L, \alpha, \beta)\) be a \( \delta \)-Bihom-Jordan-Lie algebra over a field \( \mathbb{K} \).

1. If \( L \) is solvable (nilpotent) of length \( k \), then the \( T^*\)-extension \( T^*_\omega L \) is solvable (nilpotent) of length \( r \), where \( k \leq r \leq k + 1 \) \( (k \leq r \leq 2k - 1) \).
2. If \( L \) is decomposed into a direct sum of two Bihom-ideals of \( L \), so is the trivial \( T^*\)-extension \( T^*_0 L \).
Lemma 7.8. Let \((L, q_L, \alpha, \beta)\) be a quadratic \(\delta\)-Bihom-Jordan-Lie algebra of even dimension \(n\) over a field \(K\) and \(I\) be an isotropic \(n/2\)-dimensional subspace of \(L\). If \(I\) is a Bihom-ideal of \((L, [\cdot, \cdot], \alpha, \beta)\), then \([\beta(I), \alpha(I)] = 0\).

Proof. Since \(\dim I + \dim I^\perp = n/2 + \dim I^\perp = n\) and \(I \subset I^\perp\), we have \(I = I^\perp\). If \(I\) is a Bihom-ideal of \((L, [\cdot, \cdot], \alpha, \beta)\), then \(q_L(\alpha(L), [\beta(L), \alpha(I^\perp)]) = q_L([\beta(L), \alpha(I^\perp)], \alpha(I^\perp)) \subseteq q_L([\beta(L), I], \alpha(I^\perp)) \subsetneq q_L(I, I^\perp) = 0\), which implies \([\beta(I), \alpha(I)] = [\beta(I), \alpha(I^\perp)] \subseteq (L^\perp)^\perp = 0\).

Theorem 7.9. Let \((L, q_L, \alpha, \beta)\) be a quadratic \(\delta\)-Bihom-Jordan-Lie algebra of even dimension \(n\) over a field \(K\) of characteristic not equal to two. Then \((L, q_L, \alpha, \beta)\) is isometric to a \(T^*\)-extension \((T^*_a B, q_B, \alpha', \beta')\) if and only if \(n\) is even and \((L, [\cdot, \cdot], \alpha, \beta)\) contains an isotropic Bihom-ideal \(I\) of dimension \(n/2\). In particular, \(B \cong L/I\), with \(B^*\) satisfying \(\alpha(B^*) \subseteq B^*\) and \(\beta(B^*) \subseteq B^*\).

Proof. 
\(\implies\) Since \(\dim B = \dim B^*, \dim T^*_a B = m\) is even. Moreover, it is clear that \(B^*\) is a Bihom-ideal of half the dimension of \(T^*_a B\) and by the definition of \(q_B\), we have \(q_B(B^*, B^*) = 0\), i.e., \(B^* \subseteq (B^*)^\perp\) and so \(B^*\) is isotropic.

\(\impliedby\) Suppose that \(I\) is an \(n/2\)-dimensional isotropic Bihom-ideal of \(L\). By Lemma 7.8, \([\beta(I), \alpha(I)] = 0\). Let \(B = L/I\) and \(p : L \to B\) be the canonical projection. Since \(\text{ch}K \neq 2\), we can choose an isotropic complement subspace \(B_0\) to \(I\) in \(L\), i.e., \(L = B_0 + I\) and \(B_0 \subseteq B_0^\perp\). Then \(B_0^\perp = B_0\) since \(\dim B_0 = n/2\).

Denote by \(p_0\) (resp. \(p_1\)) the projection \(L \to B_0\) (resp. \(L \to I\)) and let \(q_L^*_0\) denote the homogeneous linear map \(I \to B^*: i \mapsto q_L^*_0(i)\), where \(q_L^*_0(i)(p(x)) = q_L(i, x), \forall x \in L\). We claim that \(q_L^*_0\) is a linear isomorphism. In fact, if \(p(x) = p(y), \text{ then } x - y \in I\), hence
\( q_L(i, x - y) \in q_L(I, I) = 0 \) and so \( q_L(i, x) = q_L(i, y) \), which implies \( q_L^* \) is well-defined and it is easily seen that \( q_L^* \) is linear. If \( q_L^* (i) = q_L^* (j) \), then \( q_L^* (i)(p(x)) = q_L^* (j)(p(x)), \forall x \in L \), i.e., \( q_L(i, x) = q_L(j, x) \), which implies \( i - j \in L^\perp = 0 \), hence \( q_L^* \) is injective. Note that \( \dim I = \dim B^* \), then \( q_L^* \) is surjective.

