Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (2) (2020), 766 – 776
DOI : 10.15672/hujms.471023

Research Article

# Oscillatory behavior of $n$-th order nonlinear delay differential equations with a nonpositive neutral term

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## Abstract

We study the oscillation problem for solutions of a class of $n$-th order nonlinear delay differential equations with nonpositive neutral terms. The obtained results improve and correlate many of the known oscillation criteria in the literature for neutral and non-neutral equations.

**Mathematics Subject Classification (2010).** 34K11, 34C10

**Keywords.** oscillation, second order differential equation, neutral term

## 1. Introduction

Consider the nonlinear $n$-th order delay differential equation of the form

$$\left(a(t)\left([x(t)-p(t)x(\sigma(t))]^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)x^{\beta}(\tau(t)) = 0, \quad t \ge t_0, \tag{1.1}$$

where $n$ is even and $t_0 > 0$ is fixed. It will be assumed that

- $(i)$ $\alpha, \beta$ are the ratios of positive odd integers such that $\alpha \ge \beta$;
- $(ii)$ $a \in \mathcal{C}^1([t_0,\infty),\mathbb{R})$, $a(t) > 0$, $a'(t) \ge 0$.
- $(iii)$ $p, q \in \mathcal{C}([t_0,\infty),\mathbb{R})$, $0 < p(t) \le p_0 < 1$, $q(t) \ge 0$ and $q(t)$ is not identically zero for all large $t$;
- $(iv)$ $\tau, \sigma \in \mathcal{C}^1([t_0,\infty),\mathbb{R})$, $\tau(t) \le t$, $\sigma(t) \le t$, $\tau'(t) \ge 0, \sigma'(t) > 0$, and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\sigma(t) = \infty$.

By a solution of Eq. (1.1) we mean a function $x(t) \in \mathcal{C}^{n-1}([T_x,\infty),\mathbb{R})$, for some $t_x \ge t_0$, which has the property $a(t)([x(t)-p(t)x(\sigma(t))]^{(n-1)})^{\alpha} \in \mathcal{C}^1([t_x,\infty),\mathbb{R})$ and satisfies Eq. (1.1) on $[t_x,\infty)$. We consider only those solutions $x(t)$ of (1.1) which satisfy $\sup\{x(t) : t \ge T\} > 0$ for all $T \ge t_x$. Such a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is called nonoscillatory. Equation (1.1) is said

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Received: 16.10.2018; Accepted: 26.03.2019

to be oscillatory if all its solutions are oscillatory. We note that the equation is called half-linear when $\alpha = \beta$, and sub-half-linear when $\alpha > \beta$.

Recently, the oscillation of equations of the form (1.1) with linear and nonlinear neutral term, has been considered in [1–8, 11, 13–15, 17, 20, 21], where it is usually assumed that

$$-\infty < -p_0 \leq p(t) \leq 0.$$

We note that there are only few results dealing with the oscillation of differential equations having a nonpositive neutral term. For an important initial contribution for such equations we refer in particular to [20], where equation (1.1) was studied in the special case $n = 2$ and $\alpha = 1$ under the assumptions

$$0 \leq p(t) \leq p_0 < 1, \quad \tau(t) = t - \tau_0, \quad \sigma(t) = t - \sigma_0.$$

Further contributions for (1.1) and its particular cases can be found in [5, 11, 15, 17, 21], where the authors established sufficient conditions ensuring that every solution $x$ of (1.1) is either oscillatory or converges to zero as $t \to \infty$. Unfortunately, these results cannot distinguish solutions with different behaviors.

In this article, mainly motivated by the ideas [5, 8, 9, 19], we present new oscillation theorems for $n$-th order nonlinear differential equations with a nonpositive neutral term of type (1.1). The obtained results improve and correlate many of the known results in the literature even for the case $p(t) = 0$. The method we employ here in this work has naturally a partial resemblance for the second-order case [9], however the results and most arguments are quite different due to higher-order nature of (1.1).

In the sequel, we let

$$A(v, u) = \int_u^v \frac{1}{a^{1/\alpha}(s)} ds, \quad v \geq u \geq t_0,$$

and assume that

$$A(t, t_0) \to \infty \quad \text{as } t \to \infty. \tag{1.2}$$

It turns out that the improper integral

$$\int_{t_0}^{\infty} q(s)\, ds \tag{1.3}$$

plays a key role in our study. In case it is convergent we define

$$Q(t) = \int_t^{\infty} q(s) ds, \quad t \geq t_0.$$

The results of this paper are presented in a form which is essentially new. The paper is organized as follows. In Section 2 we provide some useful lemmas to be relied upon in the proofs of the theorems in Section 3. The last section is devoted to the illustrative examples. It may be of interest to study equation (1.1) with $\beta > \alpha$.

## 2. Lemmas

All the functional inequalities are assumed to hold eventually, that is, they are satisfied for all $t$ large enough.

