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# APPROXIMATION PROPERTIES OF MODIFIED $q$-BERNSTEIN-KANTOROVICH OPERATORS

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Abstract. In the present paper we define a q-analogue of the modified Bernstein-Kantorovich operators introduced by Özarslan and Duman(Numer. Funct. Anal. Optim. 37:92-105,2016). We establish the shape preserving properties of these operators e.g. monotonicity and convexity and study the rate of convergence by means of Lipschitz class and Peetre's K-functional and degree of approximation with the aid of a smoothing process e.g Steklov mean. Further, we introduce the bivariate case of modified q-Bernstein-Kantorovich operators and study the degree of approximation in terms of the partial and total modulus of continuity and Peetre's K-functional. Finally, we introduce the associated GBS (Generalized Boolean Sum) operators and investigate the approximation of the Bögel continuous and Bögel differentiable functions by using the mixed modulus of smoothness and Lipschitz class.

## 1. Introduction

Let $\zeta : I \to \mathbb{R}$ be an integrable function, I being [0,1] and $p_{n,k}(x)$ denote the usual Bernstein function given by

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, 0 \le x \le 1, k = 0, 1, 2, ...n.$$

Then the classical Bernstein-Kantorovich operator is defined by

$$K_n(\zeta;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \zeta(s)ds.$$

The above operator may also be expressed as follows:

$$K_n(\zeta;x) = \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 \zeta\left(\frac{k+s}{n+1}\right) ds. \tag{1.1}$$

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For $\alpha > 0$, replacing $s$ in the above equation with $s^\alpha$, the positivity and linearity of $K_n$ is preserved and it generates a new sequence of positive linear operators, so called modified Bernstein-Kantorovich operators defined by Özarslan and Duman [24] as follows:

$$K_n^\alpha(\zeta; x) = \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 \zeta\left(\frac{k+s^\alpha}{n+1}\right) ds. \tag{1.2}$$

It is observed that when $\alpha = 1$, the above operator includes the classical Bernstein-Kantorovich operators (1.1). Kajla and Goyal [19] studied modified Bernstein-Kantorovich operators for functions of one and two variables and established some approximation properties. For more significant contribution on Kantorovich type modification of positive linear operators, we refer to the papers ([1], [3], [2], [6],[16], [18], [27]). In the last decade, several researchers have studied the q-analogues of positive linear operators and established many interesting approximation properties. In 1987, first the classical Bernstein operators based on q-integers were defined by Lupas [20]. After a decade, Phillips [25] introduced another q-analogue of Bernstein polynomials which became more popular. Inspired by them, researchers introduced similar q-analogues of several positive linear operators and established many interesting approximation properties. For a detailed account of the work in this direction we refer the reader to the references [4, 7]. Before introducing our operators, for the sake of completeness we mention some basic definitions of q-calculus. For $q > 0$, and each nonnegative integer $k$, the q-integer $[k]$ and the q-factorial $[k]!$ are, respectively, given by

$$[k]_q = \begin{cases} \frac{(1-q^k)}{(1-q)}, & q \neq 1, \\ k, & q = 1, \end{cases}$$

and

$$[k]_q! = \begin{cases} [k]_q[k-1]_q....[1]_q, & k \geq 1, \\ 1, & k = 0. \end{cases}$$

For the integers $n, k$ satisfying $n \geq k \geq 0$. The q-binomial coefficients are defined by

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The q-analogue of integration, introduced by Thomae [28] in the interval [0,a] is defined by

$$\int_0^a \zeta(t) d_q t := (1-q) \sum_{n=1}^{\infty} \zeta(aq^n), 0 < q < 1. \tag{1.3}$$

For any $x \in [0, \infty), 0 < q < 1$, we adopt the convention $[x]_q = \dfrac{1-q^x}{1-q}$ (see [21]).

For each positive integer $n, 0 < q < 1$ and $\zeta \in C(I)$, endowed with the uniform norm $\|\zeta\| = \sup_{t\in I} |\zeta(t)|$, Phillips [25] proposed the following $q$-Bernstein polynomials,

$$B_{n,q}(\zeta; x) := \sum_{k=0}^{n} \zeta\left(\frac{[k]_q}{[n]_q}\right) p_{n,k}(q; x).$$

Following the above definition, for $\zeta \in C(I)$ endowed with the uniform norm $\|.\|$, we define the q-analogue of the operators (1.2) as follows:

$$K_{n,q}^{\alpha}(\zeta; x) = \sum_{k=0}^{n} \binom{n}{k}_q x^k (1-x)_q^{n-k} \int_0^1 \zeta\left(\frac{[k]_q + s^{\alpha}}{[n+1]_q}\right) d_q s. \tag{1.4}$$

The aim of the present paper is to establish the shape preserving properties and obtain the degree of approximation for the operators (1.4) by means of the Lipschitz class and the Peetre's K-functional. Next, we define the bivariate case for the operators (1.4) and investigate the rate of convergence with the aid of the modulus of continuity and the Peetre's K-functional. Lastly, we study the GBS operators of the modified $q$-Bernstein-Kantorovich type and study the approximation of Bögel continuous and Bögel differentiable functions.

## 2. Preliminaries

In order to prove the main results of the paper, we shall require the following auxiliary results.

**Lemma 1.** *The moments of the operators (1.4), are given by*

i) $K_{n,q}^{\alpha}(1; x) = 1$;

ii) $K_{n,q}^{\alpha}(t; x) = \frac{[n]_q x}{[n+1]_q} + \frac{1}{[n+1]_q[\alpha+1]_q}$;

iii) $K_{n,q}^{\alpha}(t^2; x) = \frac{1}{[n+1]_q^2}\{[n]_q x(1 + q[n-1]_q x) + \frac{2[n]_q x}{[\alpha+1]_q} + \frac{1}{[2\alpha+1]_q}\}$;

iv) $K_{n,q}^{\alpha}(t^3; x) = \frac{[n]_q x}{[n+1]_q^3}\{1 + q[n-1]_q x(2 + q + \frac{3}{[\alpha+1]_q}) + q^3[n-1]_q[n-2]_q x^2 + 3(\frac{1}{[\alpha+1]_q} + \frac{1}{[2\alpha+1]_q})\} + \frac{1}{[n+1]_q^3[3\alpha+1]_q}$;

v) $K_{n,q}^{\alpha}(t^4; x) = \frac{[n]_q x}{[n+1]_q^4}\Big[\{1 + x[n-1]_q q(q^2+3q+3) + [n-1]_q[n-2]_q x^2 q^3(q^2 + 2q+3) + [n-1]_q[n-2]_q[n-3]_q q^6 x^3\} + \frac{4}{[\alpha+1]_q}\{1 + q(q+2)[n-1]_q x + q^3[n-1]_q[n-2]_q x^2\} + \frac{6}{[2\alpha+1]_q}\{1 + q[n-1]_q x\} + \frac{4}{[3\alpha+1]_q}\Big] + \frac{1}{[4\alpha+1][n+1]_q^4}$.

*Proof.* Using the definition of $q$-analogue of integration given by (1.3), the proof of the lemma is straight forward hence we omit the details. □

As a consequence of the above lemma, we have

**Lemma 2.** *The central moments of the operators* (1.4), *for* $\phi_x^j(t) = (t-x)^j$, *where j=0,1,..4 are given by*

i) $K^{\alpha}_{n,q}(\phi^1_x;x)=x\left\{\frac{[n]_q(1-q)-1}{[n+1]_q}\right\}+\frac{1}{[n+1]_q[\alpha+1]_q}$;

ii) $K^{\alpha}_{n,q}(\phi^2_x;x)=\frac{1}{[n+1]_q^2}\Big[x^2\{[n]_q[n-1]_qq-2[n]_q[n+1]_q+[n+1]^2_q\}+\frac{x}{[\alpha+1]_q}\{[n]_q[\alpha+1]_q-2[n+1]_q+2[n]_q\}+\frac{1}{[2\alpha+1]_q}\Big]$;

iii) $K^{\alpha}_{n,q}(\phi^4_x;x)=\frac{1}{[n+1]_q^4}\Big[x^4\{[n+1]^4_q+[n]_q[n-1]_q[n-2]_q[n-3]_qq^6 - 4[n+1]_q[n]_q[n-1]_q[n-2]_qq^3+6[n+1]^2_q[n]_q[n-1]_qq-4[n+1]^3_q[n]_q\}+x^3\left\{[n]_q[n-1]_q[n-2]_qq^3\left(q^2+2q+3+\frac{4}{[\alpha+1]_q}\right)-4[n+1]_q[n]_q[n-1]_qq(2+q+\frac{3}{[\alpha+1]_q})+6[n+1]^2_q[n]_q(1+\frac{2}{[\alpha+1]_q})-4\frac{[n+1]^3_q}{[\alpha+1]_q}\right\}+x^2\left\{[n]_q[n-1]_qq(q^2+3q+3+\frac{(8+4q)}{[\alpha+1]_q}+\frac{6}{[2\alpha+1]_q})-4[n+1]_q[n]_q(1+\frac{3}{[\alpha+1]_q}+\frac{1}{[3\alpha+1]_q})+6[n+1^2]\frac{1}{[2\alpha+1]_q}\right\}+x\left\{[n]_q(1+\frac{4}{[\alpha+1]_q}+\frac{6}{[2\alpha+1]_q}+\frac{4}{[3\alpha+1]_q}-4[n+1]_q\frac{1}{[3\alpha+1]_q})\right\}+\frac{1}{[\alpha+1]_q}\Big]$.

**Remark 1.** *For every $n\in\mathbb{N},\alpha>0$ and $x\in I$, we have*

$$K^{\alpha}_{n,q}(\phi^2_x;x)=\frac{1}{[n+1]^2_q}\left\{[n]_qx(1-x)+x\left(q^{2n}-\frac{2q^n}{[\alpha+1]_q}\right)+\frac{1}{[2\alpha+1]_q}\right\}$$
$$\le\frac{1}{[n+1]^2_q}([n]_qx(1-x)+A_{q,\alpha}(n)), \tag{2.1}$$

*where $A_{q,\alpha}(n)=q^{2n}+\frac{1}{[2\alpha+1]_q}$.*

In what follows, let $(q_n)$ be a sequence in $(0,1)$ such that

$$\lim_{n\to\infty}q_n=1 \text{ and } \lim_{n\to\infty}q_n^n=c,\ \ 0\le c<1.$$

**Remark 2.** *From Lemma 2, after simple calculations, one has*

i) $\lim_{n\to\infty}[n]_{q_n}K^{\alpha}_{n,q_n}(\phi^1_x;x)=-cx+\dfrac{1}{[\alpha+1]_{q_n}}$;

ii) $\lim_{n\to\infty}[n]_{q_n}K^{\alpha}_{n,q_n}(\phi^2_x;x)=x(1-x)$;

iii) $\lim_{n\to\infty}[n]^2_{q_n}K^{\alpha}_{n,q_n}(\phi^4_x;x)=3x^2(1-x)^2+\dfrac{24x^2}{[\alpha+1]_{q_n}}$.

**Lemma 3.** *For $\zeta\in C(I)$ and $x\in I$, we have $\|K^{\alpha}_{n,q_n}(\zeta;.)\|\le\|\zeta\|$.*

*Proof.* Using Lemma 1, for every $x\in I$ we have,

$$|K^{\alpha}_{n,q_n}(\zeta;x)|=\sum_{k=0}^{n}\binom{n}{k}_{q_n}x^k(1-x)^{n-k}_{q_n}\int_0^1\left|\zeta\left(\frac{[k]_{q_n}+s^{\alpha}}{[n+1]_{q_n}}\right)\right|d_{q_n}s$$
$$\le\|\zeta\|\,K^{\alpha}_{n,q_n}(1;q_n,x)=\|\zeta\|,$$

from which the desired result is immediate. □

**Lemma 4.** *For $n \in \mathbb{N}, \alpha > 0$ and $x \in I$, we have*

$$K_{n,q_n}^{\alpha}((t-x)^2;x) \leq \frac{2}{[n+1]_{q_n}} \tau_n^2(x),$$

*where* $\tau_n^2(x) = \phi^2(x) + \frac{1}{[n+1]_{q_n}}$ *and* $\phi^2(x) = x(1-x)$.

