

Lightlike Submanifolds of a Semi-Riemannian Product Manifold with Quarter Symmetric Non-Metric Connection

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(Communicated by Cihan ÖZGÜR)

ABSTRACT

We study lightlike submanifolds of a semi-Riemannian product manifold. We introduce a class of lightlike submanifolds called screen semi-invariant lightlike submanifold. We consider lightlike submanifolds with respect to a quarter symmetric non-metric connection which is determined by the product structure. We give some equivalent conditions for integrability of distributions with respect to the Levi-Civita connection of semi-Riemannian manifolds and the quarter-symmetric nonmetric connection, and we obtain some results.

Keywords: Lightlike submanifolds, Product manifolds, Quarter symmetric connection.

AMS Subject Classification (2010): Primary 53C15, 53C25, 53C40.

1. Introduction

The theory of degenerate submanifolds of semi-Riemannian manifolds is one of a important topics of differential geometry because of the following reasons: It is well known that while the geometry of semi-Riemannian manifold is fully developed, its counter part of lightlike submanifolds (for which the local geometry is completely different from the non-degenerate case) is relatively new and in a developing stage. Second reason is that the growing importance of null geometry in mathematical physics and limited information on its geometric theory. Several papers have been written on lightlike submanifold in recent years in [3] (see also [4]) by K.L. Duggal and A. Bejancu. Differential Geometry of Lightlike Submanifolds was presented in [9] by K. L. Duggal and B. Sahin. In [5],[6], [7], [8], K. L. Duggal and B. Sahin introduced and studied geometry of classes of lightlike submanifolds in indefinite Kaehler and indefinite Sasakian manifolds which is an umbrella of CR-lightlike, SCR-lightlike, Screen real GCR-lightlike submanifolds. In [2], M. Atceken and E. Kilic introduced semi-invariant lightlike submanifolds of a semi-Riemannian product manifold. In [12], E. Kilic and B. Sahin introduced radical anti-invariant lightlike submanifolds of a semi-Riemannian product manifold and gave some examples and results for lightlike submanifolds. In [13] E. Kilic and O. Bahadır studied lightlike hypersurfaces of a semi-Riemannian product manifold with respect to quarter symmetric non-metric connection.

In [11], Hayden introduced a metric connection with nonzero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semi-symmetric (symmetric) and nonmetric connection have been studied by many authors ([15]-[14]). The idea of quarter-symmetric linear connections in a differential manifold was introduced by Golab [10]. A linear connection is said to be a quarter-symmetric connection if its torsion tensor \tilde{T} is of the form:

$$\tilde{T}(X, Y) = \pi(Y)\varphi X - \pi(X)\varphi Y \quad (1.1)$$

for any vector fields X, Y on a manifold, where π is a 1-form and φ is a tensor of type $(1, 1)$

In this paper, we study r -lightlike and coisotropic submanifolds of a semi-Riemannian product manifold. First, we introduce screen semi-invariant lightlike submanifolds of a semi-Riemannian product manifold. We

define some special distribution of screen semi-invariant lightlike submanifold. Then we give some examples and find their geometric properties of screen semi-invariant lightlike submanifold. Finally, by considering the quarter-symmetric non-metric connection, we study lightlike submanifolds of a semi-Riemannian product manifold. We obtain some results on lightlike submanifolds of a semi-Riemannian product manifold admitting the quarter symmetric non-metric connection.

2. Preliminaries

In this section, we give some notations and terminology, given by Duggal and Bejancu in [3] used throughout this paper.

Let $(\widetilde{M}, \widetilde{g})$ be a real $(m+n)$ -dimensional semi-Riemannian manifold of constant index such that $1 \leq \nu \leq m+n-1$ and (M, g) be an m -dimensional submanifold of \widetilde{M} . In case \widetilde{g} is degenerate on the tangent bundle TM of M , M is called a lightlike submanifold of \widetilde{M} . Denote by g the induced tensor field of \widetilde{g} on M and suppose g is degenerate. Then, for each tangent space $T_x M$ we consider

$$T_x M^\perp = \left\{ Y_x \in T_x \widetilde{M} \mid \widetilde{g}_x(Y_x, X_x) = 0, \forall X_x \in T_x M \right\}$$

which is a degenerate n -dimensional subspace of $T_x \widetilde{M}$. Thus, both $T_x M$ and $T_x M^\perp$ are degenerate orthogonal subspaces but no longer complementary subspaces. For this case, there exists a subspace $RadT_x M = T_x M \cap T_x M^\perp$ called *radical (null) subspace*. If the mapping

$$RadTM : x \in M \longrightarrow RadT_x M$$

defines a smooth distribution on M of rank $r > 0$, the submanifold M of \widetilde{M} is called r -lightlike (r -degenerate) submanifold and $RadTM$ is called the *radical (lightlike) distribution* on M . In the following, there are four possible cases:

Case 1. M is called a r -lightlike submanifold if $1 \leq r < \min\{m, n\}$.

Case 2. M is called a coisotropic submanifold if $1 < r = n < m$.

Case 3. M is called an isotropic submanifold if $1 < r = m < n$.

Case 4. M is called a totally lightlike submanifold if $1 < r = m = n$ [3].

In this paper, we have considered case 1 where there exists a non-degenerate screen distribution $S(TM)$ which is a complementary vector subbundle to $RadTM$ in TM . Therefore,

$$TM = RadTM \perp_s (TM). \tag{2.1}$$

in which \perp denotes orthogonal direct sum. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/RadTM$. Denote an r -lightlike submanifold by $(M, g, S(TM), S(TM^\perp))$, where $S(TM^\perp)$ is a complementary vector bundle of $RadTM$ in TM^\perp and $S(TM^\perp)$ is non-degenerate with respect to \widetilde{g} . Let us define that $tr(TM)$ is a complementary (but never orthogonal) vectors bundle to TM in $T\widetilde{M}|_M$ and

$$trTM = ltrTM \perp_s (TM^\perp), \tag{2.2}$$

where $ltr(TM)$ is an arbitrary lightlike transversal vector bundle of M . Then we have

$$\begin{aligned} T\widetilde{M}|_M &= TM \oplus tr(TM) \\ &= (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^\perp) \end{aligned} \tag{2.3}$$

where \oplus denotes direct sum, but it is not orthogonal [3].

The Gauss and Weingarten formulas given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.4}$$

$$\widetilde{\nabla}_X V = -A_V X + \nabla_X^t V, \quad \forall V \in \Gamma(trTM), \tag{2.5}$$

for any $X, Y \in \Gamma(TM)$, where $\{\nabla_X Y, A_V X\}$ belong to $\Gamma(TM)$ while $\{h(X, Y), \nabla_X^t V\}$ belong to $\Gamma(ltr(TM))$.

Suppose $S(TM^\perp) \neq 0$, that is, M is either in *Case 1* or in *Case 3*. According to the decomposition (2.3) we consider the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$, respectively. Then (2.4) – (2.5) become

$$\tilde{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \forall X, Y \in \Gamma(TM), \quad (2.6)$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^l N + D^s(X, N), \quad \forall N \in \Gamma(ltr(TM)), \quad (2.7)$$

$$\tilde{\nabla}_X W = -A_W X + \nabla_X^s W + D^l(X, W) \quad \forall W \in \Gamma(s(TM^\perp)). \quad (2.8)$$

where $h^l(X, Y) = Lh(X, Y)$, $h^s(X, Y) = Sh(X, Y)$, $\{\nabla_X^l N, D^l(X, W)\} \in \Gamma(ltr(TM))$, $\{\nabla_X^s W, D^s(X, N)\} \in \Gamma(s(TM^\perp))$ and $\{\nabla_X Y, A_N X, A_W X\} \in \Gamma(TM)$ [9]. Then, taking account of (2.6)-(2.8) and the Levi-Civita connection $\tilde{\nabla}$ is a metric, we obtain

$$\tilde{g}(h^s(X, Y), W) + g(Y, D^l(X, W)) = g(A_W X, Y), \quad (2.9)$$

$$\tilde{g}(D^s(X, N), W) = \tilde{g}(A_W X, N). \quad (2.10)$$

Let P be the projection of $S(TM)$ on M . Then according to (2.1) and (2.3) we have

$$\nabla_X PY = \nabla_X^* PY + h^*(X, PY), \quad (2.11)$$

$$\nabla_X \xi = -A_\xi^* X + \nabla_X^{*\xi} \xi, \quad (2.12)$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. By using above equations we obtain