In what follows, we put

$$y(t) = x(t) - p(t)x(\sigma(t)). \tag{2.1}$$

**Lemma 2.1** (See [12])**.** Let $u$ be a positive and $k$-times differentiable function on an interval $[t_a, \infty)$ with its $k$-th derivative $u^{(k)}$ nonpositive on $[t_a, \infty)$ and not identically zero on any subarray of $[t_a, \infty)$. Then there exists a $t_b \geq t_a$ and an integer $l$, $0 \leq l \leq k - 1$, with $k + l$ odd so that

$$\begin{cases} (-1)^{l+j} u^{(j)} > 0 \quad \text{on} \quad [t_b, \infty) \quad (j = l, \ldots, k-1), \\ u^{(i)} > 0 \quad \text{on} \quad [t_b, \infty) \quad (i = 1, \ldots, l-1), \quad \text{when} \quad l > 1. \end{cases}$$

**Lemma 2.2** (See [16])**.** Let $u$ be as in Lemma 2.1 and $t_b \geq t_a$ be assigned to $u$ by Lemma 2.1. Moreover, let $\theta$ be a number with $0 < \theta < 1$. Then there exists a $t_c \geq t_b/\theta$ such that

$$u(\theta t) \geq \frac{[\theta(1-\theta)]^{k-1}}{(k-1)!} t^{k-1} u^{(k-1)}(t), \quad \text{for all} \quad t \geq t_c. \tag{2.2}$$

In addition, when $\lim_{t\to\infty} u(t) \neq 0$, for some $t_c \geq t_a$ we have

$$u(t) \geq \frac{\theta}{(k-1)!} t^{k-1} u^{(k-1)}(t), \quad \text{for every} \quad t \geq t_c. \tag{2.3}$$

**Lemma 2.3** (See [18])**.** Let $u(t)$ be a bounded $k$-times differentiable function on an interval $[t_a, \infty)$ with

$$u(t) > 0 \quad (-1)^k u^{(k)}(t) \geq 0 \quad \text{for} \quad t \geq t_a.$$

Then there exists a $t_b \geq t_a$ such that

$$(-1)^i u^{(i)}(t) \geq 0 \quad \text{for every} \quad t \geq t_b, \quad i = 1, 2, \ldots, k$$

and

$$u(\xi) \geq \frac{(-1)^{k-1} u^{(k-1)}(\eta)}{(k-1)!} (\eta - \xi)^{k-1} \quad \text{for every} \quad t \geq t_b, \quad t_b \leq \xi \leq \eta. \tag{2.4}$$

**Lemma 2.4.** Assume that $x(t)$ is a positive solution of (1.1) for $t \geq t_1$, $t_1 \in [t_0, \infty)$. Then there exists $t_2 \in [t_1, \infty)$ such that the corresponding function $y(t)$ defined by (2.1) satisfies one of the following two cases:

$$y(t) > 0, \quad y'(t) > 0, \quad y^{(n-1)}(t) > 0, \quad \left(a(t)\left(y^{(n-1)}(t)\right)^\alpha\right)' \leq 0, \tag{$C_1$}$$

$$y(t) < 0, \quad (-1)^{i+1} y^{(i)}(t) > 0, \quad i = 1, 2, \ldots, n, \quad \left(a(t)\left(y^{(n-1)}(t)\right)^\alpha\right)' \leq 0, \tag{$C_2$}$$

for $t \geq t_2$.

***Proof.*** Let $x(t)$ be a positive solution of (1.1), say $x(t)$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. By Eq. (1.1), we have

$$\left(a(t)\left(y^{(n-1)}(t)\right)^\alpha\right)' = -\,q(t) x^\beta(\tau(t)) \leq 0, \quad t \geq t_1. \tag{2.5}$$

Hence $a(t)\left(y^{(n-1)}(t)\right)^\alpha$ is nonincreasing and of one sign eventually. That is, there exists $t_2 \geq t_1$ such that either $y^{(n-1)}(t) > 0$ or $y^{(n-1)}(t) < 0$ for $t \geq t_2$. We claim that $y^{(n-1)}(t) > 0$ for $t \geq t_2$. To see this, suppose on the contrary that $y^{(n-1)}(t) < 0$ for $t \geq t_2$. Then

$$a(t)\left(y^{(n-1)}(t)\right)^\alpha \leq a(t_2)\left(y^{(n-1)}(t_2)\right)^\alpha =: c < 0, \quad t \geq t_2.$$

Integrating the above inequality, we see that

$$y^{(n-2)}(t) \leq y^{(n-2)}(t_2) + c^{1/\alpha} \int_{t_2}^t a^{-1/\alpha}(s) \mathrm{d}s.$$