*Proof.* From relation (2.1) it follows

$$\begin{aligned} K_{n,q_n}^{\alpha}((t-x)^2;x) &\leq \frac{1}{[n+1]_{q_n}} \left( \phi^2(x) + \frac{1}{[n+1]_{q_n}} \left( q^{2n} + \frac{1}{[2\alpha+1]_{q_n}} \right) \right) \\ &\leq \frac{2}{[n+1]_{q_n}} \left( \phi^2(x) + \frac{1}{[n+1]_{q_n}} \right) = \frac{2}{[n+1]_{q_n}} \tau_n^2(x). \end{aligned}$$

□

## 3. Shape preserving properties of the operators $K_{n,q}^{\alpha}$

In this section are studied monotonicity and convexity of the q-Bernstein-Kantorovich operators. Recently, properties regarding monotonicity of certain Bernstein-type operators in quantum calculus were proved in [23].

**Definition 1.** *For any function $\zeta$, the q-differences are defined as*

$$\Delta^0 \zeta_i = \zeta_i, \ \Delta^{k+1} \zeta_i = \Delta^k \zeta_{i+1} - q^k \Delta^k \zeta_i,$$

*for $k = 0, 1, \ldots, n-i-1$, where* $\zeta_i := f\left(\frac{[i]}{[n]}\right)$.

**Definition 2.** *For any arbitrary function $\zeta$, the q-derivative is defined by*

$$(D_q \zeta)(x) = \frac{\zeta(qx) - \zeta(x)}{(q-1)x}.$$

*Define* $D_q^k \zeta = D_q(D_q^{k-1}\zeta), \ k = 1, 2, \ldots,$ *with* $D_q^0 \zeta = \zeta$.

**Definition 3.** *A function $\zeta : [a,b] \to \mathbb{R}$ is said to be q-increasing (q-decreasing) on $[a,b]$ if $\zeta(qx) \leq \zeta(x)$ $(\zeta(qx) > \zeta(x))$ whenever $x, qx \in [a,b]$.*

**Remark 3.** *A function $\zeta : [a,b] \to \mathbb{R}$ is q-increasing (q-decreasing) on $[a,b]$ if and only if $(D_q\zeta)(x) \geq 0$ $((D_q\zeta)(x) \leq 0)$ whenever $x, qx \in [a,b]$.*

**Definition 4.** *A function $\zeta : [a,b] \to \mathbb{R}$ is called q-convex on $[a,b]$ if $D_q^2 \zeta \geq 0$ on $[a,b]$.*

**Remark 4.** *A function $\zeta$ is q-convex if and only if $q\zeta(x) - (1+q)\zeta(qx) + \zeta(q^2x) \geq 0$ whenever $x, qx, q^2x \in [a,b]$.*

**Remark 5.** *If a function $\zeta : [a,b] \to \mathbb{R}$ is increasing (decreasing), then $\zeta$ is q-increasing (q-decreasing). If a function $\zeta : [a,b] \to \mathbb{R}$ is convex, then $\zeta$ is q-convex.*

The $q$-derivative of Kantorovich operator (1.4) can be written as follows:

$$D_q^i K_{n,q}^{\alpha}(\zeta;x)=[n]\dots[n-i+1]\sum_{k=0}^{n-i}\binom{n-i}{k}_q x^k\prod_{\nu=i}^{n-k-1}(1-q^{\nu}x)\int_0^1\Delta^i\zeta\left(\frac{[k]+s^{\alpha}}{[n+1]}\right)d_qs. \tag{3.1}$$

**Theorem 1.** *i) If $\zeta$ is increasing (decreasing) function on $I$, then the operator $K_{n,q_n}^{\alpha}$ is q-increasing (q-decreasing).*

*ii) If $\zeta$ is convex function on $I$, then the operator $K_{n,q_n}^{\alpha}$ is q-convex.*

*Proof.* i) Since the function $\zeta$ is increasing, using relation (3.1) it follows that $K_{n,q_n}^{\alpha}$ is $q$-increasing.

ii) We have

$$\Delta^2\zeta\left(\frac{[k]_{q_n}+s^{\alpha}}{[n+1]_{q_n}}\right)=\left\{\zeta\left(\frac{[k+2]_{q_n}+s^{\alpha}}{[n+1]_{q_n}}\right)-q\zeta\left(\frac{[k+1]_{q_n}+s^{\alpha}}{[n+1]_{q_n}}\right)+q\zeta\left(\frac{[k]_{q_n}+s^{\alpha}}{[n+1]_{q_n}}\right)\right\}$$
$$-\zeta\left(\frac{[k+1]_{q_n}+s^{\alpha}}{[n+1]_{q_n}}\right)\geq\zeta\left(\frac{[k+2]_{q_n}+s^{\alpha}}{[n+1]_{q_n}}-q\frac{[k+1]_{q_n}+s^{\alpha}}{[n+1]_{q_n}}+q\frac{[k]_{q_n}+s^{\alpha}}{[n+1]_{q_n}}\right)$$
$$-\zeta\left(\frac{[k+1]_{q_n}+s^{\alpha}}{[n+1]_{q_n}}\right)=0.$$

Therefore, $D_{q_n}^2 K_{n,q_n}^{\alpha}(\zeta;x)\geq 0$, and the proof is complete. □

## 4. Direct Estimates

First we prove the uniform convergence theorem for the operators (1.4).

**Theorem 2.** *Let $\zeta\in C(I)$. Then $\lim_{n\to\infty}K_{n,q_n}^{\alpha}(\zeta,x)=\zeta(x)$, uniformly in $I$.*

*Proof.* From Lemma 1, it follows that $K_{n,q_n}^{\alpha}(e_i,x)\to e_i(x)$, as n $\to\infty$ uniformly in $I$, for $i=0,1,2$. Hence applying Bohman-Korovkin Theorem [15], we obtain the required result. □

**Example 1.** *Let $f(x)=14x^3-20x^2+8x-1$, $\alpha=0.8$, $q_n=1-\dfrac{1}{n}$ and $n\in\{20,40,60\}$. Denote $E_{n,q_n}^{\alpha}(f;x)=\left|K_{n,q_n}^{\alpha}(f;x)-f(x)\right|$, the error function of approximation by $K_{n,q_n}^{\alpha}$ operators. The convergence of the operators $K_{n,q_n}^{\alpha}$ to the function $f$ is illustrated in Figure 1. The error of approximation $E_{n,q_n}^{\alpha}$ is given in Figure 2. Also, in Table 1 we computed the error of approximation for $K_{20,q_n}^{\alpha}$, $K_{40,q_n}^{\alpha}$ and $K_{60,q_n}^{\alpha}$ at certain points. From Figure 1, it is clear that as n increases, the approximation of $K_{n,q_n}^{\alpha}(\zeta,x)$ to $\zeta$ becomes better. Further from Figure 2 and Table 1, it follows that the error of approximation decreases as n increases.*

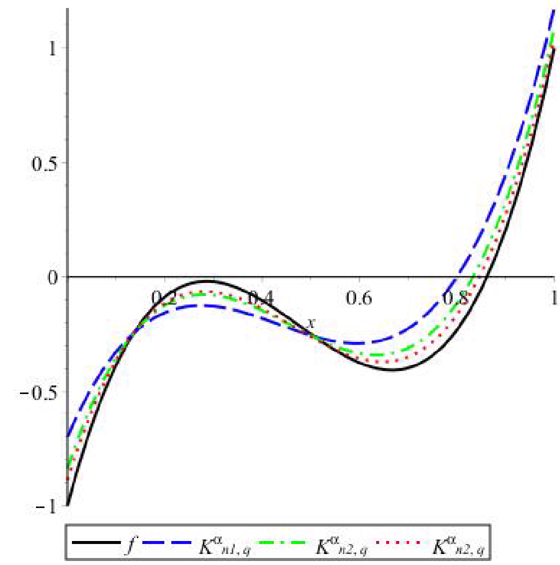


FIGURE 1. Approximation process by $K^{\alpha}_{n,q_n}$

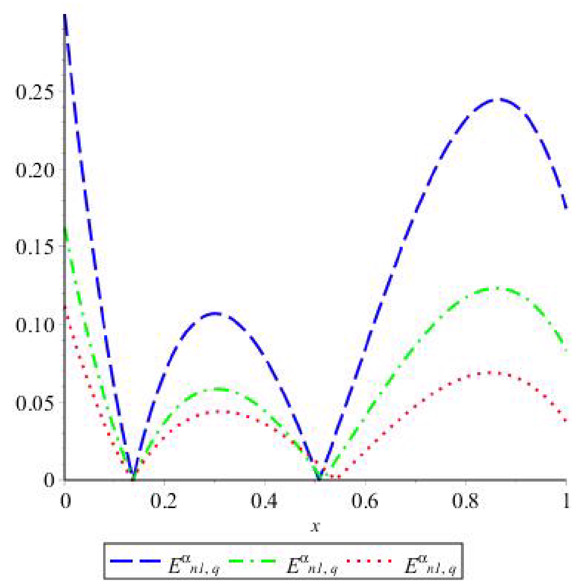


FIGURE 2. Error of approximation

**Table 1.** *Error of approximation* $E^{\alpha}_{n,q_n}$ *for* $n = 20, 40, 60$

| $x$ | $E^{\alpha}_{20,q_n}(f;x)$ | $E^{\alpha}_{40,q_n}(f;x)$ | $E^{\alpha}_{60,q_n}(f;x)$ |
|---|---|---|---|
| 0.1 | 0.0584383292 | 0.0320928000 | 0.0211775395 |
| 0.2 | 0.0692935187 | 0.0373613049 | 0.0278362499 |
| 0.3 | 0.1069712939 | 0.0585258713 | 0.0439339728 |
| 0.4 | 0.0781251448 | 0.0440465298 | 0.0357756110 |
| 0.5 | 0.0062852221 | 0.0065689107 | 0.0120211465 |
| 0.6 | 0.0850183289 | 0.0412613554 | 0.0186694386 |
| 0.7 | 0.1722553569 | 0.0867986370 | 0.0476361628 |
| 0.8 | 0.2318957119 | 0.1173973050 | 0.0662190429 |
| 0.9 | 0.2404092469 | 0.1204117290 | 0.0657580977 |

Next, we establish a Voronovskaja type asymptotic result for the operators $K^{\alpha}_{n,q_n}$. Let $C^r(I)$ denote the space of $r$-times continuously differentiable functions on $I$.

**Theorem 3.** *If* $\zeta \in C^2(I)$, *then*

$$\lim_{n\to\infty} [n]_{q_n}[K^{\alpha}_{n,q_n}(\zeta, x) - \zeta(x)] = \left\{-cx + \frac{1}{[\alpha+1]_{q_n}}\right\}\zeta'(x) + x(1-x)\zeta''(x),$$

*uniformly in* $x \in I$.

*Proof.* By the Taylor's expansion of $\zeta$, we have

$$\zeta(s) = \zeta(x) + \zeta'(x)(s-x) + \frac{1}{2}\zeta''(x)(s-x)^2 + \frac{(s-x)^2}{2}\{\zeta''(\theta) - \zeta''(x)\}, \quad (4.1)$$

where $\theta$ lies between $s$ and $x$. Operating by $K^{\alpha}_{n,q_n}$ to the equation (4.1), we obtain

$$K^{\alpha}_{n,q_n}(\zeta, x) - \zeta(x) = K^{\alpha}_{n,q_n}((s-x); x)\zeta'(x) + K^{\alpha}_{n,q_n}\left((s-x)^2; x\right)\frac{1}{2}\zeta''(x) + \frac{1}{2}K^{\alpha}_{n,q_n}\left((s-x)^2\left(\zeta''(\theta) - \zeta''(x)\right); x\right). \quad (4.2)$$

Using the well known properties of modulus of continuity, we have

$$|\zeta''(\theta) - \zeta''(x)| \le \omega\left(\zeta'', |\theta - x|\right) \le \left(1 + \frac{|s-x|}{\delta}\right)\omega\left(\zeta'', \delta\right),\ \delta > 0.$$

Therefore,

$$|K^{\alpha}_{n,q_n}\left((s-x)^2\left(\zeta''(\theta) - \zeta''(x)\right); x\right)| \le K^{\alpha}_{n,q_n}\left((s-x)^2|\zeta''(\theta) - \zeta''(x)|; x\right) \le \omega(\zeta'', \delta)K^{\alpha}_{n,q_n}\left(((s-x)^2 + \frac{1}{\delta}|s-x|^3); x\right).$$

Using Cauchy-Schwarz inequality, properties $(ii)$ and $(iii)$ of Remark 2 and choosing $\delta = [n]_{q_n}^{-\frac{1}{2}}$, we get

$$|K^{\alpha}_{n,q_n}\left((s-x)^2\left(\zeta''(\theta) - \zeta''(x)\right); x\right)| \le \omega(\zeta'', \delta)\left[K^{\alpha}_{n,q_n}\left((s-x)^2; x\right) + \frac{1}{\delta}\sqrt{K^{\alpha}_{n,q_n}((s-x)^2; x)}\sqrt{K^{\alpha}_{n,q_n}((s-x)^4; x)}\right]$$

$$= \omega(\zeta'', \delta)\left[O\left(\frac{1}{[n]_{q_n}}\right) + \frac{1}{\delta}O\left(\frac{1}{[n]_{q_n}^{\frac{1}{2}}}\right)O\left(\frac{1}{[n]_{q_n}}\right)\right]$$

$$= \omega(\zeta'', [n]_{q_n}^{-\frac{1}{2}})O\left(\frac{1}{[n]_{q_n}}\right),$$

uniformly in $x \in I$. Hence,

$$[n]_{q_n}|K_{n,q_n}^{\alpha}\left((s-x)^2\left(\zeta''(\theta)-\zeta''(x)\right);x\right)| = \omega\left(\zeta''; [n]_{q_n}^{-\frac{1}{2}}\right)O(1),$$

uniformly $x \in I$.