$$g(h^l(X, Y), \xi) + g(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) = 0, \quad (2.13)$$

$$g(h^*(X, PY), N) = g(A_N X, PY), \quad (2.14)$$

$$g(h^l(X, PY), \xi) = g(A_\xi^* X, PY), \quad (2.15)$$

$$g(A_N X, PY) = g(N, \tilde{\nabla}_X PY), \quad (2.16)$$

$$g(h^l(X, \xi), \xi) = 0, \quad A_\xi^* \xi = 0, \quad (2.17)$$

In general, the induced connection ∇ on M is not metric connection. Since $\tilde{\nabla}$ is a metric connection, by using (2.6) we get

$$(\nabla_X g)(Y, Z) = \tilde{g}(h^l(X, Y), Z) + \tilde{g}(h^l(X, Z), Y), \quad (2.18)$$

However, it is important to note that ∇^* is a metric connection on $S(TM)$ [9].

3. Product manifolds

In this section, we give basic concept for product manifold.

Let M be an n - dimensional diferentiable manifold with a tensor field F of type $(1, 1)$ on \tilde{M} such that

$$F^2 = I. \quad (3.1)$$

Then M is called an almost product manifold with almost product structure F . If we put

$$\pi = \frac{1}{2}(I + F), \quad \sigma = \frac{1}{2}(I - F)$$

then we have

$$\pi + \sigma = I, \quad \pi^2 = \pi, \quad \sigma^2 = \sigma, \quad \pi\sigma = \sigma\pi = 0, \quad F = \pi - \sigma.$$

Thus π and σ define two complementary distributions and F has the eigenvalue of $+1$ or -1 . If an almost product manifold \tilde{M} admits a semi-Riemannian metric \tilde{g} such that

$$g(FX, FY) = g(X, Y), \quad (3.2)$$

for any X, Y vector fields on \tilde{M} , then \tilde{M} is called a semi-Riemannian almost product manifold. From (3.1) and (3.2), we have

$$\tilde{\nabla}_X FY = F\tilde{\nabla}_X Y, \quad (3.3)$$

If, for any vector fields X, Y on \tilde{M} ,

$$\tilde{\nabla} F = 0, \quad \text{that is } \tilde{\nabla}_X FY = F\tilde{\nabla}_X Y, \quad (3.4)$$

then \tilde{M} is called a semi-Riemannian product manifold, where $\tilde{\nabla}$ is the Levi-Civita connection on \tilde{M} [16].

4. Lightlike Submanifolds of Semi-Riemannian Product Manifolds

In this section we defined screen semi-invariant lightlike submanifolds of a semi-Riemannian product manifold. We give some examples and study their geometric properties.

Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g}, F)$. For each X tangent to M , FX can be written as follows:

$$FX = fX + wX, \tag{4.1}$$

where fX and wX are the tangential and the transversal parts of FX , respectively. In addition, for any $V \in \Gamma(tr(TM))$, FV can be written as:

$$FV = BV + CV, \tag{4.2}$$

where BV and CV are the tangential and the transversal parts of FV , respectively .

Definition 4.1. Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g}, F)$. If $FRadTM \subset s(TM)$, $FltrTM \subset s(TM)$ and $Fs(TM^\perp) \subset s(TM)$ then we say that M is a screen semi-invariant lightlike submanifold.

Now, let M be a screen semi-invariant lightlike submanifold of a semi-Riemannian product manifold. If we set $L_1 = FRadTM$, $L_2 = FltrTM$ and $L_3 = Fs(TM^\perp)$ then we can write

$$s(TM) = L_0 \perp \{L_1 \oplus L_2\} \perp L_3, \tag{4.3}$$

where L_0 is a $(m - r - 4)$ - dimensional distribution. Hence we have the following decomposition:

$$TM = \{L_1 \oplus L_2\} \perp L_3 \perp L_0 \perp RadTM, \tag{4.4}$$

$$\widetilde{TM} = \{L_1 \oplus L_2\} \perp L_3 \perp L_0 \perp \{RadTM \oplus ltrTM\}. \tag{4.5}$$

According to this definition we can write

$$L = L_0 \perp L_1 \perp RadTM \tag{4.6}$$

and

$$L' = L_2 \perp L_3. \tag{4.7}$$

Thus we have

$$TM = L \oplus L' \tag{4.8}$$

From (4.7) and (4.8) we have the following proposition

Proposition 4.1. Let $(M, g, S(TM), S(TM^\perp))$ be a screen semi-invariant lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g}, F)$. Then L and L' are F -invariant and anti-invariant distributions, respectively.