In addition, \( q_L^* \) has the following property:

\[
q_L^* (\{\beta(x), \alpha(i)\})(p(\alpha(y))) = q_L^* (\{\beta(x), \alpha(i)\} L, \alpha(y)) = -q_L^* (\{\beta(x), \alpha(i)\} L, \alpha(y)) = -q_L^* (\{\beta(x), \alpha(i)\} p(\alpha(x)) p(\alpha(y))) = \delta(\pi(p(\beta(x))) q_L^* (\{\alpha(i)\})(p(\alpha(x)))) = p(\beta(x)), q_L^* (\{\alpha(i)\} L, \alpha(y)),
\]

where \( x, y \in L, i \in I \). A similar computation shows that

\[
q_L^* (\{\beta(x), \alpha(i)\}) = [p(\beta(x)), q_L^* (\{\alpha(i)\})_{L \oplus L^*}, q_L^* (\{\beta(x), \alpha(i)\}) = [q_L^* (\beta(x)), p(\beta(x))]_{L \oplus L^*}.
\]

Define a homogeneous bilinear map

\[
\omega : B \times B \rightarrow B^*
\]

\[
(p(b_0), p(b_0')) \mapsto q_L^* (p_1([b_0, b_0'])),
\]

where \( b_0, b_0' \in B_0 \). Then \( \omega \) is well-defined since the restriction of the projection \( p \) to \( B_0 \) is a linear isomorphism.

Let \( \varphi \) be the linear map \( L \rightarrow B \oplus B^* \) defined by \( \varphi(b_0 + i) = p(b_0) + q_L^* (i), \forall b_0 + i \in B_0 + I = L \). Since the restriction of \( p \) to \( B_0 \) and \( q_L^* \) are linear isomorphisms, \( \varphi \) is also a linear isomorphism. Note that

\[
\varphi([\beta(b_0 + i), \alpha(b_0')]) = \varphi([\beta(b_0), \alpha(b_0')] L + [\beta(b_0), \alpha(b_0')]_{L} + [\beta(i), \alpha(b_0')]_{L}
\]

\[
\varphi(\{p_0(\beta(b_0), \alpha(b_0'))_{L} + p_1([\beta(b_0), \alpha(b_0')]_{L} + [\beta(b_0), \alpha(b_0')]_{L} + [\beta(i), \alpha(b_0')]_{L}
\]

\[
\{p(\beta(b_0)), p(\alpha(b_0'))_{L} + \omega(p(\beta(b_0)), p(\alpha(b_0'))) + [p(\beta(b_0)), q_L^* (\alpha(i'))\}_{L}
\]

\[
-\delta(\pi(p(\beta(b_0))) q_L^* (\{\alpha(i')\})) = [\varphi(\{\beta(b_0 + i), \alpha(b_0' + i')\})_{L \oplus L^*}.
\]

Then \( \varphi \) is an isomorphism of algebras, and so \( (B \oplus B^*, [\cdot, \cdot]_{B \oplus B^*}, \alpha, \beta) \) is a \( \delta \)-Bihom-Jordan-Lie algebra. Furthermore, we have

\[
q_B(\varphi(b_0 + i), \varphi(b_0' + i')) = q_B(p(b_0) + q_L^* (i), p(b_0') + q_L^* (i'))
\]

\[
q_L(i, b_0') = q_L(i, b_0' + i),
\]

then \( \varphi \) is isometric. The relation

\[
q_B([\beta'(\varphi(x)), \alpha'(\varphi(\alpha(y)))), \alpha'(\varphi(\alpha(z)))) = q_B([\varphi(\beta(x)), \varphi(\alpha(y))), \varphi(\alpha(z))] = q_L([\beta(x), \alpha(y)], \alpha(z))
\]

\[
q_B([\varphi(\beta(x)), \varphi(\alpha(y))), \varphi(\alpha(z))] = q_B(\varphi(\alpha(x)), [\varphi(\beta(y)), \varphi(\alpha(z))]
\]

\[
q_B(\alpha'(\varphi(x)), [\beta'(\varphi(y)), \alpha'(\varphi(z))])
\]
which implies that \( q_B \) is a nondegenerate invariant symmetric bilinear form, and so 
\((B \oplus B^*, q_B, \alpha', \beta')\) is a quadratic \( \delta \)-Bihom-Jordan-Lie algebra. In this way, we get a 
\( T^* \)-extension \( T^*_\omega B \) of \( B \) and consequently, \((L, q_L, \alpha, \beta)\) and \((T^*_\omega B, q_B, \alpha', \beta')\) are isomorphic 
as required. □