Consequently,

$$\lim_{n\to\infty}[n]_{q_n}K_{n,q_n}^{\alpha}\left((s-x)^2\left(\zeta''(\theta)-\zeta''(x)\right);x\right) = 0,$$

uniformly in $x \in I$. Thus, from equation (4.2) and Remark 2, we get

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{q_n}[K_{n,q_n}^{\alpha}(\zeta,x)-\zeta(x)]\\
&= \lim_{n\to\infty}[n]_{q_n}\Big\{K_{n,q_n}^{\alpha}((s-x),x)\zeta'(x) + K_{n,q_n}^{\alpha}\left((s-x)^2,x\right)\frac{1}{2}\zeta''(x)\\
&+ K_{n,q_n}^{\alpha}\left((s-x)^2\{\zeta''(\theta)-\zeta''(x)\};x\right)\Big\}\\
&= \left(-cx + \frac{1}{[\alpha+1]_{q_n}}\right)\zeta'(x) + \frac{x(1-x)}{2}\zeta''(x),
\end{aligned}$$

uniformly in $x \in I$. This completes the proof. □

## 5. Global Approximation Theorems

First we obtain a global approximation theorem for the operators (1.4) in terms of the second order modulus of continuity by using a smoothing process, e.g. Steklov mean. For $\zeta \in C(I)$, the Steklov mean is defined as

$$\zeta_h(x) = \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}[2\zeta(x+u+v)-\zeta(x+2(u+v))]dudv. \tag{5.1}$$

It is known that for the function $\zeta_h(x)$, there hold the following properties:

a) $\|\zeta_h - \zeta\| \le \omega_2(\zeta, h)$

b) $\zeta_h', \zeta_h'' \in C(I)$ *and* $\|\zeta_h'\| \le \frac{5}{h}\omega(\zeta, h)$, $\|\zeta_h''\| \le \frac{9}{h^2}\omega_2(\zeta, h)$, where $\omega(\zeta; h)$ and $\omega_2(\zeta; h)$ denote the first and second order modulus of continuity.

In what follows, let $\mu_{n,q_n}^{\alpha,m}(x) = K_{n,q_n}^{\alpha}\left((s-x)^m;x\right), m \in \mathbb{N}\cup\{0\}$.

**Theorem 4.** *Let $\zeta \in C(I)$. Then the following inequality holds*

$$\|K_{n,q_n}^{\alpha}(\zeta;\cdot)-\zeta\| \le 5\omega\left(\zeta; \sqrt{\sup_{x\in I}\mu_{n,q_n}^{\alpha,2}(x)}\right) + \frac{13}{2}\omega_2\left(\zeta; \sqrt{\sup_{x\in I}\mu_{n,q_n}^{\alpha,2}(x)}\right).$$

*Proof.* Using the Steklov mean $\zeta_h(x)$ given by (5.1), we may write

$$|K^{\alpha}_{n,q_n}(\zeta;x)-\zeta(x)|\leq K^{\alpha}_{n,q_n}(|\zeta-\zeta_h|;x)+|K^{\alpha}_{n,q_n}(\zeta_h;x)-\zeta_h(x)|+|\zeta_h(x)-\zeta(x)|. \quad (5.2)$$

Hence using Lemma 3 and property (a) of Steklov mean, we have

$$K^{\alpha}_{n,q_n}(|\zeta-\zeta_h|;x)\leq \|\zeta-\zeta_h\|\leq \omega_2(\zeta,h). \quad (5.3)$$

Since $\zeta_h''\in C(I)$, by Taylor's expansion we have

$$\zeta_h(s)=\zeta_h(x)+(s-x)\zeta_h'(x)+\frac{(s-x)^2}{2!}\zeta_h''(\theta),$$

where $\theta$ lies between $s$ and $x$. Then, applying Cauchy-Schwarz inequality

$$|K^{\alpha}_{n,q_n}(\zeta_h(s)-\zeta_h(x);x)|\leq \|\zeta_h'\|\sqrt{\sup_{x\in I}\mu^{\alpha,2}_{n,q_n}(x)}+\frac{1}{2}\|\zeta_h''\|\sup_{x\in I}\mu^{\alpha,2}_{n,q_n}(x).$$

Now, applying property (b) of Steklov mean, we obtain

$$|K^{\alpha}_{n,q_n}(\zeta_h(s)-\zeta_h(x);x)|\leq\frac{5}{h}\omega(\zeta,h)\sqrt{\sup_{x\in I}\mu^{\alpha,2}_{n,q_n}(x)}+\frac{9}{2h^2}\omega_2(\zeta,h)\sup_{x\in I}\mu^{\alpha,2}_{n,q_n}(x). \quad (5.4)$$

Choosing $h=\sqrt{\sup_{x\in I}\mu^{\alpha,2}_{n,q_n}(x)}$ and combining the equations (5.2) – (5.4), we get the desired result. □

**Theorem 5.** *For any $\zeta\in C^1(I)$ and $x\in I$, we have*

$$\|K^{\alpha}_{n,q_n}(\zeta;\cdot)-\zeta\|\leq\sup_{x\in I}|\mu^{\alpha,1}_{n,q_n}(x)|\|\zeta'\|+2\sqrt{\sup_{x\in I}\mu^{\alpha,2}_{n,q_n}(x)}\,\omega\left(\zeta',\frac{1}{2}\sqrt{\sup_{x\in I}\mu^{\alpha,2}_{n,q_n}(x)}\right).$$

*Proof.* Let $\zeta\in C^1(I)$. For any $s\in I$, we have

$$\zeta(s)-\zeta(x)=\zeta'(x)(s-x)+\int_x^s(\zeta'(u)-\zeta'(x))du.$$

Applying the operator $K^{\alpha}_{n,q_n}$ on both sides of the above relation, we get

$$K^{\alpha}_{n,q_n}(\zeta(s)-\zeta(x);x)=\zeta'(x)K^{\alpha}_{n,q_n}((s-x);x)+K^{\alpha}_{n,q_n}\left(\int_x^s(\zeta'(u)-\zeta'(x))du;x\right).$$

Using the well known property of modulus of continuity

$$|\zeta'(u)-\zeta'(x)|\leq\omega(\zeta',\delta)\left(\frac{|u-x|}{\delta}+1\right),\delta>0,$$

we obtain

$$\left|\int_x^s(\zeta'(u)-\zeta'(x))du\right|\leq\left|\int_x^s|\zeta'(u)-\zeta'(x)|du\right|\leq\left|\int_x^s\left(1+\frac{|u-x|}{\delta}\right)\omega(\zeta',\delta)du\right|$$
$$=\omega(\zeta';\delta)\left(|s-x|+\frac{(s-x)^2}{2\delta}\right).$$

Therefore,

$$|K^{\alpha}_{n,q_n}(\zeta;x)-\zeta(x)|\le|\zeta'(x)||K^{\alpha}_{n,q_n}((s-x),x)| \\ +\omega(\zeta',\delta)\left\{\frac{1}{2\delta}K^{\alpha}_{n,q_n}((s-x)^2,x)+K^{\alpha}_{n,q_n}(|s-x|;x)\right\}.$$

Using Cauchy-Schwarz inequality, we have

$$|K^{\alpha}_{n,q_n}(\zeta;x)-\zeta(x)|\le|\zeta'(x)||K^{\alpha}_{n,q_n}((s-x);x)| \\ +\omega(\zeta';\delta)\left\{\frac{1}{2\delta}\sqrt{K^{\alpha}_{n,q_n}((s-x)^2,x)}+1\right\}\sqrt{K^{\alpha}_{n,q_n}((s-x)^2,x)}.$$

Choosing $\delta=\frac{1}{2}\sqrt{\sup_{x\in I}\mu^{\alpha,2}_{n,q_n}(x)}$, the required result is obtained. □

**Theorem 6.** *For all $\zeta\in C(I)$ and $n\in N$, there exists a constant $C>0$ such that*

$$\|K^{\alpha}_{n,q_n}(\zeta;\cdot)-\zeta\|\le C\omega_2\left(\zeta;\frac{\sqrt{\phi^{\alpha}_{n,q_n}(x)}}{2}\right)+\omega\left(\zeta,\sup_{x\in I}\left|\mu^{\alpha,1}_{n,q_n}(x)\right|\right),$$

*where* $\phi^{\alpha}_{n,q_n}(x)=\dfrac{1}{2}\left\{\sup_{x\in I}\mu^{\alpha,2}_{n,q_n}(x)+\sup_{x\in I}(\mu^{\alpha,1}_{n,q_n}(x))^2\right\}.$

*Proof.* For $x\in I$, consider the auxiliary operators defined by

$$K^{\alpha*}_{n,q_n}(\zeta;x)=K^{\alpha}_{n,q_n}(\zeta;x)+\zeta(x)-\zeta\left(\frac{[n]_{q_n}x}{[n+1]_{q_n}}+\frac{1}{[n+1]_{q_n}[\alpha+1]_{q_n}}\right). \tag{5.5}$$

It is clear that the operator $K^{\alpha*}_{n,q_n}(\zeta;x)$ is linear, in view of Lemma 1, $K^{\alpha*}_{n,q_n}(1;x)=1$ and

$$K^{\alpha*}_{n,q_n}(s-x;x)=K^{\alpha}_{n,q_n}(s-x;,x)-\left(\frac{[n]_{q_n}x}{[n+1]_{q_n}}+\frac{1}{[n+1]_{q_n}[\alpha+1]_{q_n}}-x\right)=0.$$

For every $g\in W^2(I)=\{g\in C(I):g''\in C(I)\}$ and $s\in I$, from the Taylor's theorem we have

$$g(s)=g(x)+(s-x)g'(x)+\int_x^s(s-u)g''(u)du.$$

Operating by $K^{\alpha*}_{n,q_n}(.;x)$ on the above equation and using (5.5), we have

$$K^{\alpha*}_{n,q_n}(g;x)=g(x)+K^{\alpha}_{n,q_n}\left(\int_x^s(s-u)g''(u)du;x\right) \\ -\int_x^{\frac{[n]_{q_n}x}{[n+1]_{q_n}}+\frac{1}{[n+1]_{q_n}[\alpha+1]_{q_n}}}\left(\frac{[n]_{q_n}x}{[n+1]_{q_n}}+\frac{1}{[n+1]_{q_n}[\alpha+1]_{q_n}}-u\right)g''(u)du.$$

Thus,

$$|K^{\alpha*}_{n,q_n}(g;x)-g(x)|\le\left|K^{\alpha}_{n,q_n}\left(\int_x^s(s-u)g''(u)du;x\right)\right|$$

$$+\left|\int_x^{\frac{[n]_{q_n}x}{[n+1]_{q_n}}+\frac{1}{[n+1]_{q_n}[\alpha+1]_{q_n}}}\left(\frac{[n]_{q_n}x}{[n+1]_{q_n}}+\frac{1}{[n+1]_{q_n}[\alpha+1]_{q_n}}-u\right)g''(u)du\right|$$
$$\leq K_{n,q_n}^{\alpha}\left(\left|\int_x^s |s-u||g''(u)|du\right|;x\right)$$
$$+\left|\int_x^{\frac{[n]_{q_n}x}{[n+1]_{q_n}}+\frac{1}{[n+1]_{q_n}[\alpha+1]_{q_n}}}\left|\frac{[n]_{q_n}x}{[n+1]_{q_n}}+\frac{1}{[n+1]_{q_n}[\alpha+1]_{q_n}}-u\right|\left|g''(u)\right|du\right|$$
$$\leq \frac{\|g''\|}{2}\left[K_{n,q_n}^{\alpha}((s-x)^2;x)+\left(\frac{[n]_{q_n}x}{[n+1]_{q_n}}+\frac{1}{[n+1]_{q_n}[\alpha+1]_{q_n}}-x\right)^2\right]$$
$$\leq \|g''^{\alpha}_{n,q_n}, \forall x\in I. \tag{5.6}$$

