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# On Partial Sums of Normalized Error Function

*Normalize Edilmiş Hata Fonksiyonunun Kısmi Toplamları Üzerine*

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**Abstract**
The main purpose of this paper is to determine some lower bounds for real parts of the quotient of normalized error function and its partial sum. In addition, the some upper bounds for absolute values of normalized error function and its derivative are also given.

**Keywords:** Analytic Function, Error Function, Partial Sums

***Abstract***
*Bu makalenin temel amacı normalize edilmiş hata fonksiyonunun kısmi toplamlarına oranının reel kısımları için bazı alt sınırlar belirlemektir. Ek olarak, normalize edilmiş hata fonksiyonu ve türevinin mutlak değerleri için bazı üst sınırlar da verilmiştir.*

***Anahtar kelimeler**: Analitik Fonksiyon, Hata Fonksiyonu, Kısmi Toplamlar*

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## 1. Introduction

The error function is defined by (Abromowitz and Stegun, 1965)

$$\mathrm{erf(z)} = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)n!}. \quad (1)$$

This function appears widely in mathematics and related disciplines. Especially, it has various applications in statistics, probability theory, partial differential equations, special functions and physics. It is important to mention here that the error function is also known as *probability integral* in the literature. Because of its remarkable properties, some interesting studies has been done on the error function. For some interesting properties including completely monotonicity, functional inequalities and differential inequalities of the error function one can refer to the papers (Alzer, 2003, 2009, 2010) and references therein. On the other hand, Kreyszig and Todd (1959a, 1959b) studied on the univalence of the error function and a related function while, Ramachandran et al. (2018, p.365-367) gave certain results for $q$ −starlike and $q$ −convex error functions. In additon, Silverman (1997) and Silvia (1985) gave some results on the partial sums of starlike and convex functions. Also, Çağlar and Deniz (2015), Aktaş (2019) and, Aktaş and Orhan (2016, 2018) obtained some lower bounds for the quotient of some special functions and their partial sums. Moreover, Çağlar and Orhan (2017) studied on neighborhood and partial sum problem for generalized Sakaguchi type functions. However, other geometric properties (like starlikeness, convexity, close-to convexity, uniform convexity and so forth) of the error function has not been studied yet. The importance of this study is that: determining other geometric properties of the error function can be useful for other disciplines such as mathematical physics, engineering, probabilty and statistics. Because some functions with the positive real part are frequently used in geometric function theory and related areas. Motivated by the earlier works on analytic univalent functions, our main aim is to present some lower bounds for real parts of the quotient of normalized error function and its partial sum. In addition, we give upper bounds for absolute values of normalized error function and its derivative.

Now, we would like to recall some basic notions concerning geometric function theory. Let $\mathcal{A}$ denote the class of function of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (2)$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z: z \in \mathbb{C}, |z| < 1\}.$$

We denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$, which are univalent in $\mathcal{U}$. It is clear that the function $\mathrm{erf}\, z$ does not belong to the class $\mathcal{A}$. For this reason, we consider the following normalized form:

$$E_r f(z) = \frac{\sqrt{\pi z}}{2}\mathrm{erf}(\sqrt{z})$$
$$= z + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(n-1)!(2n-1)} z^n. \quad (3)$$

As a result, the function $z \mapsto E_r f(z)$ is in the class $\mathcal{A}$. We would like to mention here that the following well-known series sums which will be used in the sequel hold true:

$$\sum_{n=1}^{\infty}\frac{1}{n2^n} = \ln 2 \quad (4)$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n!} = e - 1. \quad (5)$$

Also, the well-known triangle inequality

$$|z_1 + z_2| \le |z_1| + |z_2| \qquad (z_1, z_2 \in \mathbb{C}) \quad (6)$$

and the following known inequalities

$$2^{n-1} \le n! \qquad (n \ge 1) \quad (7)$$

and

$$2n < 2n + 1 \qquad (n \ge 1) \quad (8)$$

will be used in order to derive our main results.

Let $w(z)$ denote an analytic function in $\mathcal{U}$. It is worth to remember here that the following well-known result which will be frequently used in the sequel plays a vital role to prove our main results:

$$\Re\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0 \text{ iff } |w(z)| < 1, z \in \mathcal{U}. \quad (9)$$

## 2. Main Results

In this section, we firstly prove the following lemma which will be used in order to derive our main results.