Let \((L, [\cdot, \cdot]_L, \alpha, \beta)\) be a \( \delta \)-Bihom-Jordan-Lie algebra over a field \( K \), and let \( \omega_1 : L \times L \to L^* \) and \( \omega_2 : L \times L \to L^* \) be two different Jordan cyclic \( 2 \)-cocycles. The \( T^* \)-extensions \( T^*_\omega L \) and \( T^*_\omega L \) of \( L \) are said to be equivalent if there exists an isomorphism of \( \delta \)-Bihom-Jordan-Lie algebras \( \phi : T^*_\omega L \to T^*_\omega L \) which is the identity on the Bihom-ideal \( L^* \) and which induces the identity on the factor \( \delta \)-Bihom-Jordan-Lie algebra \( T^*_\omega L/L^* \cong L \cong T^*_\omega L/L^* \). The two \( T^* \)-extensions \( T^*_\omega L \) and \( T^*_\omega L \) are said to be isometrically equivalent if they are equivalent and \( \phi \) is an isometry.

**Proposition 7.10.** Let \( L \) be a \( \delta \)-Bihom-Jordan-Lie algebra over a field \( K \) of characteristic 
not equal to 2, and \( \omega_1, \omega_2 \) be two Jordan cyclic \( 2 \)-cocycles \( L \times L \to L^* \). Then we have

(i) \( T^*_\omega L \) is equivalent to \( T^*_\omega L \) if and only if there is \( z \in C^1(L, L^*) \) such that

\[
\omega_1(x, y) - \omega_2(x, y) = \delta \pi(x) z(y) - \pi(\alpha^{-1} \beta(y) \tilde{\alpha} \tilde{\beta}^{-1} z(x) - z(x, y), \forall x, y \in L. \quad (7.7)
\]

If this is the case, then the symmetric part \( z_s \) of \( z \), defined by \( z_s(x)(y) := \frac{1}{2} (z(x)(y) + z(y)(x)) \), for all \( x, y \in L \), induces a symmetric invariant bilinear form on \( L \).

(ii) \( T^*_\omega L \) is isometrically equivalent to \( T^*_\omega L \) if and only if there is \( z \in C^1(L, L^*) \) such 
that \((29)\) holds for all \( x, y \in L \) and the symmetric part \( z_s \) of \( z \) vanishes.

**Proof.** (i) \( T^*_\omega L \) is equivalent to \( T^*_\omega L \) if and only if there is an isomorphism of \( \delta \)-Bihom-Jordan-Lie algebras \( \Phi : T^*_\omega L \to T^*_\omega L \) satisfying \( \Phi|_{L^*} = 1_{L^*} \) and \( x - \Phi(x) \in L^* \), \( \forall x \in L \).

Suppose that \( \Phi : T^*_\omega L \to T^*_\omega L \) is an isomorphism of \( \delta \)-hom-Jordan-Lie algebra and define a linear map \( z : L \to L^* \) by \( z(x) := \Phi(x) - x \), then \( z \in C^1(L, L^*) \) and for all 
\( x + f, y + g \in T^*_\omega L \), we have

\[
\begin{align*}
\Phi([x + f, y + g]) & = \Phi([x, y]) + \omega_1(x, y) + \delta \pi(x) g - \pi(\alpha^{-1} \beta(y) \tilde{\alpha} \tilde{\beta}^{-1} f) \\
& = [x, y] + z([x, y]) + \omega_1(x, y) + \delta \pi(x) g - \pi(\alpha^{-1} \beta(y) \tilde{\alpha} \tilde{\beta}^{-1} f).
\end{align*}
\]

On the other hand,

\[
\begin{align*}
\Phi(x + f, \Phi(y + g)) & = [x + z(x) + f, y + z(y) + g] \\
& = [x, y] + \omega_2(x, y) + \delta \pi(x) g + \delta \pi(x) z(y) - \pi(\alpha^{-1} \beta(y) \tilde{\alpha} \tilde{\beta}^{-1} z(x) - z(\beta(x), y)]_{L^*} (\alpha(\beta(\alpha^{-1} \beta(y) \tilde{\alpha} \tilde{\beta}^{-1} f)).
\end{align*}
\]

Since \( \Phi \) is an isomorphism, \((7.7)\) holds.