For all $\zeta\in C(I)$, in view of Lemma 3 and (5.5), we get

$$|K_{n,q_n}^{\alpha *}(\zeta;x)|\leq |K_{n,q_n}^{\alpha}(\zeta;x)|+|\zeta(x)|+\left|\zeta\left(\frac{[n]_{q_n}x}{[n+1]_{q_n}}+\frac{1}{[n+1]_{q_n}[\alpha+1]_{q_n}}\right)\right|$$
$$\leq 3\|\zeta\|, \forall x\in I. \tag{5.7}$$

Now, for $\zeta\in C(I)$ and any $g\in W^2(I)$, using (5.6) and (5.7)

$$|K_{n,q_n}^{\alpha}(\zeta;x)-\zeta(x)|\leq |K_{n,q_n}^{\alpha *}(\zeta-g;x)-(\zeta-g)(x)|$$
$$+\left|\zeta\left(\frac{[n]_{q_n}x}{[n+1]_{q_n}}+\frac{1}{[n+1]_{q_n}[\alpha+1]_{q_n}}\right)-\zeta(x)\right|+|K_{n,q_n}^{\alpha *}(g;x)-g(x)|$$
$$\leq 4\left(\|\zeta-g\|+\frac{1}{4}\phi_{n,q_n}^{\alpha}(x)\|g''\|\right)+\omega\left(\zeta;\sup_{x\in I}|\mu_{n,q_n}^{\alpha,1}(x)|\right).$$

Now, taking the infimum on the right hand side over all $g\in W^2(I)$,

$$\|K_{n,q_n}^{\alpha}(\zeta)-\zeta\|\leq 4K_2\left(\zeta;\frac{\phi_{n,q_n}^{\alpha}}{4}\right)+\omega\left(\zeta,\sup_{x\in I}|\mu_{n,q_n}^{\alpha,1}(x)|\right),$$

where

$$K_2(f,\delta):=\inf\left\{\|f-g\|+\delta\,\|g''\|:g\in W^2(I)\right\}.$$

Finally, using the relation between K-functional and second order modulus of smoothness [14], we get

$$\|K_{n,q_n}^{\alpha}(\zeta)-\zeta\|\leq C\omega_2\left(\zeta;\frac{\sqrt{\phi_{n,q_n}^{\alpha}}}{2}\right)+\omega\left(\zeta;\sup_{x\in I}|\mu_{n,q_n}^{\alpha,1}(x)|\right),$$

for some positive constant C. This completes the proof. □

Let

$$Lip_M(r)=\{\zeta\in C(I):|\zeta(s)-\zeta(y)|\leq M|s-y|^r,\ for\,all\,s,y\in I, 0<r\leq 1, M>0\}$$

be the space of Lipschitz continuous functions. In the following result we establish the degree of approximation by the operators $K_{n,q_n}^{\alpha}$ for the functions in $Lip_M(r)$.

**Theorem 7.** *Let $\zeta \in Lip_M(r)$. Then*

$$\|K^{\alpha}_{n,q_n}(\zeta;\cdot)-\zeta\| \le M\left(\sup_{x\in I}\mu^{\alpha,2}_{n,q_n}(x)\right)^{\frac{r}{2}}.$$

*Proof.* By the definition of Lipschitz function of order $r$, we have

$$\begin{aligned}|K^{\alpha}_{n,q_n}(\zeta;x)-\zeta(x)| &\le K^{\alpha}_{n,q_n}(|\zeta(s)-\zeta(x)|;x)\\ &\le \sum_{k=0}^{n} p^{\alpha}_{n,k}(q_n;x)\int_0^1\left|\zeta\left(\frac{[k]_{q_n}+s^{\alpha}}{[n+1]_{q_n}}\right)-\zeta(x)\right| d_{q_n}s\\ &\le \sum_{k=0}^{n} p^{\alpha}_{n,k}(q_n;y)\int_0^1 M\left|\frac{[k]_{q_n}+s^{\alpha}}{[n+1]_{q_n}}-x\right|^r d_{q_n}s.\end{aligned}$$

Applying Hölder's inequality and Lemma 1, we get

$$\begin{aligned}|K^{\alpha}_{n,q_n}(\zeta;x)-\zeta(x)| &\le M\left(\sum_{k=0}^{n} p^{\alpha}_{n,k}(q_n;x)\int_0^1\left|\frac{[k]_{q_n}+s^{\alpha}}{[n+1]_{q_n}}-x\right|^2 d_{q_n}s\right)^{\frac{r}{2}}\\ &\le M\{K^{\alpha}_{n,q_n}((s-x)^2;x)\}^{\frac{r}{2}} \le M\left(\mu^{\alpha,2}_{n,q_n}(x)\right)^{\frac{r}{2}},\end{aligned}$$

from which the required result is straight forward. □

## 6. Construction of the bivariate operators

Now we proceed to study the approximation of continuous functions of two variables by the bivariate extension of the operators $K^{\alpha}_{n,q_n}$. For $I^2 = I\times I$, let $C(I^2)$ be the space of all real valued continuous functions on $I^2$, endowed with the norm given by $\|\zeta\|_{C(I^2)} = \sup_{(t,s)\in I^2}|\zeta(t,s)|$. For $\zeta \in C(I^2)$, the bivariate extension of the operator (1.4) is defined as

$$\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},x,y) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} p^{\alpha_1,\alpha_2}_{n_1,n_2,k_1,k_2}(q_{n_1},q_{n_2},x,y)\int_0^1\int_0^1 \zeta\left(\frac{[k_1]_{q_{n_1}}+t^{\alpha_1}}{[n_1+1]_{q_{n_1}}},\frac{[k_2]_{q_{n_2}}+s^{\alpha_2}}{[n_2+1]_{q_{n_2}}}\right)d_{q_{n_1}}t\,d_{q_{n_2}}s, \tag{6.1}$$

where

$$p^{\alpha_1,\alpha_2}_{n_1,n_2,k_1,k_2}(q_{n_1},q_{n_2},x,y) = \binom{n_1}{k_1}_{q_{n_1}} x^{k_1}(1-x)^{n_1-k_1}_{q_{n_1}}\cdot\binom{n_2}{k_2}_{q_{n_2}} y^{k_2}(1-y)^{n_2-k_2}_{q_{n_2}},\ x,y\in I.$$

**Lemma 5.** *Let $e_{ij}(x,y) = x^i y^j$, $(i,j)\in\mathbb{N}_0\times\mathbb{N}_0$, with $i+j\le 2$ and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$ be the bivariate test functions. Then, for all $(x,y)\in I^2$*

i) $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(e_{00};q_{n_1},q_{n_2},x,y) = 1,$

ii) $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(e_{10};q_{n_1},q_{n_2},x,y) = \frac{[n_1]_{q_{n_1}}x}{[n_1+1]_{q_{n_1}}} + \frac{1}{[n_1+1]_{q_{n_1}}[\alpha_1+1]_{q_{n_1}}},$

iii) $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(e_{01};q_{n_1},q_{n_2},x,y) = \frac{[n_2]_{q_{n_2}}y}{[n_2+1]_{q_{n_2}}} + \frac{1}{[n_2+1]_{q_{n_2}}[\alpha_2+1]_{q_{n_2}}},$

iv) $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(e_{20};q_{n_1},q_{n_2},x,y) = \frac{1}{[n_1+1]^2_{q_{n_1}}}\{[n_1]_{q_{n_1}}x(1+q_{n_1}[n_1-1]_{q_{n_1}}x)+\frac{2[n_1]_{q_{n_1}}x}{[\alpha_1+1]_{q_{n_1}}}+\frac{1}{[2\alpha_1+1]_{q_{n_2}}}\}$

v) $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(e_{02};q_{n_1},q_{n_2},x,y) = \frac{1}{[n_2+1]^2_{q_{n_2}}}\{[n_2]_{q_{n_2}}y(1+q_{n_2}[n_2-1]_{q_{n_2}}y)+\frac{2[n_2]_{q_{n_2}}y}{[\alpha_2+1]_{q_{n_2}}}+\frac{1}{[2\alpha_2+1]_{q_{n_2}}}\}$.

*Proof.* This lemma follows easily by using Lemma 1 and the definition of bivariate Bernstein-Kantorovich operators, therefore the details are omitted. □

## 7. Approximation properties of bivariate operators

In this section, we establish the degree of approximation of the operators given by (6.1) in the space of continuous functions on compact set $I^2 := I \times I$. For $\zeta \in C(I^2)$, endowed with sup norm $||\zeta||_{C(I^2)} = \sup_{(t,s)\in I^2}|\zeta(t,s)|$ the total modulus of continuity for the bivariate case is defined as follows:

$$\omega(\zeta;\delta_1,\delta_2) = \sup\left\{|\zeta(t,s)-\zeta(x,y)| :\ t,s\in I,\ |t-x|\le\delta_1, |s-y|\le\delta_2\right\},$$

where $\delta_1,\delta_2 > 0$. Further $\omega(\zeta;\delta_1,\delta_2)$ satisfies the following properties:

a) $\omega(\zeta;\delta_1,\delta_2)\to 0, \quad as \quad \delta_1\to 0, \delta_2\to 0$

b) $|\zeta(t,s)-\zeta(x,y)| \le \omega(\zeta;\delta_1,\delta_2)(1+\frac{|t-x|}{\delta_1})(1+\frac{|s-y|}{\delta_2})$.

The partial moduli of continuity with respect to $x$ and $y$ is given by

$$\omega^1(\zeta;\delta) = \sup\left\{|\zeta(x_1,y)-\zeta(x_2,y)| : y\in I \text{ and } |x_1-x_2|\le\delta\right\},$$

$$\omega^2(\zeta;\delta) = \sup\left\{|\zeta(x,y_1)-\zeta(x,y_2)| : x\in I \text{ and } |y_1-y_2|\le\delta\right\}.$$

**Lemma 6.** [26] *Let $J_1$ and $J_2$ be compact intervals of the real line. Let $L_{m,n} : C(J_1\times J_2)\to C(J_1\times J_2)$ be linear positive operators. If*

$$\lim_{m,n\to\infty} L_{m,n}(e_{ij};x,y) = x^iy^j, (i,j)\in\{(0,0),(1,0),(0,1)\} \textit{ and}$$

$$\lim_{m,n\to\infty} L_{m,n}(e_{20}+e_{02};x,y) = x^2+y^2,$$

*uniformly in $J_1\times J_2$, then the sequence $L_{m,n}(g)$ converges to $g$ uniformly on $J_1\times J_2$ for any $g\in C(J_1\times J_2)$.*

In what follows, let $\mu^{\alpha_1,m}_{n_1,q_{n_1}}(x) = \tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((e_{10}-x)^m;x,y)$ and $\mu^{\alpha_2,m}_{n_2,q_{n_2}}(y) = \tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((e_{01}-y)^m;x,y)$, $m\in N\cup\{0\}$.

**Theorem 8.** *For $\zeta\in C(I^2)$, we have*

$$\|\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},.,.)-\zeta\|_{C(I^2)} \le 4\omega\left(\zeta;\sqrt{\sup_{x\in I}\mu^{\alpha_1,2}_{n_1,q_{n_1}}(x)}\sqrt{\sup_{y\in I}\mu^{\alpha_2,2}_{n_2,q_{n_2}}(y)}\right).$$

**Example 2.** *Let* $f(x,y)=x^2y^3-2x^3y+3xy$, $\alpha_i=2$ *and* $q_{n_i}=1-\dfrac{1}{n_i}$, $i=1,2$. *Denote* $E^{\alpha_1,\alpha_2}_{n_1,n_2}(f;x,y)=\left|\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\right|$, *the error function of approximation by* $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}$ *operators. The error of approximation* $E^{\alpha_1,\alpha_2}_{n_1,n_2}$ *is given in Figure 3. It is observed that the operators* $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(f)$ *converge towards the function* $f$ *by increasing the values of* $n_i$, $i=1,2$.