**Lemma.** The normalized error function $z \mapsto E_r f(z)$ which is given by (3) satisfies the following two inequalities for $z \in \mathcal{U}$:

$\boldsymbol{i}$. $|E_r f(z)| \leq 1 + \ln 2,$ (10)

$\boldsymbol{ii}$. $\left|\left(E_r f(z)\right)'\right| \leq \frac{1+\mathrm{e}+2\ln 2}{2}.$ (11)

**Proof.** $\boldsymbol{i}$. By using the inequalities which are given by (6), (7) and (8), we can write that

$$|E_r f(z)| = \left|z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)!(2n-1)} z^n\right|$$
$$= \left|z + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{n+1}\right|$$
$$\leq |z| + \left|\sum_{n=1}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{n+1}\right|$$
$$\leq 1 + \sum_{n=1}^{\infty} \frac{1}{n!(2n+1)}$$
$$\leq 1 + \sum_{n=1}^{\infty} \frac{1}{n2^n}$$

for $z \in \mathcal{U}$. Now consider the series sum which is given by (4) in the last inequality, we deduce that

$$|E_r f(z)| \leq 1 + \ln 2,$$

which is desired.

$\boldsymbol{ii}$. By considering the inequalities which are given by (6), (7) and (8), it can be written that

$$\left|\left(E_r f(z)\right)'\right| = \left|1 + \sum_{n=1}^{\infty} \frac{(n+1)(-1)^n}{n!(2n+1)} z^n\right|$$
$$\leq 1 + \sum_{n=1}^{\infty} \frac{(n+1)}{n!(2n+1)}$$
$$= 1 + \sum_{n=1}^{\infty} \frac{1}{(n-1)!(2n+1)}$$
$$+ \sum_{n=1}^{\infty} \frac{1}{n!(2n+1)}$$
$$\leq 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n!} + \sum_{n=1}^{\infty} \frac{1}{n2^n}.$$

Now using the series sums which are given by (4) and (5), we have

$$\left|\left(E_r f(z)\right)'\right| \leq \frac{1+\mathrm{e}+2\ln 2}{2}.$$

So the proof is completed.

**Theorem.** Let $E_r f: \mathcal{U} \to \mathbb{C}$ be defined by

$$E_r f(z) = z + \sum_{n=1}^{\infty} A_n z^{n+1}$$

and its sequence of partial sum defined by

$$(E_r f)_m(z) = z + \sum_{n=1}^{m} A_n z^{n+1},$$

where $A_n = \frac{(-1)^n}{n!(2n+1)}$. Then, the following two inequalities are valid for $z \in \mathcal{U}$:

$\boldsymbol{i}$. $\Re\left(\frac{E_r f(z)}{(E_r f)_m(z)}\right) \geq 1 - \ln 2 \cong 0.307$ (12)

and

$\boldsymbol{ii}$. $\Re\left(\frac{(E_r f)_m(z)}{E_r f(z)}\right) \geq \frac{1}{1+\ln 2} \cong 0.591.$ (13)

**Proof.** $\boldsymbol{i}$. From the inequality (10) in Lemma, it can be written that

$$|E_r f(z)| = |z + \sum_{n=1}^{\infty} A_n z^{n+1}|$$
$$\leq 1 + \sum_{n=1}^{\infty} |A_n|$$
$$\leq 1 + \ln 2$$

which is equivalent to

$$\frac{1}{\ln 2}\sum_{n=1}^{\infty} |A_n| \leq 1. \quad (14)$$

In order to prove the inequality (12), consider the function $w(z)$ defined by

$$\frac{1+w(z)}{1-w(z)} = \frac{1}{\ln 2}\left\{\frac{E_r f(z)}{(E_r f)_m(z)} - (1 - \ln 2)\right\}$$
$$= \frac{1}{\ln 2}\left\{\frac{z+\sum_{n=1}^{\infty} A_n z^{n+1}}{z+\sum_{n=1}^{m} A_n z^{n+1}} - (1 - \ln 2)\right\}$$
$$= \left\{\frac{1+\sum_{n=1}^{m} A_n z^n + \frac{1}{\ln 2}\sum_{n=m+1}^{\infty} A_n z^n}{1+\sum_{n=1}^{m} A_n z^n}\right\}.$$