For Case (2), we know that $s(TM^\perp) = \{0\}$. Therefore we have the following decomposition

$$s(TM) = \{L_1 \oplus L_2\} \perp L_0, \tag{4.9}$$

$$TM = \{L_1 \oplus L_2\} \perp L_0 \perp RadTM, \tag{4.10}$$

$$\widetilde{TM} = \{L_1 \oplus L_2\} \perp L_0 \perp \{RadTM \oplus ltrTM\}, \tag{4.11}$$

$$TM = L \oplus L_2. \tag{4.12}$$

Proposition 4.2. Let $(M, g, S(TM), S(TM^\perp))$ be a screen semi-invariant lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g}, F)$. The distribution L_0 is a invariant distribution with respect to F .

Example 4.1. Let M be a submanifold of semi- Riemannian product manifold $\widetilde{M} = R_2^5 \times R_1^3$ with metric tensor $\widetilde{g} = \pi^*g_1 + \sigma^*g_2$, where g_1 and g_2 are standard metric tensors of R_2^5 and R_1^3 , respectively. Consider the

submanifold M in \widetilde{M} defined by the equations:

$$\begin{aligned} x_1 &= t_1 + t_2 + 2t_3 + 2t_4 + (1 + 2\sqrt{2})t_5 - 3t_6, \\ x_2 &= 2t_1 + 2t_2 + t_3 + (2 - \sqrt{2})t_5 + (2 - \sqrt{2})t_6, \\ x_3 &= t_3 + t_4 + 3\sqrt{2}t_5 - 6t_6, \\ x_4 &= t_1 + t_2 + 2t_3 + \sqrt{2}t_4 + t_5 + t_6, \\ x_5 &= 2t_1 + 2t_2 + t_3 + t_4 + 2t_5 - \sqrt{2}t_6, \\ x_6 &= \sqrt{2}t_1 - \sqrt{2}t_2 - \sqrt{2}t_3 - 3t_4, \\ x_7 &= -\sqrt{2}t_3 - t_4, \\ x_8 &= 2t_1 - 2t_2 - t_3 - 2\sqrt{2}t_4, \end{aligned}$$

where $t_i, 1 \leq i \leq 4$, are real parameters. Then the tangent bundle TM is spanned by $\{U_1, U_2, U_3, U_4, U_5, U_6\}$, where

$$\begin{aligned} U_1 &= \frac{\partial}{\partial x_1} + 2\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4} + 2\frac{\partial}{\partial x_5} + \sqrt{2}\frac{\partial}{\partial x_6} + 2\frac{\partial}{\partial x_8}, \\ U_2 &= \frac{\partial}{\partial x_1} + 2\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4} + 2\frac{\partial}{\partial x_5} - \sqrt{2}\frac{\partial}{\partial x_6} - 2\frac{\partial}{\partial x_8}, \\ U_3 &= 2\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + 2\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} - \sqrt{2}\frac{\partial}{\partial x_6} - \sqrt{2}\frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_8}, \\ U_4 &= 2\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + \sqrt{2}\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} - 3\frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_7} - 2\sqrt{2}\frac{\partial}{\partial x_8}, \\ U_5 &= (1 + 2\sqrt{2})\frac{\partial}{\partial x_1} + (2 - \sqrt{2})\frac{\partial}{\partial x_2} + 3\sqrt{2}\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + 2\frac{\partial}{\partial x_5}, \\ U_6 &= -3\frac{\partial}{\partial x_1} + (2 - \sqrt{2})\frac{\partial}{\partial x_2} - 6\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} - \sqrt{2}\frac{\partial}{\partial x_5}. \end{aligned}$$