By virtue of (1.2), we have $\lim_{t\to\infty} y(t) = -\infty$. Since $y(t) > -x(\sigma(t))$, $x(t)$ must be unbounded, and so there exists a sequence $\{T_k\}_{k=0}^\infty$ such that $x(T_k) = \max\{x(s) : T_0 \leq s \leq T_k\}$ with $\lim_{k\to\infty} T_k = \infty$ and $\lim_{k\to\infty} x(T_k) = \infty$. Furthermore, since $\sigma(T_k) > T_0$ for all $k$ sufficiently large and $\sigma(t) \leq t$, we see that

$$x(\sigma(T_k)) \leq \max\{x(s) : T_0 \leq s \leq T_k\} = x(T_k).$$

Therefore, for all large $k$,

$$y(T_k) = x(T_k) - p(T_k)x(\sigma(T_k)) \geq (1 - p(T_k))x(T_k) > 0$$

which contradicts the fact that $\lim_{t\to\infty} y(t) = -\infty$. Hence, we have proven the claim. In view of (2.5) and (ii), we also have $y^{(n)}(t) < 0$ for $t \geq t_2$. There are two possibilities to consider: either $y(t) > 0$ or $y(t) < 0$ for $t \geq t_2$. If $y(t) > 0$, then it follows from Lemma 2.1 that $y(t)$ satisfies $(C_1)$. If $y(t) < 0$, then we see that

$$x(t) \leq p(t)x(\sigma(t)) \leq x(\sigma(t)), \tag{2.6}$$

which implies that $x(t)$ and hence $y(t)$ are bounded functions. Using Lemma 2.3 with $u = -y$, we obtain that $y(t)$ satisfies $(C_2)$. The proof is complete. □

**Remark 2.1.** For any positive solution $x(t)$ of (1.1), the case $(C_2)$ is completely caused by presence of the neutral term. If $p(t) = 0$, such a case never occurs.

## 3. Oscillation of solutions

For the sake of clarity, we put

$$k(t) = \begin{cases} 1, & \text{when } \beta = \alpha \\ c\left(t^{n-2}A(t,t_1)\right)^{\alpha-\beta}, & \text{when } \beta < \alpha, \end{cases}$$

$$l(t) = \begin{cases} \left(\frac{4^{1-n}}{(n-1)!}\right)^{\beta}, & \text{when } \beta = \alpha \\ \tilde{c}\left(t^{n-2}A(t,t_1)\right)^{\alpha-\beta}, & \text{when } \beta < \alpha, \end{cases}$$

and

$$R(t) = \frac{\tau^{n-2}(t)\tau'(t)}{\left(a(\tau(t))k(t)\right)^{1/\alpha}}, \quad h(t) = \sigma^{-1}(\tau(t))$$

where $c, \tilde{c}, t_1 \in \mathbb{R}$.

We start with the following theorem.

**Theorem 3.1.** Let conditions (*i*)–(*iv*) and (1.2) hold, and let the integral (1.3) be convergent. If there exists a function $\rho \in \mathcal{C}^1([t_0,\infty),(0,\infty))$ with $\rho'(t) \geq 0$ such that, for all sufficiently large $c, \tilde{c}, t_1$, and for some $T > t_1$,

$$\limsup_{t\to\infty}\left[\rho(t)Q(t) + \int_T^t \left[\rho(s)q(s) - \mu\frac{a(\tau(s))k(s)(\rho'(s))^{\alpha+1}}{\left(\tau^{n-2}(s)\tau'(s)\rho(s)\right)^{\alpha}}\right]\mathrm{d}s\right] = \infty, \tag{3.1}$$

where

$$\mu = \frac{\alpha^\alpha}{(1+\alpha)^{\alpha+1}}\left(\frac{2(n-2)!}{\beta\, 4^{2-n}}\right)^{\alpha},$$

and

$$\limsup_{t\to\infty}\left[a^{-1}(h(t))\int_{h(t)}^t\left(\frac{(h(t)-h(s))^{n-1}}{(n-1)!}\right)^{\beta}\frac{q(s)}{p^{\beta}(h(s))}\mathrm{d}s\right] > \begin{cases} 1 & \text{when } \beta = \alpha, \\ 0 & \text{when } \beta < \alpha, \end{cases} \tag{3.2}$$

then (1.1) is oscillatory.

***Proof.*** Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. It follows from Lemma 2.4 that there exists $t_2 \in [t_1,\infty)$ such that the function $y$ defined by (2.1) satisfies either $(C_1)$ or $(C_2)$ for $t \geq t_2$. We will consider both cases separately.

At first, assume that $(C_1)$ holds. In view of (2.5) and $x(t) \geq y(t)$, we may write that

$$\left(a(t)\left(y^{(n-1)}(t)\right)^{\alpha}\right)' \leq -q(t)y^{\beta}(\tau(t)) \leq -q(t)y^{\beta}\left(\tau(t)/2\right). \tag{3.3}$$

Define

$$w(t) := \rho(t)\frac{a(t)(y^{(n-1)}(t))^{\alpha}}{y^{\beta}(\tau(t)/2)}, \quad t \geq t_2. \tag{3.4}$$