As a result of the last equality, we have

$$w(z) = \frac{\frac{1}{\ln 2}\sum_{n=m+1}^{\infty} A_n z^n}{2 + 2\sum_{n=1}^{m} A_n z^n + \frac{1}{\ln 2}\sum_{n=m+1}^{\infty} A_n z^n}$$

and

$$|w(z)| < \frac{\frac{1}{\ln 2}\sum_{n=m+1}^{\infty} |A_n|}{2 - 2\sum_{n=1}^{m} |A_n| - \frac{1}{\ln 2}\sum_{n=m+1}^{\infty} |A_n|}.$$

The inequality

$$\sum_{n=1}^{m} |A_n| + \frac{1}{\ln 2}\sum_{n=m+1}^{\infty} |A_n| \leq 1 \quad (15)$$

implies that $|w(z)| \leq 1$. It suffices to show that the left hand side of the inequality (15) is bounded above by
$\frac{1}{\ln 2}\sum_{n=1}^{\infty} |A_n|,$

which is equivalent to

$$\frac{1-\ln 2}{\ln 2}\sum_{n=1}^{m} |A_n| \geq 0.$$

$\boldsymbol{ii}$. In order to prove the inequality (13), consider the function $p(z)$ defined by

$$\frac{1+p(z)}{1-p(z)} = \left(1+\frac{1}{\ln 2}\right)\left\{\frac{z+\sum_{n=1}^{m} A_n z^{n+1}}{z+\sum_{n=1}^{\infty} A_n z^{n+1}} - \frac{1}{1+\ln 2}\right\}$$

$$= \frac{1+\ln 2}{\ln 2}\left\{\frac{1+\sum_{n=1}^{m} A_n z^{n}}{1+\sum_{n=1}^{\infty} A_n z^{n}} - \frac{1}{1+\ln 2}\right\}$$

$$= \frac{1+\sum_{n=1}^{m} A_n z^{n} - \frac{1}{\ln 2}\sum_{n=m+1}^{\infty} A_n z^{n}}{\sum_{n=1}^{\infty} A_n z^{n}}.$$

A simple calculation yields that

$$p(z) = \frac{-\left(\frac{1+\ln 2}{\ln 2}\right)\sum_{n=m+1}^{\infty} A_n z^n}{2+2\sum_{n=1}^{m} A_n z^n - \left(\frac{1-\ln 2}{\ln 2}\right)\sum_{n=m+1}^{\infty} A_n z^n}$$

and

$$|p(z)| < \frac{\frac{1+\ln 2}{\ln 2}\sum_{n=m+1}^{\infty}|A_n|}{2-2\sum_{n=1}^{m}|A_n| - \left(\frac{1-\ln 2}{\ln 2}\right)\sum_{n=m+1}^{\infty}|A_n|}.$$

To show that $|p(z)| \leq 1$, it is enough to prove that

$$\frac{\frac{1+\ln 2}{\ln 2}\sum_{n=m+1}^{\infty}|A_n|}{2-2\sum_{n=1}^{m}|A_n| - \left(\frac{1-\ln 2}{\ln 2}\right)\sum_{n=m+1}^{\infty}|A_n|} \leq 1,$$

which is equivalent to

$$\sum_{n=1}^{m}|A_n| + \frac{1}{\ln 2}\sum_{n=m+1}^{\infty}|A_n| \leq 1.$$

But the last inequality is bounded above by

$$\frac{1}{\ln 2}\sum_{n=1}^{\infty}|A_n|.$$

This implies that $|p(z)| \leq 1$. The proof is thus completed.

## 3. Conclusion

Geometric properties of special functions and their zeros are very important for engineers and physicists. It is known that, some criterions which depend on positive real part of the functions has been developed to determine geometric properties of analytic functions. In this investigation, by making use of some earlier results for analytic function, we obtain some lower bounds for real part of the quotient of normalized error function and its partial sum. Moreover, with the help of the some well-known inequalities and series sums in mathematics, we present some upper bounds for the absolute values of the normalized error function and its derivative.

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