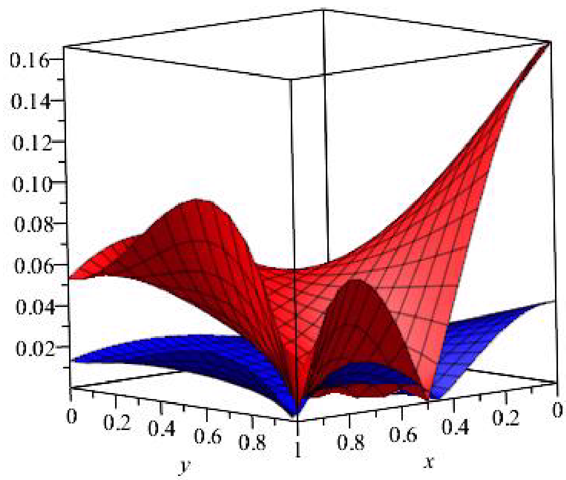


FIGURE 3. Error of approximation $E^{\alpha_1,\alpha_2}_{n_1,n_2}$ (red for $n_1=n_2=10$, blue for $n_1=n_2=40$)

**Theorem 9.** *For* $\zeta\in C(I^2)$, *there holds the following inequality*

$$\|\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},.,.)-\zeta\|_{C(I^2)}\leq 2\left(\omega^1(\zeta;\sup_{x\in I}\mu^{\alpha_1,2}_{n_1,q_{n_1}}(x))+\omega^2(\zeta;\sup_{y\in I}\mu^{\alpha_2,2}_{n_2,q_{n_2}}(y))\right),$$

*Proof.* Using the definition of the partial moduli of continuity, Lemma 5 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&|\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},x,y)-\zeta(x,y)|\leq\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(|\zeta(t,s)-\zeta(x,y)|;q_{n_1},q_{n_2},x,y)\\&\leq\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(|\zeta(t,s)-\zeta(x,s)|;q_{n_1},q_{n_2},x,y)+\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(|\zeta(x,s)-\zeta(x,y)|;q_{n_1},q_{n_2},x,y)\\&\leq\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\omega^1(\zeta;|t-x|);q_{n_1},q_{n_2},x,y\right)+\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\omega^2(\zeta;|s-y|);q_{n_1},q_{n_2},x,y\right)\end{aligned}$$

$$\leq \omega^1(\zeta;\mu^{\alpha_1,2}_{n_1,q_{n_1}}(x))\left[1+\frac{1}{\sqrt{\mu^{\alpha_1,2}_{n_1,q_{n_1}}(x)}}\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(|t-x|;q_{n_1},q_{n_2},x,y)\right]$$
$$+\,\omega^2(\zeta;\mu^{\alpha_2,2}_{n_2,q_{n_2}}(x))\left[1+\frac{1}{\sqrt{\mu^{\alpha_2,2}_{n_2,q_{n_2}}(y)}}\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(|s-y|;q_{n_1},q_{n_2},x,y)\right]$$
$$\leq \omega^1(\zeta;\mu^{\alpha_1,2}_{n_1,q_{n_1}}(x))\left[1+\frac{1}{\sqrt{\mu^{\alpha_1,2}_{n_1,q_{n_1}}(x)}}\left(\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((e_{10}-x)^2;q_{n_1},q_{n_2},x,y)\right)^{1/2}\right]$$
$$+\,\omega^2(\zeta;\mu^{\alpha_2,2}_{n_2,q_{n_2}}(y))\left[1+\frac{1}{\sqrt{\mu^{\alpha_2,2}_{n_2,q_{n_2}}(y)}}\left(\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((e_{01}-y)^2;q_{n_1},q_{n_2},x,y)\right)^{1/2}\right],$$

which yields the desired result. □

Now, we establish the degree of approximation for the bivariate operators (6.1) with the aid of Lipschitz class. For $0<\theta_1\leq 1$ and $0<\theta_2\leq 1$ and $\zeta\in C(I^2)$ we define the Lipschitz class $Lip_M(\theta_1,\theta_2)$ for the bivariate case as follows:

$$|\zeta(t_1,t_2)-\zeta(s_1,s_2)|\leq M|t_1-s_1|^{\theta_1}|t_2-s_2|^{\theta_2},\ M>0.$$

**Theorem 10.** *Let $\zeta\in Lip_M(\theta_1,\theta_2)$. Then, we have*

$$\|\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},.,.)-\zeta\|_{C(I^2)}\leq M\left(\sup_{x\in I}\mu^{\alpha_1,2}_{n_1,q_{n_1}}(x)\right)^{\frac{\theta_1}{2}}\left(\sup_{y\in I}\mu^{\alpha_2,2}_{n_2,q_{n_2}}(y)\right)^{\frac{\theta_2}{2}}.$$

*Proof.* Since $\zeta\in Lip_M(\theta_1,\theta_2)$, we may write
$|\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},x,y)-\zeta(x,y)|$

$$\begin{aligned}
&\leq \tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(|\zeta(t,s)-\zeta(x,y)|;q_{n_1},q_{n_2},x,y)\\
&\leq \tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(M|t-x|^{\theta_1}|s-y|^{\theta_2};q_{n_1},q_{n_2},x,y\right)\\
&\leq MK^{\alpha_1}_{n_1}\left(|t-x|^{\theta_1};q_{n_1},x,y\right)\ K^{\alpha_2}_{n_2}\left(|s-y|^{\theta_2};q_{n_2},x,y\right).
\end{aligned}$$

Applying the Hölder's inequality with $(p_1,q_1)=\left(\frac{2}{\theta_1},\frac{2}{2-\theta_1}\right)$ and $(p_2,q_2)=\left(\frac{2}{\theta_2},\frac{2}{2-\theta_2}\right)$, we have

$$\begin{aligned}
&|\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},x,y)-\zeta(x,y)|\\
&\leq MK^{\alpha_1}_{n_1}\left((e_{1,0}-x)^2;q_{n_1},x\right)^{\theta_1/2}\ K^{\alpha_1}_{n_1}\left(e_{0,0};q_{n_1},x\right)^{(2-\theta_1)/2}\\
&\times\ K^{\alpha_2}_{n_2}\left((e_{0,1}-y)^2;q_{n_2},y\right)^{\theta_2/2}\ K^{\alpha_2}_{n_2}\left(e_{0,0};q_{n_2},y\right)^{(2-\theta_2)/2}\\
&\leq M(\sup_{x\in I}\mu^{\alpha_1,2}_{n_1,q_{n_1}}(x))^{\frac{\theta_1}{2}}\ (\sup_{y\in I}\mu^{\alpha_2,2}_{n_2,q_{n_2}}(y))^{\frac{\theta_2}{2}}.
\end{aligned}$$

This completes the proof. □

Let $C^1(I^2)$ be the space of continuous functions in $I^2$ whose first order partial derivatives are continuous in $I^2$. In the next theorem, we obtain the degree of approximation by the operators defined in (6.1), for the functions in $C^1(I^2)$.

**Theorem 11.** *Let $\zeta \in C^1(I^2)$. Then, we have*

$$\|\tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2}(\zeta; q_{n_1}, q_{n_2}, .) - \zeta\|_{C(I^2)} \leq \|\zeta'_x\|_{C(I^2)} \left(\sup_{x\in I} \mu_{n_1,q_{n_1}}^{\alpha_1,2}(x)\right)^{\frac{1}{2}} + \|\zeta'_y\|_{C(I^2)} \left(\sup_{y\in I} \mu_{n_2,q_{n_2}}^{\alpha_2,2}(y)\right)^{\frac{1}{2}}.$$

*Proof.* From the hypothesis we can write

$$\zeta(t,s) - \zeta(x,y) = \int_x^t \zeta'_w(w;s)dw + \int_y^s \zeta'_u(x;u)du. \tag{7.1}$$

Applying $\tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2}(.; q_{n_1}, q_{n_2}, x, y)$ on both sides of (7.1), we get

$$|\tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2}(\zeta; q_{n_1}, q_{n_2}, x, y) - \zeta(x,y)| \leq \tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\left|\int_x^t |\zeta'_w(w;s)|dw\right|; q_{n_1}, q_{n_2}, x, y\right) + \tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\left|\int_y^s |\zeta'_u(x;u)|du\right|; q_{n_1}, q_{n_2}, x, y\right).$$

Using mean value theorem, we have

$$\left|\int_x^t |\zeta'_w(w;s)|dw\right| \leq \|\zeta'_x\|_{C(I^2)}|t-x| \text{ and } \left|\int_y^s |\zeta'_u(x;u)|du\right| \leq \|\zeta'_y\|_{C(I^2)}|s-y|.$$

Therefore

$$|\tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2}(\zeta; q_{n_1}, q_{n_2}, x, y) - \zeta(x,y)| \leq \|\zeta'_x\|_{C(I^2)}\tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2}(|t-x|; q_{n_1}, q_{n_2}, x, y) + \|\zeta'_y\|_{C(I^2)}\tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2}(|s-y|; q_{n_1}, q_{n_2}, x, y).$$

Applying Cauchy-Schwarz inequality and using Lemma 5, we obtain

$$\begin{aligned}
&|\tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2}(\zeta; q_{n_1}, q_{n_2}, x, y) - \zeta(x,y)| \\
&\leq \|\zeta'_x\|_{C(I^2)}\left(\tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2}\left((t-x)^2; q_{n_1}, q_{n_2}, x, y\right)\right)^{1/2}\left(\tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2}\left(e_{0,0}; q_{n_1}, q_{n_2}, x, y\right)\right)^{1/2} \\
&+ \|\zeta'_y\|_{C(I^2)}\left(\tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2}\left((s-y)^2; q_{n_1}, q_{n_2}, x, y\right)\right)^{1/2}\left(\tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2}\left(e_{0,0}; q_{n_1}, q_{n_2}, x, y\right)\right)^{1/2} \\
&\leq \|\zeta'_x\|_{C(I^2)}\left(\sup_{x\in I}\mu_{n_1,q_{n_1}}^{\alpha_1,2}(x)\right)^{\frac{1}{2}} + \|\zeta'_y\|_{C(I^2)}\left(\sup_{y\in I}\mu_{n_2,q_{n_2}}^{\alpha_2,2}(y)\right)^{\frac{1}{2}},
\end{aligned}$$

which leads us to the required result. □

Let $C^2\left(I^2\right)=\{\zeta\in C\left(I^2\right):\zeta^{(i,j)}\in C\left(I^2\right),1\leq i,j\leq 2\}$, where $\zeta^{(i,j)}$ is the $(i,j)^{th}$ order partial derivative with respect to $x,y$ of $\zeta$, endowed with the norm

$$\|\zeta\|_{C^{(2)}(I^2)}=\|\zeta\|_{C(I^2)}+\|\zeta\|^1_{C(I^2)}+\|\zeta\|^2_{C(I^2)},$$

where

$$\|\zeta\|^1_{C(I^2)}=\sup_{(x,y)\in I^2}\left\{|\zeta(x,y)|,|\zeta'_x(x,y)|,|\zeta'_y(x,y)|\right\}$$

and

$$\|\zeta\|^2_{C(I^2)}=\sup_{(x,y)\in I^2}\left\{|\zeta(x,y)|,|\zeta'_x(x,y)|,|\zeta'_y(x,y)|,|\zeta''_{xx}(x,y)|,|\zeta''_{xy}(x,y)|,|\zeta''_{yy}(x,y)|\right\}.$$

Now, we proceed to determine the order of approximation of the sequence $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;.,.)$ to the function $\zeta\in C\left(I^2\right)$ in terms of the Peetre's K-functional given by

$$K(\zeta;\delta)=\inf_{g\in C^2(I^2)}\left\{\|\zeta-g\|_{C(I^2)}+\delta\|g\|_{C^2(I^2)},\delta>0\right\}.$$

It is known [13] that the inequality

$$K(\zeta;\delta)\quad\leq M_1\left\{\bar{\omega_2}(\zeta;\sqrt{\delta})+\min\{1,\delta\}\|\zeta\|_{C(I^2)}\right\}\tag{7.2}$$

holds for all $\delta>0$, where the constant $M_1$ is independent of $\zeta$, $\delta$ and

$$\bar{\omega}_2(\zeta;\sqrt{\delta})=\sup\left\{\left|\sum_{\nu=0}^{2}(-1)^{2-\nu}\zeta(x+\nu h,y+\nu k)\right|:(x,y),(x+2h,y+2k)\in I^2,\right.$$
$$\left.|h|\leq\delta,|k|\leq\delta\right\}$$

is the second order modulus of continuity for the bivariate case.