Therefore, $w(t) > 0$. By differentiating (3.4) and using (3.3), we get

$$\begin{aligned} w'(t) &= \left(\frac{\rho(t)}{y^{\beta}(\tau(t)/2)}\right)' (a(t)(y^{(n-1)}(t))^{\alpha} + \left(a(t)(y^{(n-1)}(t))^{\alpha}\right)' \frac{\rho(t)}{y^{\beta}(\tau(t)/2)} \\ &\le -\rho(t)q(t) + \left(\frac{\rho'(t)}{\rho(t)}\right) w(t) - \beta\rho(t)\frac{a(t)(y^{(n-1)}(t))^{\alpha} y'(\tau(t)/2)\tau'(t)}{2y^{\beta+1}(\tau(t)/2)}. \end{aligned} \tag{3.5}$$

Employing the inequality (2.2) in Lemma 2.2 with $u = y'$, it follows that there exists $t_3 \ge t_2$ such that

$$y'(\tau(t)/2) \ge M_{1/2}\tau^{n-2}(t)y^{(n-1)}(\tau(t)), \quad M_{1/2} = \frac{4^{2-n}}{(n-2)!}, \quad \text{for} \quad t \ge t_3. \tag{3.6}$$

Using (3.5), (3.6) and the fact that $a^{1/\alpha}(t)y^{(n-1)}(t)$ is decreasing, we have

$$w'(t) \le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\beta M_{1/2}}{2}\frac{\tau^{n-2}(t)\tau'(t)\rho(t)}{a^{1/\alpha}(\tau(t))}\frac{\left(a^{1/\alpha}(t)y^{(n-1)}(t)\right)^{\alpha+1}}{y^{\beta+1}(\tau(t)/2)}, \tag{3.7}$$

and hence

$$w' \le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w - \frac{\beta M_{1/2}}{2}\frac{\tau^{n-2}(t)\tau'(t)}{(a(\tau(t))\rho(t))^{1/\alpha}}y^{(\beta-\alpha)/\alpha}(\tau(t)/2)w^{(\alpha+1)/\alpha}(t).$$

If $\beta = \alpha$, then $y^{(\beta-\alpha)/\alpha}(t) = 1$ while for the case $\beta < \alpha$ and since $a(t)(y^{(n-1)}(t))^{\alpha}$ is decreasing, there exists a constant $c_1 > 0$ such that

$$a(t)(y^{(n-1)}(t))^{\alpha} \le c_1 \quad \text{for } t \ge t_2,$$

which by integrating $(n-1)$-times from $t_2$ to $t$ leads to

$$y(t) \le c_2 t^{n-2} A(t, t_2) \text{ for } t \ge t_4$$

for some constant $c_2 > 0$ and $t_4 \ge t_2$ . Then,

$$y^{(\beta-\alpha)/\alpha}(\tau(t)/2) \ge y^{(\beta-\alpha)/\alpha}(t) \ge c_2^{(\beta-\alpha)/\alpha}t^{(n-2)(\beta-\alpha)/\alpha}A^{(\beta-\alpha)/\alpha}(t, t_2).$$

Using the two cases and the definition of $k(t)$ in (3.7), we get

$$w' \le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w - \frac{\beta M_{1/2}}{2}\frac{\tau^{n-2}(t)\tau'(t)}{(a(\tau(t))k(t)\rho(t))^{1/\alpha}}w^{(\alpha+1)/\alpha}. \tag{3.8}$$

Setting

$$B_1 := \frac{\rho'(t)}{\rho(t)}, \quad B_2 := \frac{\beta M_{1/2}}{2}\frac{\tau^{n-2}(t)\tau'(t)}{(a(\tau(t))k(t)\rho(t))^{1/\alpha}}$$

and employing the inequality

$$B_1 u - B_2 u^{(1+\alpha)/\alpha} \le \frac{\alpha^{\alpha}}{(1+\alpha)^{\alpha+1}}B_1^{\alpha+1}B_2^{-\alpha},$$

(see [10]), we have from (3.8),

$$w'(t) \le -\rho(t)q(t) + \mu\frac{a(\tau(t))k(t)}{(\tau^{n-2}(t)\tau'(t))^{\alpha}}\frac{(\rho'(t))^{\alpha+1}}{\rho^{\alpha}(t)}.$$

Integrating this inequality from $t_4$ to $t$ we get

$$w(t) \le w(t_4) - \int_{t_4}^{t}\left[\rho(s)q(s) - \mu\frac{a(\tau(s))k(s)}{(\tau^{n-2}(s)\tau'(s))^{\alpha}}\frac{(\rho'(s))^{\alpha+1}}{\rho^{\alpha}(s)}\right]\mathrm{d}s. \tag{3.9}$$

On the other hand, it follows from (3.5) that

$$w'(t) \le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t), \tag{3.10}$$

that is,

$$\left(\frac{w(t)}{\rho(t)}\right)' \le -q(t).$$

Integrating the above inequality from $t$ to $t'$, we get

$$\frac{w(t')}{\rho(t')} \le \frac{w(t)}{\rho(t)} - \int_t^{t'} q(s)\mathrm{d}s,$$

and hence

$$w(t) \ge \rho(t)Q(t). \tag{3.11}$$

By using (3.11) in (3.9), we find that

$$w(t_4) \ge \rho(t)Q(t) + \int_{t_4}^{t}\left[\rho(s)q(s) - \mu\frac{a(\tau(s))k(s)}{(\tau^{n-2}(s)\tau'(s))^{\alpha}}\frac{(\rho'(s))^{\alpha+1}}{\rho^{\alpha}(s)}\right]\mathrm{d}s,$$

which clearly contradicts (3.1).