Conversely, if there exists \( z \in C^1(L, L^*) \) satisfying \((7.7)\), then we can define \( \Phi : T^*_\omega L \to T^*_\omega L \) by \( \Phi(x + f) := x + z(x) + f \). It is easy to prove that \( \Phi \) is an isomorphism of \( \delta \)-Bihom-Jordan-Lie algebras such that \( \Phi|_{L^*} = 1_{L^*} \) and \( x - \Phi(x) \in L^* \), \( \forall x \in L \), i.e. \( T^*_\omega L \) is equivalent to \( T^*_\omega L \).

Consider the symmetric bilinear form \( q_L : L \times L \to K, (x, y) \mapsto z_s(x)(y) \) induced by \( z_s \). Note that

\[
\begin{align*}
\omega_1(\beta(x), \alpha(y))(\alpha(m)) - \omega_2(\beta(x), \alpha(y))(\alpha(m)) & = \delta \pi(\beta(x)) z(\alpha(y))(\alpha(m)) - \pi(\alpha^{-1} \beta(y) \tilde{\alpha} \tilde{\beta}^{-1} z(\beta(x))(\alpha(m)) - z(\beta(x), \alpha(y)]_{L^*} (\alpha(\alpha^{-1} \beta(y) \tilde{\alpha} \tilde{\beta}^{-1}(x))) \\
& = \delta \pi(\beta(x)) z(\alpha(y))(\alpha(m)) - \pi(\alpha(y)) z(\alpha(x))(\alpha(m)) - z(\beta(x), \alpha(y)]_{L^*} (\alpha(\alpha^{-1} \beta(y) \tilde{\alpha} \tilde{\beta}^{-1}(x))) \\
& = -\delta z(\alpha(y))(\alpha(m)]_{L} + z(\alpha(x))(\alpha(m)]_{L} - z(\beta(x), \alpha(y)]_{L^*} (\alpha(\alpha^{-1} \beta(y) \tilde{\alpha} \tilde{\beta}^{-1}(x))).
\end{align*}
\]
and
\[
\omega_1(\beta(y), \alpha(m))(\alpha(x)) - \omega_2(\beta(y), \alpha(m))(\alpha(x))
\]
\[
= \delta \pi(\beta(y))z(\alpha(m))(\alpha(x)) - \pi(\alpha(m))z(\alpha(y))(\alpha(x)) - z(\beta(y), \alpha(m)|_L)(\alpha(x))
\]
\[
= -\delta \pi(\alpha(m))(\beta(y), \alpha(x)|_L) + z(\alpha(y))(\beta(m), \alpha(x)|_L) - z(\beta(y), \alpha(m)|_L)(\alpha(x))
\]
\[
= z(\alpha(m))(\beta(x), \alpha(y)|_L) - \delta z(\alpha(y))(\beta(x), \alpha(m)|_L) - z(\beta(y), \alpha(m)|_L)(\alpha(x)).
\]
Since both \(\omega_1\) and \(\omega_2\) are Jordancyclic, the right hand sides of above two equations are equal. Hence
\[
-\delta z(\alpha(y))(\beta(x), \alpha(m)|_L) + z(\alpha(x))(\beta(y), \alpha(m)|_L) - z(\beta(x), \alpha(y)|_L)(\alpha(m))
\]
\[
= z(\alpha(m))(\beta(x), \alpha(y)|_L) - \delta z(\alpha(y))(\beta(x), \alpha(m)|_L) - z(\beta(y), \alpha(m)|_L)(\alpha(x)).
\]
That is
\[
z(\alpha(x))(\beta(y), \alpha(m)|_L) + z(\beta(y), \alpha(m)|_L)(\alpha(x))
\]
\[
= z(\beta(x), \alpha(y)|_L)(\alpha(m)) + z(\alpha(m))(\beta(x), \alpha(y)|_L).
\]
Since \(\text{ch} K \neq 2\), \(q_L(\alpha(x), [\beta(y), \alpha(m)]) = q_L([\beta(x), \alpha(y)], \alpha(m))\), which proves the invariance of the symmetric bilinear form \(q_L\) induced by \(z_s\).

(ii) Let the isomorphism \(\Phi\) be defined as in (i). Then for all \(x + f, y + g \in L \oplus L^*\), we have
\[
q_B(\Phi(x + f), \Phi(y + g)) = q_B(x + z(x) + f, y + z(y) + g)
\]
\[
= z(x)(y) + f(y) + z(y)(x) + g(x)
\]
\[
= z(x)(y) + z(y)(x) + f(y) + g(x)
\]
\[
= 2z_s(x)(y) + q_B(x + f, y + g).
\]
Thus, \(\Phi\) is an isometry if and only if \(z_s = 0\). \(\Box\)

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References