**Theorem 12.** *If $\zeta\in C\left(I^2\right)$, then we have*

$$\|\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},.,.)-\zeta\|_{C(I^2)}$$
$$\leq M\left\{\bar{\omega_2}(\zeta;\sqrt{\sup_{x,y\in I^2}F^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)}+\min\{1,\sup_{x,y\in I^2}F^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)\}\|\zeta\|_{C(I^2)}\right\}$$
$$+\omega(\zeta;\sup_{x\in I}|\mu^{\alpha_1,1}_{n_1,q_{n_1}}(x)|,\sup_{y\in I}|\mu^{\alpha_2,1}_{n_2,q_{n_2}}(y)|),$$

*where* $F^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)=\mu^{\alpha_1,2}_{n_1,q_{n_1}}(x)+\mu^{\alpha_2,2}_{n_2,q_{n_2}}(y)+\left(\mu^{\alpha_1,1}_{n_1,q_{n_1}}(x)\right)^2+\left(\mu^{\alpha_2,1}_{n_2,q_{n_2}}(y)\right)^2$ *and $M>0$ is a constant.*

*Proof.* We define the following auxiliary operator

$$\widehat{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},x,y)=\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},x,y)+\zeta(x,y)$$

$$-\zeta\left(\frac{[n_1]_{q_{n_1}}x}{[n_1+1]_{q_{n_1}}}+\frac{1}{[n_1+1]_{q_{n_1}}[\alpha_1+1]_{q_{n_1}}},\frac{[n_2]_{q_{n_2}}y}{[n_2+1]_{q_{n_2}}}+\frac{1}{[n_2+1]_{q_{n_2}}[\alpha_2+1]_{q_{n_2}}}\right). \tag{7.3}$$

Since $\widehat{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(e_{10};q_{n_1},q_{n_2},x,y)=x$ , $\widehat{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(e_{01};q_{n_1},q_{n_2},x,y)=y$, we get

$$\widehat{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((t-x);q_{n_1},q_{n_2},x,y)=0, \widehat{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((s-y);q_{n_1},q_{n_2},x,y)=0.$$

Let $g\in C^2\left(I^2\right)$ and $(t,s)\in I^2$. Then by Taylor's formula

$$\begin{aligned}g(t,s)-g(x,y)&=g(t,y)-g(x,y)+g(t,s)-g(t,y)\\&=\frac{\partial g(x,y)}{\partial x}(t-x)+\int_x^t(t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du\\&+\frac{\partial g(t,y)}{\partial y}(s-y)+\int_y^s(s-v)\frac{\partial^2 g(t,v)}{\partial v^2}dv.\end{aligned}$$

Applying the operator $\widehat{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(.;q_{n_1},q_{n_2},x,y)$ on both sides of the above relation and using (7.3), we obtain

$$\begin{aligned}&\widehat{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(g;q_{n_1},q_{n_2},x,y)-g(x,y)\\&=\widehat{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\int_x^t(t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;q_{n_1},q_{n_2},x,y\right)\\&+\widehat{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\int_y^s(s-v)\frac{\partial^2 g(t,v)}{\partial v^2}dv;q_{n_1},q_{n_2},x,y\right)\\&=\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\int_x^t(t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;q_{n_1},q_{n_2},x,y\right)\\&-\int_x^{\frac{[n_1]_{q_{n_1}}x}{[n_1+1]_{q_{n_1}}}+\frac{1}{[n_1+1]_{q_{n_1}}[\alpha_1+1]_{q_{n_1}}}}\left(\frac{[n_1]_{q_{n_1}}x}{[n_1+1]_{q_{n_1}}}+\frac{1}{[n_1+1]_{q_{n_1}}[\alpha_1+1]_{q_{n_1}}}-u\right)\frac{\partial^2 g(u,y)}{\partial u^2}du\\&+\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\int_y^s(s-v)\frac{\partial^2 g(t,v)}{\partial v^2}dv;q_{n_1},q_{n_2},x,y\right)\\&-\int_y^{\frac{[n_2]_{q_{n_2}}y}{[n_2+1]_{q_{n_2}}}+\frac{1}{[n_2+1]_{q_{n_2}}[\alpha_2+1]_{q_{n_2}}}}\left(\frac{[n_2]_{q_{n_2}}y}{[n_2+1]_{q_{n_2}}}+\frac{1}{[n_2+1]_{q_{n_2}}[\alpha_2+1]_{q_{n_2}}}-v\right)\frac{\partial^2 g(t,v)}{\partial v^2}dv.\end{aligned}$$

Hence using Lemma 5, we have

$$\begin{aligned}&|\widehat{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(g;q_{n_1},q_{n_2},x,y)-g(x,y)|\leq\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\left|\int_x^t|(t-u)|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right|;q_{n_1},q_{n_2},x,y\right)\\&+\int_x^{\frac{[n_1]_{q_{n_1}}x}{}[n_1+1]_{q_{n_1}}+\frac{1}{[n_1+1]_{q_{n_1}}[\alpha_1+1]_{q_{n_1}}}}\left|\left(\frac{[n_1]_{q_{n_1}}x}{[n_1+1]_{q_{n_1}}}+\frac{1}{[n_1+1]_{q_{n_1}}[\alpha_1+1]_{q_{n_1}}}-u\right)\right|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\end{aligned}$$

$$+\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\left|\int_y^s |(s-v)|\left|\frac{\partial^2 g(t,v)}{\partial v^2}\right|dv\right|;q_{n_1},q_{n_2},x,y\right)$$

$$+\int_y^{\frac{[n_2]_{q_{n_2}}y}{[n_2+1]_{q_{n_2}}}+\frac{1}{[n_2+1]_{q_{n_2}}[\alpha_2+1]_{q_{n_2}}}}\left|\left(\frac{[n_2]_{q_{n_2}}y}{[n_2+1]_{q_{n_2}}}+\frac{1}{[n_2+1]_{q_{n_2}}[\alpha_2+1]_{q_{n_2}}}-v\right)\right|\left|\frac{\partial^2 g(t,v)}{\partial v^2}\right|dv$$

$$\leq\left\{\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left((t-x)^2;q_{n_1},q_{n_2},x,y\right)+\left(\frac{[n_1]_{q_{n_1}}x}{[n_1+1]_{q_{n_1}}}+\frac{1}{[n_1+1]_{q_{n_1}}[\alpha_1+1]_{q_{n_1}}}-x\right)^2\right\}\|g\|_{C^2(I^2)}$$

$$+\left\{\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left((s-y)^2;q_{n_1},q_{n_2},x,y\right)+\left(\frac{[n_2]_{q_{n_2}}y}{[n_2+1]_{q_{n_2}}}+\frac{1}{[n_2+1]_{q_{n_2}}[\alpha_2+1]_{q_{n_2}}}-y\right)^2\right\}\|g\|_{C^2(I^2)}$$

$$\leq\left\{\mu^{\alpha_1,2}_{n_1,q_{n_1}}(x)+\left(\mu^{\alpha_1,1}_{n_1,q_{n_1}}(x)\right)^2\right\}||g||_{C^2(I^2)}+\left\{\mu^{\alpha_2,2}_{n_2,q_{n_2}}(y)+\left(\mu^{\alpha_2,1}_{n_2,q_{n_2}}(y)\right)^2\right\}||g||_{C^2(I^2)}$$

$$=\left\{\mu^{\alpha_1,2}_{n_1,q_{n_1}}(x)+\mu^{\alpha_2,2}_{n_2,q_{n_2}}(y)+\left(\mu^{\alpha_1,1}_{n_1,q_{n_1}}(x)\right)^2+\left(\mu^{\alpha_2,1}_{n_2,q_{n_2}}(y)\right)^2\right\}||g||_{C^2(I^2)}$$

$$=F^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)\|g\|_{C^2(I^2)}. \tag{7.4}$$

Also, from equation (7.3), we have

$$|\widehat{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},x,y)|\leq|\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},x,y)|+|\zeta(x,y)|$$

$$+\left|\zeta\left(\frac{[n_1]_{q_{n_1}}x}{[n_1+1]_{q_{n_1}}}+\frac{1}{[n_1+1]_{q_{n_1}}[\alpha_1+1]_{q_{n_1}}},\frac{[n_2]_{q_{n_2}}y}{[n_2+1]_{q_{n_2}}}+\frac{1}{[n_2+1]_{q_{n_2}}[\alpha_2+1]_{q_{n_2}}}\right)\right|$$

$$\leq 3\|\zeta\|_{C(I^2)}. \tag{7.5}$$

Hence for all $\zeta\in C(I^2)$ and $g\in C^2(I^2)$, using (7.4) and (7.5) we get

$$|\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},x,y)-\zeta(x,y)|\leq\Bigg|\widehat{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},x,y)-\zeta(x,y)$$

$$+\zeta\left(\frac{[n_1]_{q_{n_1}}x}{[n_1+1]_{q_{n_1}}}+\frac{1}{[n_1+1]_{q_{n_1}}[\alpha_1+1]_{q_{n_1}}},\frac{[n_2]_{q_{n_2}}y}{[n_2+1]_{q_{n_2}}}+\frac{1}{[n_2+1]_{q_{n_2}}[\alpha_2+1]_{q_{n_2}}}\right)-\zeta(x,y)\Bigg|$$

$$\leq|\widehat{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((\zeta-g);q_{n_1},q_{n_2},x,y)|+|\widehat{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(g;q_{n_1},q_{n_2},x,y)-g(x,y)|+|g(x,y)-\zeta(x,y)|$$

$$+\left|\zeta\left(\frac{[n_1]_{q_{n_1}}x}{[n_1+1]_{q_{n_1}}}+\frac{1}{[n_1+1]_{q_{n_1}}[\alpha_1+1]_{q_{n_1}}},\frac{[n_2]_{q_{n_2}}y}{[n_2+1]_{q_{n_2}}}+\frac{1}{[n_2+1]_{q_{n_2}}[\alpha_2+1]_{q_{n_2}}}\right)-\zeta(x,y)\right|$$

$$\leq 4\|\zeta-g\|_{C(I^2)}+|\widehat{K}_{n_1,n_2}(g;q_{n_1},q_{n_2},x,y)-g(x,y)|$$

$$+\left|\zeta\left(\frac{[n_1]_{q_{n_1}}x}{[n_1+1]_{q_{n_1}}}+\frac{1}{[n_1+1]_{q_{n_1}}[\alpha_1+1]_{q_{n_1}}},\frac{[n_2]_{q_{n_2}}y}{[n_2+1]_{q_{n_2}}}+\frac{1}{[n_2+1]_{q_{n_2}}[\alpha_2+1]_{q_{n_2}}}\right)-\zeta(x,y)\right|$$

$$\leq 4\|\zeta-g\|_{C(I^2)}+F^{\alpha_1,\alpha_2}_{n_1,n_2,q_{n_1},q_{n_2}}(x,y)\|g\|_{C^2(I^2)}+\omega(\zeta;|\mu^{\alpha_1,1}_{n_1,q_{n_1}}(x)|,|\mu^{\alpha_2,1}_{n_2,q_{n_2}}(y)|),$$

Taking the infimum on the right-hand side over all $g \in C^2(I^2)$ and using inequality (7.2), we obtain

$$|\tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2}(\zeta; q_{n_1}, q_{n_2}, x, y) - \zeta(x, y)|$$
$$\leq 4K(\zeta; \sup_{x,y\in I^2} F_{n_1,n_2,q_{n_1},q_{n_2}}^{\alpha_1,\alpha_2}(x, y)) + \omega(\zeta; |\mu_{n_1,q_{n_1}}^{\alpha_1,1}(x)|, |\mu_{n_2,q_{n_2}}^{\alpha_2,1}(y)|),$$
$$\leq M\left\{\bar{\omega_2}\left(\zeta; \sqrt{\sup_{x,y\in I^2} F_{n_1,n_2,q_{n_1},q_{n_2}}^{\alpha_1,\alpha_2}(x,y)}\right) + \min\{1, \sup_{x,y\in I^2} F_{n_1,n_2,q_{n_1},q_{n_2}}^{\alpha_1\alpha_2}(x,y)\}\|\zeta\|_{C^2(I^2)}\right\}$$
$$+\omega(\zeta; \sup_{x\in I}|\mu_{n_1,q_{n_1}}^{\alpha_1,1}(x)|, \sup_{y\in I}|\mu_{n_2,q_{n_2}}^{\alpha_2,1}(y)|).$$

This completes the proof. □

## 8. Construction of GBS operators of $q$-Bernstein-Kantorovich type

The continuity and the differentiability of a function in Bögel space were first discussed by Bögel in [10] and [12]. After that, several researchers have studied $B$-continuity and $B$-differentiability to obtain some approximation results of GBS of bivariate positive linear operators. For related papers we refer the reader to [17, 5, 9] and the references therein.