Consider now case $(C_2)$. If we put $z = -y$, then Eq. (1.1) gives

$$\left(a(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)' \ge q(t)x^{\beta}(\tau(t)).$$

Using the inequality $z(t) \le p(t)x(\sigma(t))$, we get

$$\left(a(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)' \ge \frac{q(t)}{p^{\beta}(h(t))}z^{\beta}(h(t)). \tag{3.12}$$

In view of Lemma 2.3, we have

$$z(h(s)) \ge \frac{(h(t)-h(s))^{n-1}}{(n-1)!}\left(-z^{(n-1)}(h(t))\right), \quad t \ge s \ge t_2. \tag{3.13}$$

Integrating (3.12) from $h(t)$ to $t$ and using (3.13) in the resulting inequality gives

$$\left(-z^{(n-1)}(h(t))\right)^{\alpha} \ge \frac{\left(-z^{(n-1)}(h(t))\right)^{\beta}}{a(h(t))}\int_{h(t)}^{t}\frac{q(s)}{p^{\beta}(h(s))}\left(\frac{(h(t)-h(s))^{n-1}}{(n-1)!}\right)^{\beta}\mathrm{d}s$$

or

$$\left(-z^{(n-1)}(h(t))\right)^{\alpha-\beta} \ge a^{-1}(h(t))\int_{h(t)}^{t}\frac{q(s)}{p^{\beta}(h(s))}\left(\frac{(h(t)-h(s))^{n-1}}{(n-1)!}\right)^{\beta}\mathrm{d}s,$$

which contradicts (3.2). Note that $z^{(n-1)}(t) \to 0$ as $t \to \infty$ is used when $\alpha > \beta$. □

**Remark 3.1.** As it will be shown in Example 4.1, the additional term $\rho(t)Q(t)$ in (3.1) plays an important role in case

$$\limsup_{t\to\infty}\int_T^t\left[\rho(s)q(s) - \frac{\alpha^{\alpha}}{(1+\alpha)^{\alpha+1}}\left(\frac{2(n-2)!}{\beta\, 4^{2-n}}\right)^{\alpha}\frac{a(\tau(s))k(s)(\rho'(s))^{\alpha+1}}{(\tau^{n-2}(s)\tau'(s)\rho(s))^{\alpha}}\right]\mathrm{d}s < \infty. \tag{3.14}$$

**Theorem 3.2.** Let conditions $(i)$–$(iv)$, (1.2), and (3.2) hold. If the integral (1.3) is divergent, then (1.1) is oscillatory.

***Proof.*** Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. It follows from Lemma 2.4 that there exists $t_2 \in [t_1, \infty)$ such that $y$ satisfies either $(C_1)$ or $(C_2)$ for $t \ge t_2$.

If we assume that $(C_1)$ holds, then by letting $t' \to \infty$ in (3.10), we obtain a contradiction to the positivity of $w(t)$. The rest of the proof is similar to that of Theorem 3.1 and hence is omitted. □

In the following results we use different approaches to replace (3.1) in Theorem 3.1.

**Theorem 3.3.** Assume that $\alpha \leq 1$ and the hypotheses of Theorem 3.1 hold with (3.1) replaced by

$$\limsup_{t\to\infty} \left[\rho(t)Q(t) + \int_T^t \left(\rho(s)q(s) - \frac{(n-2)!}{\beta 4^{2-n}} \frac{(a(\tau(s))k(s))^{1/\alpha} (\rho'(s))^2}{\tau^{n-2}(s)\tau'(s)\rho(s)Q^{(1-\alpha)/\alpha}(s)}\right) \mathrm{d}s\right] = \infty. \tag{3.15}$$

Then (1.1) is oscillatory.

***Proof.*** Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. It follows from Lemma 2.4 that there exists $t_2 \in [t_1, \infty)$ such that $y$ satisfies either $(C_1)$ or $(C_2)$ for $t \geq t_2$. If $(C_1)$ holds, then as in the proof of Theorem 3.1, we obtain (3.8). Thus, in view of (3.11), we have

$$\begin{aligned} w'(t) &\leq -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\beta M_{1/2}}{2}\frac{\tau^{n-2}(t)\tau'(t)}{(a(\tau(t))k(t)\rho(t))^{1/\alpha}}w^{(\alpha+1)/\alpha}(t) \\ &\leq -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\beta M_{1/2}}{2}\frac{\tau^{n-2}(t)\tau'(t)}{(a(\tau(t))k(t))^{1/\alpha}\rho(t)}Q^{(1-\alpha)/\alpha}(t)w^2(t) \\ &\leq -\rho(t)q(t) + \frac{1}{\beta M_{1/2}}\frac{(a(\tau(t))k(t))^{1/\alpha}(\rho'(t))^2}{\tau^{n-2}(t)\tau'(t)\rho(t)Q^{(1-\alpha)/\alpha}(t)}. \end{aligned}$$

The rest of the proof is similar to that of Theorem 3.1 and hence is omitted. □

**Theorem 3.4.** Assume that the hypotheses of Theorem 3.1 hold with (3.1) replaced by

$$\liminf_{t\to\infty}\left(\frac{1}{Q(t)}\int_t^\infty R(s)Q^{(\alpha+1)/\alpha}(s)\mathrm{d}s\right) > \frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}\frac{2(n-2)!}{\beta 4^{2-n}}. \tag{3.16}$$

Then (1.1) is oscillatory.

***Proof.*** Let $x(t)$ be a nonoscillatory solution of equation (1.1) , say $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. It follows from Lemma 2.4 that there exists $t_2 \in [t_1, \infty)$ such that $y$ satisfies either $(C_1)$ or $(C_2)$ for $t \geq t_2$. We will consider both cases separately.

At first, assume that $(C_1)$ holds. Define $w(t)$ as in (3.4) with $\rho(t) = 1$, i.e.,

$$w(t) := \frac{a(t)(y^{(n-1)}(t))^\alpha}{y^\beta(\tau(t)/2)}, \quad t \geq t_2. \tag{3.17}$$

Then as in proof of Theorem 3.1 we get

$$w'(t) \leq -q(t) - \frac{\beta M_{1/2}}{2}\frac{\tau^{n-2}(t)\tau'(t)}{(a(\tau(t))k(t))^{1/\alpha}}w^{(\alpha+1)/\alpha}(t), \tag{3.18}$$

Integrating (3.18) from $t$ to $t'$, we see that

$$\begin{aligned} w(t') &\leq w(t) - \int_t^{t'} q(s)\mathrm{d}s - \frac{\beta M_{1/2}}{2}\int_t^{t'}\frac{\tau^{n-2}(s)\tau'(s)}{(a(\tau(s))k(s))^{1/\alpha}}w^{(\alpha+1)/\alpha}(s)\mathrm{d}s \\ &= w(t) - \int_t^{t'} q(s)\mathrm{d}s - \frac{\beta M_{1/2}}{2}\int_t^{t'} R(s)w^{(\alpha+1)/\alpha}(s)\mathrm{d}s. \end{aligned} \tag{3.19}$$

As in the proof of Theorem 3.1, we can show $Q(t) < \infty$ and $\int_t^\infty R(s)w^{(\alpha+1)/\alpha}(s)\mathrm{d}s < \infty$ for $t \geq t_3$. Letting $t' \to \infty$ in (3.19), we get

$$w(t) \geq Q(t) + \frac{\beta 4^{2-n}}{2(n-2)!}\int_t^\infty R(s)w^{(\alpha+1)/\alpha}(s)\mathrm{d}s. \tag{3.20}$$

Hence,

$$\frac{w(t)}{Q(t)} \geq 1 + \frac{\beta 4^{2-n}}{2(n-2)!}\frac{1}{Q(t)}\int_t^\infty R(s)Q^{(\alpha+1)/\alpha}(s)\left(\frac{w(s)}{Q(s)}\right)^{(\alpha+1)/\alpha}\mathrm{d}s. \tag{3.21}$$

Let $\lambda = \inf_{t\geq T}(w(t)/Q(t))$. Then it is easy to see that $\lambda \geq 1$ and, from (3.16) and (3.21),

$$\lambda \geq 1 + \alpha\left(\frac{\lambda}{\alpha+1}\right)^{(\alpha+1)/\alpha},$$

which contradicts the admissible value of $\lambda$ and $\alpha$.

Consider now case $(C_2)$. Similar to the proof of Theorem 3.1, one can get a contradiction to (3.2). The proof is complete. □

Let the integral (1.3) be convergent. We define the sequence $\{u_n(t)\}_{n=0}^{\infty}$ by

$$u_0(t) = Q(t),$$
$$u_n(t) = \int_t^{\infty} R(s)u_{n-1}^{(\alpha+1)/\alpha}(s)\mathrm{d}s + u_0(t), \quad n = 1, 2, \dots$$

for $t \geq T \geq t_1 \geq t_0$. By induction, it is easy to see that $u_n(t) \leq u_{n+1}(t)$, $n = 0, 1, 2, \dots$.

**Theorem 3.5.** Assume that the hypotheses of Theorem 3.1 except (3.1) hold. If there exists any $u_i(t)$ such that

$$\limsup_{t\to\infty} \frac{l(t)\tau^{\beta(n-1)}(t)}{a^{\beta/\alpha}(\tau(t))}u_i(t) > 1, \tag{3.22}$$

then (1.1) is oscillatory.