First of all we study some basic definitions and notations:
A real-valued function $\zeta$ over the rectangle $I^2$ is said to be B-continuous if for every $(t, s) \in I^2$ there holds

$$\lim_{(t,s)\to(x,y)} \Delta_{(t,s)}\zeta(x, y) = 0,$$

where $\Delta_{(t,s)}\zeta(x, y) = \zeta(x, y) - \zeta(x, s) - \zeta(t, y) + \zeta(t, s)$.

Let $C_b\left(I^2\right)$ denote the space of all $B$-continuous functions over $I^2$. Let $B\left(I^2\right), C\left(I^2\right)$ be the spaces of all bounded functions and the space of all continuous (in the usual sense) functions respectively over $I^2$ equipped with the sup-norm $\|.\|_\infty$. It is known that $C\left(I^2\right) \subset C_b\left(I^2\right)$ ([11], page 52).

The mixed modulus of smoothness of $\zeta \in C_b\left(I^2\right)$ is defined as

$$\omega_{mixed}\left(\zeta; \delta_1, \delta_2\right) := \sup\{|\Delta_{(x+h_1,y+h_2)}\zeta(x, y)|\},$$

where the supremum is taken over all $(t, s) \in I^2$, $(h_1, h_2) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$, in such a way that $(x + h_1, y + h_2) \in I^2$, $0 < |h_1| \leq \delta_1,\ 0 < |h_2| \leq \delta_2$. The mixed modulus of continuity including the upper bounds and the total modulus of continuity were first discussed by Marchaud [22].

A real valued function defined over the rectangle $I^2$ is said to be uniformly B-continuous function if and only if

$$\lim_{\delta_1,\delta_2\to 0} \omega_{mixed}(\zeta; \delta_1, \delta_2) = 0.$$

Furthermore, for all non-negative numbers $\lambda_1, \lambda_2$ there holds

$$\omega_{mixed}(\zeta; \lambda_1\delta_1, \lambda_2\delta_2) \leq (1+]\lambda_1[)\,(1+]\lambda_2[)\,\omega_{mixed}(\zeta; \delta_1, \delta_2),$$

where ].[ denotes the greatest integer function.

A function $\zeta : I^2 \longrightarrow \mathbb{R}$ is said to be Bögel differentiable if for every $(t,s) \in I^2$,

$$\lim_{(t,s)\to(x,y)} \frac{\Delta_{(t,s)}\zeta(x,y)}{(t-x)(s-y)} = D_B\zeta(x,y) < \infty.$$

Here, $D_B$ is called the B-derivative of $\zeta$ and $D_b(I^2)$ denotes the space of all B-differentiable functions over $I^2$.

In this section, we introduce the GBS case for the operators defined in (6.1). For every $\zeta \in C_b(I^2)$, the GBS operator associated with the operator $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}$ is defined as follows:

$$\tilde{U}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta; q_{n_1}, q_{n_2}, x, y) = \tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta(x,s)+\zeta(t,y)-\zeta(t,s); q_{n_1}, q_{n_2}, x, y). \tag{8.1}$$

Evidently $\tilde{U}^{\alpha_1,\alpha_2}_{n_1,n_2} : C_b(I^2) \to C(I^2)$ is a linear operator
Also,

$$\tilde{U}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta; q_{n_1}, q_{n_2}, x, y) = \tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\zeta(x,y) - \Delta_{(t,s)}\zeta(x,y); q_{n_1}, q_{n_2}, x, y\right).$$

**Example 3.** *Let* $f(x,y) = y^2 - 2\sqrt{3}(3xy^2 + x^2 + y^2 - 2x - y + 1) - 6xy$, $n_i = 10$, $\alpha_i = 2$, $q_i = 1 - \dfrac{1}{n_i}$, $i = 1, 2$. *In Figure 4 we compare* $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}$ *and its GBS type operator. We note that* $\tilde{U}^{\alpha_1,\alpha_2}_{n_1,n_2}$ *gives a better approximation than the operator* $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}$. *Let us denote* $E^{\alpha_1,\alpha_2}_{n_1,n_2}(f; x, y) = \left|\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(f; q_1, q_2, x, y) - f(x,y)\right|$ *and* $\tilde{E}^{\alpha_1,\alpha_2}_{n_1,n_2}(f; x, y) = \left|\tilde{U}^{\alpha_1,\alpha_2}_{n_1,n_2}(f; q_1, q_2, x, y) - f(x,y)\right|$, *the error functions of approximation by* $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}$ *and* $\tilde{U}^{\alpha_1,\alpha_2}_{n_1,n_2}$. *In Figure 5 we compare the error of approximation for these two operators. Also, in Table 2 we compute the error of approximation for* $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}$ *and* $\tilde{U}^{\alpha_1,\alpha_2}_{n_1,n_2}$ *at certain points.*

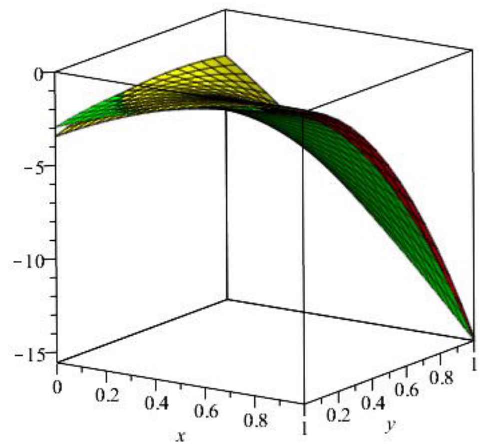


FIGURE 4. The convergence of $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}$ (green) and $\tilde{U}^{\alpha_1,\alpha_2}_{n_1,n_2}$ (yellow) to $f$ (red)

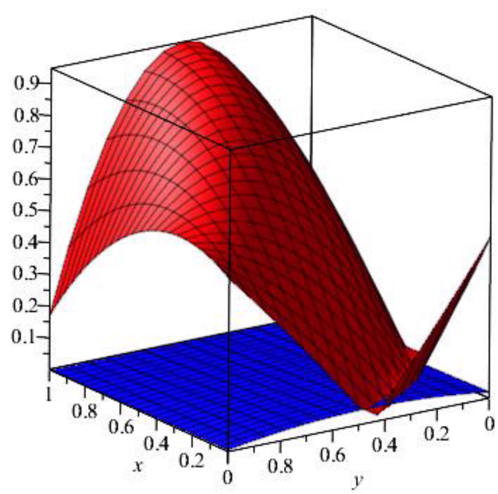


FIGURE 5. The errors of approximation $E^{\alpha_1,\alpha_2}_{n_1,n_2}$ (red) and $\tilde{E}^{\alpha_1,\alpha_2}_{n_1,n_2}$ (blue)

**Table 2.** *Error of approximation* $E_{n_1,n_2}^{\alpha_1,\alpha_2}$ *and* $\tilde{E}_{n_1,n_2}^{\alpha_1,\alpha_2}$

| $x$ | $y$ | $E_{n_1,n_2}^{\alpha_1,\alpha_2}(f;x,y)$ | $\tilde{E}_{n_1,n_2}^{\alpha_1,\alpha_2}(f;x,y)$ |
|---|---|---|---|
| 0.1 | 0.1 | 0.2276350064 | 0.02781616682 |
| 0.1 | 0.4 | 0.1215441729 | 0.04184227259 |
| 0.1 | 0.6 | 0.3097633341 | 0.03918083942 |
| 0.1 | 0.8 | 0.4623289273 | 0.02690967180 |
| 0.2 | 0.6 | 0.4426519342 | 0.03509223074 |
| 0.2 | 0.8 | 0.5504140271 | 0.02410158530 |
| 0.4 | 0.2 | 0.3092122211 | 0.02397017177 |
| 0.4 | 0.8 | 0.6772472603 | 0.01848541202 |
| 0.5 | 0.2 | 0.4135011700 | 0.02032890335 |
| 0.5 | 0.5 | 0.7081630738 | 0.02430147765 |
| 0.8 | 0.2 | 0.6276940917 | 0.00940510786 |
| 0.8 | 0.9 | 0.5794053624 | 0.00462801353 |
| 0.9 | 0.2 | 0.6662004167 | 0.00576384107 |
| 0.9 | 0.8 | 0.7065313925 | 0.00444498079 |

**Theorem 13.** [8] *Let* $\tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2} : C_b(I^2) \to B(I^2)$, $n_1, n_2 \in N$ *be a sequence of bivariate linear positive operators,* $\tilde{U}_{n_1,n_2}^{\alpha_1,\alpha_2}(.; q_{n_1}, q_{n_2}, x, y)$ *be the GBS-operators associated with* $\tilde{K}_{n_1,n_2}^{\alpha_1,\alpha_2}$ *and the following conditions are satisfied:*

(1) $\tilde{U}_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{00}; q_{n_1}, q_{n_2}, x, y) = 1$;
(2) $\tilde{U}_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{10}; q_{n_1}, q_{n_2}, x, y) = x + u_{m,n}(x, y)$;
(3) $\tilde{U}_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{01}; q_{n_1}, q_{n_2}, x, y) = y + v_{m,n}(x, y)$;
(4) $\tilde{U}_{n_1,n_2}^{\alpha_1,\alpha_2}(e_{20} + e_{02}; q_{n_1}, q_{n_2}, x, y) = x^2 + y^2 + w_{m,n}(x, y)$;
*for all* $(x, y) \in I^2$. *If all the sequences* $u_{n_1,n_2}(x, y)$, $v_{n_1,n_2}(x, y)$ *and* $w_{n_1,n_2}(x, y)$ *converge to zero as* $n_i \to \infty, i = 1, 2$ *uniformly in* $I^2$, *then the sequence* $\{\tilde{U}_{n_1,n_2}^{\alpha_1,\alpha_2}(\zeta; q_{n_1}, q_{n_2}, x, y)\}$ *converges to* $\zeta$, *as* $n_i \to \infty, i = 1, 2$ *uniformly in* $I^2$ *for all* $\zeta \in C_b(I^2)$.

As a consequence of the above theorem, we have

**Theorem 14.** *For* $\zeta \in C_B(I^2)$, *the operator* $\tilde{U}_{n_1,n_2}^{\alpha_1,\alpha_2}(\zeta; q_{n_1}, q_{n_2}, x, y)$ *converges to* $\zeta$, *as* $n \to \infty$ *uniformly in* $I^2$.

**Theorem 15.** *For every* $\zeta \in C_b\left(I^2\right)$, *at each point* $(x, y) \in I^2$, *the operator (8.1) verifies the following inequality*

$$\|\tilde{U}_{n_1,n_2}^{\alpha_1,\alpha_2}(\zeta; q_{n_1}, q_{n_2}, \cdot, \cdot) - \zeta\| \leq 4\, \omega_{mixed}(\zeta; \sup_{x\in I} \mu_{n_1,q_{n_1}}^{\alpha_1,2}(x), \sup_{y\in I} \mu_{n_2,q_{n_2}}^{\alpha_2,2}(y)).$$

In our next theorem, we obtain the approximation properties of the operators $\tilde{U}_{n_1,n_2}^{\alpha_1,\alpha_2}(\zeta; ., .)$ by means of the Lipschitz-class for Bögel continuous function which is defined as follows:

If $\zeta \in C_b(I^2)$, for two parameters $0<\theta_1\leq 1$ and $0<\theta_2\leq 1$, $Lip_M(\theta_1,\theta_2)$, $M>0$ is defined by

$$Lip_M(\theta_1,\theta_2)=\left\{\zeta\in C_b(I^2):\left|\Delta_{(t,s)}\zeta(x,y)\right|\leq M\left|t-x\right|^{\theta_1}\left|s-y\right|^{\theta_2},(t,s),(x,y)\in I^2\right\}.$$

**Theorem 16.** *For $\zeta \in Lip_M(\theta_1,\theta_2)$ and $(x,y)\in I^2$, we have*

$$\|\tilde{U}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},\cdot,\cdot)-\zeta\|\leq M(\sup_{x\in I}\mu^{\alpha_1,2}_{n_1,q_{n_1}}(x))^{\theta_1/2}(\sup_{y\in I}\mu^{\alpha_2,2}_{n_2,q_{n_2}}(y))^{\theta_2/2}.$$

*Proof.* By the definition of $\tilde{U}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},\cdot,\cdot)$ and using the linearity of the operator $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},\cdot,\cdot)$, we may write

$$\begin{aligned}
&\tilde{U}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},x,y)=\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta(x,s)+\zeta(t,y)-\zeta(t,s);q_{n_1},q_{n_2},x,y)\\
&=\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\zeta(x,y)-\Delta_{(t,s)}\zeta(x,y);q_{n_1},q_{n_2},x,y\right)\\
&=\zeta(x,y)\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(e_{00};q_{n_1},q_{n_2},x,y)-\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Delta_{(t,s)}\zeta(x,y);q_{n_1},q_{n_2},x,y\right).
\end{aligned}$$

By our hypothesis, we get

$$\begin{aligned}
&\left|\tilde{U}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},x,y)-\zeta(x,y)\right|\leq\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\left|\Delta_{(t,s)}\zeta(x,y)\right|;q_{n_1},q_{n_2},x,y\right)\\
&\leq M\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(|t-x|^{\theta_1}|s-y|^{\theta_2};q_{n_1},q_{n_2},x,y\right)\\
&=M\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(|t-x|^{\theta_1};q_{n_1},q_{n_2},x,y\right)\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\left(|s-y|^{\theta_2};q_{n_1},q_{n_2},x,y\right).
\end{aligned}$$

Now, applying the Hölder's inequality with $(p_1,q_1)=(2/\theta_1,2/(2-\theta_1))$ and $(p_2,q_2)=(2/\theta_2,2/(2-\theta_2))$, we obtain

$$\begin{aligned}
|\tilde{U}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},x,y)-\zeta(x,y)|\leq &MK^{\alpha_1}_{n_1}\left((t-x)^2;q_{n_1},x\right)^{\theta_1/2}K^{\alpha_1}_{n_1}(e_0;q_{n_1},x)^{(2-\theta_1)/2}\\
&\times K^{\alpha_2}_{n_2}\left((s-y)^2;q_{n_2},y\right)^{\theta_2/2}K^{\alpha_2}_{n_2}(e_0;q_{n_2},y)^{(2-\theta_2)/2},
\end{aligned}$$

which leads us to the required result. □

**Theorem 17.** *Let the function $\zeta\in D_b(I^2)$ with $D_B\zeta\in B(I^2)$. Then, for each $(x,y)\in I^2$, we have*

$$\begin{aligned}
&\|\tilde{U}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},\cdot,\cdot)-\zeta\|\\
&\leq C\left(\frac{1}{[n_1]_{q_{n_1}}}\right)^{\frac{1}{2}}\left(\frac{1}{[n_2]_{q_{n_2}}}\right)^{\frac{1}{2}}\left(||D_B\zeta||_\infty+\omega_{mixed}(D_B\zeta;[n_2]^{-\frac{1}{2}}_{q_{n_1}},[n_2]^{-\frac{1}{2}}_{q_{n_2}})\right).
\end{aligned}$$

*where $C$ is a certain positive constant.*

*Proof.* Since $\zeta\in D_b(I^2)$, we have the identity

$$\Delta_{(t,s)}\zeta(x,y)=(t-x)(s-y)D_B\zeta(\beta_1,\beta_2),$$

where $\beta_1,\beta_2$ lie between $t$ and $x$, and $s$ and $y$ respectively. It is clear that

$$D_B\zeta(\beta_1,\beta_2)=\Delta_{(\beta_1,\beta_2)}D_B\zeta(x,y)+D_B\zeta(\beta_1,y)+D_B\zeta(x,\beta_2)-D_B\zeta(x,y).$$

Since $D_B\zeta \in B(I^2)$, by above relations, we can write

$$
\begin{aligned}
&|\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\Delta_{(t,s)}\zeta(x,y);q_{n_1},q_{n_2},x,y)| \\
&= |\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((t-x)(s-y)D_B\zeta(\beta_1,\beta_2);q_{n_1},q_{n_2},x,y)| \\
&\leq \tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(|t-x||s-y||\Delta_{(\beta_1,\beta_2)}D_B\zeta(x,y)|;q_{n_1},q_{n_2},x,y) \\
&+ \tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(|t-x||s-y|(|D_B\zeta(\beta_1,y)| \\
&+ |D_B\zeta(x,\beta_2)| + |D_B\zeta(x,y)|);q_{n_1},q_{n_2},x,y) \\
&\leq \tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(|t-x||s-y|\omega_{mixed}(D_B\zeta;|\beta_1-x|,|\beta_2-y|);q_{n_1},q_{n_2},x,y) \\
&+3||D_B\zeta||_\infty \tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(|t-x||s-y|;q_{n_1},q_{n_2},x,y).
\end{aligned}
$$

By the above inequality, using the linearity of $\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;.,.)$ and applying the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
&|\tilde{U}^{\alpha_1,\alpha_2}_{n_1,n_2}(\zeta;q_{n_1},q_{n_2},x,y)-\zeta(x,y)| = |\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}\Delta_{(t,s)}\zeta(x,y);q_{n_1},q_{n_2},x,y| \\
&\leq 3||D_B\zeta||_\infty \sqrt{\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((t-x)^2(s-y)^2;q_{n_1},q_{n_2},x,y)} \\
&+ \Bigg( \tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(|t-x||s-y|;q_{n_1},q_{n_2},x,y) \\
&+ \mu_{n_1}^{-1}\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((t-x)^2|s-y|;q_{n_1},q_{n_2},x,y) \\
&+ \mu_{n_2}^{-1}\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}(|t-x|(s-y)^2;q_{n_1},q_{n_2},x,y) \\
&+ \mu_{n_1}^{-1}\mu_{n_2}^{-1}\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((t-x)^2(s-y)^2;q_{n_1},q_{n_2},x,y)\Bigg)\omega_{mixed}(D_B\zeta;\mu_{n_1},\mu_{n_2}) \\
&\leq 3||D_B\zeta||_\infty \sqrt{\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((t-x)^2(s-y)^2;q_{n_1},q_{n_2},x,y)} \\
&+ \Bigg( \sqrt{\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((t-x)^2(s-y)^2;q_{n_1},q_{n_2},x,y)} \\
&+ \mu_{n_1}^{-1}\sqrt{\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((t-x)^4(s-y)^2;q_{n_1},q_{n_2},x,y)} \\
&+ \mu_{n_2}^{-1}\sqrt{\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((t-x)^2(s-y)^4;q_{n_1},q_{n_2},x,y)} \\
&+ \mu_{n_1}^{-1}\mu_{n_2}^{-1}\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((t-x)^2(s-y)^2;q_{n_1},q_{n_2},x,y)\Bigg)\omega_{mixed}(D_B\zeta;\mu_{n_1},\mu_{n_2}). \quad (8.2)
\end{aligned}
$$

We observe that for $(t,s),(x,y) \in I^2$ and $i,j = 1,2$

$$
\begin{aligned}
\tilde{K}^{\alpha_1,\alpha_2}_{n_1,n_2}((t-x)^{2i}(s-y)^{2j};q_{n_1},q_{n_2},x,y) &= \tilde{K}^{\alpha_1}_{n_1}((t-x)^{2i};q_{n_1},x,y)\tilde{K}^{\alpha_2}_{n_2}((s-y)^{2j};q_{n_2},x,y). \\
&\leq \frac{c_1}{[n_1]^i_{q_{n_1}}}\frac{c_2}{[n_2]^j_{q_{n_2}}}, \quad (8.3)
\end{aligned}
$$

for some constants $c_1, c_2 > 0$.

Replacing $\mu_{n_k}$ by $\left(\frac{1}{[n_k]_{q_{n_k}}}\right)^{\frac{1}{2}}, k = 1, 2$ in (8.2) and combining (8.2)-(8.3) we obtain the desired result. □

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## References

[1] Acu, A.M., Stancu-Schurer-Kantorovich operators based on $q$-integers, *Appl. Math. and Comput.*, 259, (2015), 896-907.
[2] Acu, AM, Sofonea, F, Barbosu, D., Note on a $q$-analogue of Stancu-Kantorovich operators, *Miskolc Mathematical Notes,* 16(1), (2015), 3-15.
[3] Acu, A.M., Manav, N., Sofonea, D.F., Approximation properties of $\lambda$-Kantorovich operators, *J. Inequal. Appl.,* (2018), 2018:202.
[4] Acu, A.M., Muraru, C.V., Sofonea, D.F., Radu, V.A.:, Some approximation properties of a Durrmeyer variant of $q$-Bernstein-Schurer operators, *Mathematical Methods in Applied Science*, 39(18), (2016), 5636-5650.
[5] Acu, A.M., Acar, T., Muraru, C.V., Radu, V.A., Some approximation properties by a class of bivariate operators, *Mathematical Methods in the Applied Sciences*, 42 (2019), 1-15, https://doi.org/10.1002/mma.5515
[6] Altomare, F., Cappelletti, M. M. and Leonessa, V., On a generalization of Szasz-Mirakjan-Kantorovich operators, *Results Math.* 63(3-4), (2013), 837-863.
[7] Aral, A., Gupta, V. and Agarwal, R. P., Applications of q-Calculus in Operator Theory, Springer, 2013.
[8] Badea, C., Badea, I. and Gonska, H. H., Notes on the degree of approximation of $B$-continuous and $B$-differentiable functions, *J. Approx. Theory Appl.* 4, (1988), 95-108.
[9] Başcanbaz-Tunca, G., Erençin, A., Ince-Ilarslan, H.G., Bivariate Cheney-Sharma operators on simplex, *Hacettepe Journal of Mathematics and Statistics,* 47(4), (2018), 793-804.
[10] Bögel, K., Mehrdimensionale Differentiation von Funtionen mehrerer Veränderlicher, *J. Reine Angew. Math.* 170, (1934), 197-217.
[11] Bögel, K., Über die mehrdimensionale differentiation, *Jahresber. Deutsch. Math.-Verein.*, 65 (1962), 45-71.
[12] Bögel, K., Über die mehrdimensionale differentiation, integration und beschränkte variation, *J. Reine Angew. Math.,* 173, (1935), 5-29
[13] Butzer, P.L., Berens, H., Semi-groups of Operators and Approximation, Springer, New York, 1967.
[14] Ditzian, Z. and Totik, V., Moduli of smoothness, volume 9 of Springer Series in Computational Mathematics, SpringerVerlag, New York, 1987.
[15] Gadjiev, A. D., Theorems of the type of P. P. Korovkin's theorems, *Mat. Zametki* 20 (5) (1976), 781-786. ((in Russian), Math. Notes 20(5-6) (1976), 995-998 (Engl. Trans.)).
[16] Gonska, H., Heilmann M. and Rasa I., Kantorovich operators of order k, *Numer. Funct. Anal. Optim.,* 32(7), (2011), 717-738.
[17] Gupta, V., Rassias, T. M., Agarwal, P.M. and Acu, A. M., Recent Advances in Constructive Approximation Theory, Springer, Cham, 2018.
[18] Kajla, A., Agrawal, P. N., Szász-Kantorovich type operators based on Charlier polynomials, *Kyungpook Math. J.,* 56(3), (2016), 877-897.

[19] Kajla, A., Goyal, M., Modified Bernstein-Kantorovich operators for functions of one and two variables, Rendiconti del Circolo Matematico di Palermo, (2017), DOI10.1007/s12215-017-0320-z.

[20] Lupas, A., A $q$-analogue of the Bernstein operators, Semianr on Numerical and Statistical Calculus, *University of Cluj-Napoca*, 9 (1987), 85-98.

[21] Lupaş, A., $q$-Analogues of Stancu operators, *Mathematical Analysis and Approximation Theory*, *Burg Verlag* , 2002, 145-154.

[22] Marchaud, A., Sur les Derivées et sur les Différences des Fonctions de Variables Réclles, *J. Math. Pures et Appl.* 6 (1927), 337-426.

[23] Muraru, C.V., Acu, A.M., Radu, V.A., On the monotonicity of $q$-Schurer-Stancu type polynomials, *Miskolc Mathematical Notes*, 19(1), (2018), 19-28.

[24] Özarslan, M. A., Duman, O., Smoothnesss properties of modified Bernstein-kantorovich operators, *Numer. Func. Anal. Opt.* 37 (1) (2016), 92-105.

[25] Phillips, G. M., Bernstein polynomials based on q-integers, *Ann. Numer. Math.,* 4 (1997), 511-518.

[26] Shisha, O. and Mond, P., The degree of convergence of linear positive operators, *Proc. Nat. Acad. Sci.* USA 60, (1968), 1196-1200.

[27] Sucu, S., Ibikli, E., Approximation by means of Kantorovich-Stancu type operators, *Numer. Funct. Anal. Optim.,* 34(5), (2013), 557-575.

[28] Thomae, J., Beiträge zue Theorie der durch die Heineshe Reihe., *J. Reine Angew Math.* 70, (1869), 258-281.

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