***Proof.*** Let $x(t)$ be a nonoscillatory solution of equation (1.1) , say $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. It follows from Lemma 2.4 that there exists $t_2 \in [t_1, \infty)$ such that $y$ satisfies either $(C_1)$ or $(C_2)$ for $t \geq t_2$.

If $y(t)$ satisfies $(C_1)$, then as in the proof of Theorem 3.4, we get that (3.20) holds for $w(t)$ defined by (3.17) and some $T \geq t_0$ large enough, and thus, $w(t) \geq Q(t) = u_0(t)$. By induction, we can see that

$$w(t) \geq u_i(t), \quad t \geq T, \quad i = 1, 2, \dots. \tag{3.23}$$

Since the sequence $\{u_i(t)\}_{i=0}^{\infty}$ is monotone increasing and bounded above, there exists a function $u(t)$ such that $u(t) = \lim_{i\to\infty} u_i(t)$. By Lebesgue monotone theorem,

$$u(t) = \frac{\beta 4^{2-n}}{2(n-2)!}\int_t^{\infty} R(s)u^{(\alpha+1)/\alpha}(s)\mathrm{d}s + Q(t).$$

On the other hand, using (2.2) and the fact that $a(t)(y^{(n-1)}(t))^{\alpha}$ is decreasing in (3.17), we arrive at

$$\begin{aligned}\frac{1}{w(t)} &= \frac{y^{\beta}(\tau(t)/2)}{a(t)\left(y^{(n-1)}(t)\right)^{\alpha}}\\ &\geq \left(\frac{4^{1-n}}{(n-1)!}\tau^{n-1}(t)\right)^{\beta}\frac{\left(y^{(n-1)}(\tau(t))\right)^{\beta}}{a(t)\left(y^{(n-1)}(t)\right)^{\alpha}}\\ &\geq \left(\frac{4^{1-n}}{(n-1)!}\tau^{n-1}(t)\right)^{\beta}\frac{\left(a^{1/\alpha}(t)y^{(n-1)}(t)\right)^{\beta-\alpha}}{a^{\beta/\alpha}(\tau(t))}.\end{aligned} \tag{3.24}$$

If $\alpha = \beta$, then evidently

$$\frac{1}{w(t)} \geq \frac{l(t)\tau^{\beta(n-1)}(t)}{a^{\beta/\alpha}(\tau(t))}. \tag{3.25}$$

If $\alpha > \beta$, then there exists a constant $c > 0$ such that

$$a^{1/\alpha}(t)y^{(n-1)}(t) \leq c \quad \text{for } t \geq T.$$

Thus, in view of $(i)$ and (3.24), we also get (3.25). Combining (3.23) with (3.25), we see that

$$\frac{l(t)\tau^{\beta(n-1)}(t)}{a^{\beta/\alpha}(\tau(t))}u_i(t) \le 1,$$

which contradicts (3.22).

Consider now case $(C_2)$. Similar to the proof of Theorem 3.1, one can get a contradiction to (3.2). The proof is complete. □

**Theorem 3.6.** Assume that the hypotheses of Theorem 3.1 except (3.1) hold. If

$$\limsup_{t\to\infty} \tau^{(n-1)\beta}(t)a^{-1}(\tau(t))Q(t) > ((n-1)!)^{\beta}, \quad \beta = \alpha \tag{3.26}$$

and

$$\limsup_{t\to\infty} \tau^{(n-1)\beta}(t)a^{-\beta/\alpha}(\tau(t))Q(t) = \infty, \quad \beta < \alpha, \tag{3.27}$$

then (1.1) is oscillatory.

***Proof.*** Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. It follows from Lemma 2.4 that there exists $t_2 \in [t_1, \infty)$ such that $y$ satisfies either $(C_1)$ or $(C_2)$ for $t \ge t_2$. We will consider both cases separately.

At first, assume that $(C_1)$ holds. Now, set $w(t) := a(t)\left(y^{(n-1)}(t)\right)^{\alpha}$. Integrating (1.1) from $t$ to $\infty$ and using $(iii)$, we have

$$w(t) \ge \int_t^{\infty} q(s)y^{\beta}(\tau(s))\mathrm{d}s \ge Q(t)y^{\beta}(\tau(t)). \tag{3.28}$$

By virtue of Lemma 2.2, we get

$$y(\tau(t)) \ge \frac{\theta}{(n-1)!}\tau^{n-1}(t)y^{(n-1)}(\tau(t)) \tag{3.29}$$

for every $\theta \in (0,1)$. Thus,

$$\begin{aligned} w(t) &\ge Q(t)\left(\frac{\theta}{(n-1)!}\right)^{\beta}\tau^{\beta(n-1)}(t)\left(y^{(n-1)}(\tau(t))\right)^{\beta} \\ &= Q(t)\left(\frac{\theta}{(n-1)!}\right)^{\beta}\frac{\tau^{\beta(n-1)}(t)}{a^{\beta/\alpha}(\tau(t))}w^{\beta/\alpha}(\tau(t)). \end{aligned} \tag{3.30}$$

Using the fact that $w(t)$ is decreasing, we have

$$w(t) \ge Q(t)\left(\frac{\theta}{(n-1)!}\right)^{\beta}\frac{\tau^{\beta(n-1)}(t)}{a^{\beta/\alpha}(\tau(t))}w^{\beta/\alpha}(t)$$

or

$$w^{1-\beta/\alpha}(t) \ge Q(t)\left(\frac{\theta}{(n-1)!}\right)^{\beta}\frac{\tau^{\beta(n-1)}(t)}{a^{\beta/\alpha}(\tau(t))}.$$

Taking lim sup of both sides of this inequality as $t \to \infty$, we arrive at a contradiction to (3.26). when $\beta = \alpha$ and (3.27) when $\beta < \alpha$.

Consider now case $(C_2)$. Similar to the proof of Theorem 3.1, one can get a contradiction to (3.2). The proof is complete. □

If the equation is not of neutral type, then we can drop the condition (3.2). Without this condition, a weaker result is still possible.

**Theorem 3.7.** Assume that excluding (3.2) all the assumptions of Theorems 3.1 or Theorems 3.3 or Theorems 3.4 or Theorems 3.5 or Theorems 3.6 hold. Then every solution $x(t)$ of (1.1) is oscillatory when p(t)=0, and is either oscillatory or approaches zero as t tends to infinity.

***Proof.*** It suffices to show that if $x(t)$ is a positive solution of (1.1) and $y(t)$ satisfies $(C_2)$, then $\lim_{t\to\infty} x(t) = 0$. To see this we observe from $y(t) < 0$ and (2.6) that $x(t)$ is bounded. Therefore, we have

$$\limsup_{t\to\infty} x(t) = a \geq 0.$$

We claim that $a = 0$. If not, then there exists a sequence $\{T_k\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} T_k = \infty$ and $\lim_{k\to\infty} x(T_k) = a > 0$. Let $\epsilon = a(1-p_0)/(2p_0)$; then, for all large $k$, we have $x(\sigma(T_k)) < a + \epsilon$. From this and the definition of $y$, we obtain

$$0 \geq \lim_{k\to\infty} y(T_k) \geq \lim_{k\to\infty} x(T_k) - p_0(a+\epsilon) = \frac{a(1-p_0)}{2} > 0,$$

a contradiction. Thus $a = 0$ and $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

## 4. Examples

The following examples are illustrative.

**Example 4.1.** Consider the neutral equation

$$\left(\left((x(t) - p_0 x(\sigma_0 t))'''\right)^{1/4}\right)' + \frac{q_0}{t^{7/4}} x^{1/4}\left(\tfrac{4}{5}t\right) = 0, \tag{4.1}$$

where $q_0$ is a positive constant, $\sigma_0 \in (0,1)$ and $p_0 \in [0,1)$. If we set $\rho(t) := t$, then condition (3.1) reduces to

$$q_0 > 3\sqrt{10}/5 \approx 1.89737, \tag{4.2}$$

while (3.14) gives only $q_0 > 4\sqrt{10}/5 \approx 2.52828$. This improvement is due to the additional term $\rho(t)Q(t)$ in (3.1). In view of Theorems 3.1 and 3.7, we conclude that Eq. (1.1) is oscillatory for $p_0 = 0$. For $p_0 > 0$ and, e.g., $\sigma_0 = 10/9$, it is easy to see that $h(t) = (8/9)t \leq t$, and by Theorem 3.1, we have that Eq. (4.1) is oscillatory if

$$q_0 > 122.8072\, p_0.$$

**Example 4.2.** Consider the neutral equation

$$\left(x(t) - \tfrac{1}{2}x(t - \tfrac{\pi}{2})\right)'' + 8x(t-\pi) = 0. \tag{4.3}$$

Clearly, $\sigma(t) = t - \frac{\pi}{2}$ and $\sigma^{-1}(t) = t + \frac{\pi}{2}$, $\tau(t) = t - \pi$, and so $h(t) := t - \frac{\pi}{2}$. All conditions of Theorem 3.2 are satisfied and hence the Eq. (4.3) is oscillatory. One such solution is $x(t) = \sin(4t)$.

**Example 4.3.** Consider the neutral equation

$$\left(\left(\left(x(t) - \tfrac{1}{2}x(\sqrt{t})\right)'\right)^3\right)' + \frac{q_0}{t^{5/4}+1} x(t^{1/4}) = 0, \tag{4.4}$$

where $q_0$ is a positive constant. Here, $\sigma(t) = \sqrt{t}$ and $\sigma^{-1}(t) = t^2$, $\tau(t) = t^{1/4}$, and so $h(t) = \sqrt{t}$. All conditions of Theorem 3.1 are satisfied for every $q_0$ and all large $t$ and hence the Eq. (4.4) is oscillatory.

**Acknowledgment.** We thank the anonymous reviewer whose comments have greatly improved this manuscript.

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