It is easy to check that M is a lightlike submanifold and U_1 is a degenerate vector. Then we have $Rad TM = Span\{U_1\}$ and $S(TM) = Span\{U_2, U_3, U_4, U_5, U_6\}$. By direct calculations we obtain

$$ltrTM = Span\{N = 2\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + \sqrt{2}\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + 3\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7} + 2\sqrt{2}\frac{\partial}{\partial x_8}\},$$

and

$$s(TM^\perp) = Span\{u = 2\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + 2\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \sqrt{2}\frac{\partial}{\partial x_6} + \sqrt{2}\frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8}\}.$$

Hence M is a 6– dimensional 1– lightlike submanifold. Furthermore, we get

$$\begin{aligned} F\xi &= U_2 \in \Gamma(s(TM)), \quad FN = U_4 \in \Gamma(s(TM)), \\ Fu &= U_3 \in \Gamma(s(TM)), \quad FU_5 = U_5, \quad FU_6 = U_6. \end{aligned}$$

Thus we have

$$\begin{aligned} L_0 &= Span\{U_5, U_6\}, \quad L_1 = Span\{U_2\}, \\ L_2 &= Span\{U_4\}, \quad L_3 = Span\{U_3\}. \end{aligned}$$

Therefore M is screen semi-invariant lightlike submanifold of \widetilde{M} .

Example 4.2. Let $(\widetilde{M} = R_2^5, \widetilde{g})$ be a 5-dimensional semi-Euclidean space with signature $(-, -, +, +, +)$ and $(x_1, x_2, x_3, x_4, x_5)$ be the standard coordinate system of R_2^5 . If we set

$$F(x_1, x_2, x_3, x_4, x_5) = (x_2, x_1, x_4, x_3, x_5),$$

then $F^2 = I$ and F is a product structure on R_2^5 . Consider a submanifold M in \widetilde{M} by the equations:

$$\begin{aligned} x_1 &= 2t_1 + t_2 + t_3 - \arcsin t_4, \\ x_2 &= t_1 + 2t_2 + 2t_3 + \arcsin t_4, \\ x_3 &= -t_2 + t_3 + 2\arcsin t_4, \\ x_4 &= -t_1, \\ x_5 &= 2t_1 + 2t_2 + 2t_3 - \frac{1}{2}\arcsin t_4. \end{aligned}$$

where $t_i, 1 \leq i \leq 4$, are real parameters. Then the tangent bundle TM is spanned by $\{U_1, U_2, U_3, U_4\}$, where

$$\begin{aligned} U_1 &= 2\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_4} + 2\frac{\partial}{\partial x_5}, \\ U_2 &= \frac{\partial}{\partial x_1} + 2\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + 2\frac{\partial}{\partial x_5}, \\ U_3 &= \frac{\partial}{\partial x_1} + 2\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + 2\frac{\partial}{\partial x_5}, \\ U_4 &= -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3} - \frac{1}{2}\frac{\partial}{\partial x_5}, \end{aligned}$$

It is easy to check that M is a 1- lightlike submanifold and U_1 is a degenerate vector. Then we have $RadTM = Span\{\xi\}$ and $S(TM) = Span\{U_2, U_3, U_4\}$. Then by direct calculations we get

$$ltrTM = Sp\{N = 2\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4} + 2\frac{\partial}{\partial x_5}\}.$$

Moreover we can write $L_0 = Sp\{U_4\}, L_1 = Sp\{U_2\}, L_2 = Sp\{U_3\}$. Thus M is a screen semi-invariant coisotropic submanifold of \widetilde{M} .

Now, let $(M, g, S(TM), S(TM^\perp))$ be a screen semi-invariant lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g}, F)$. For any $X, Y \in \Gamma(TM)$ we obtain

$$\widetilde{\nabla}_X FY = \nabla_X fY + h(X, fY) - A_{wY}X + \nabla_X^t wY \tag{4.13}$$

and

$$F\widetilde{\nabla}_X Y = f\nabla_X Y + w\nabla_X Y + Bh(X, Y). \tag{4.14}$$

Since F is parallel on M , from (4.13) and (4.14) we get

$$(\nabla_X f)Y = A_{wY}X + Bh(X, Y), \tag{4.15}$$

$$w\nabla_X Y = \nabla_X^t wY + h(X, fY). \tag{4.16}$$

Theorem 4.1. Let $(M, g, S(TM), S(TM^\perp))$ be a screen semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g}, F)$. Then the following assertions are equivalent:

- (i) M is mixed geodesic.
- (ii) A_ξ^*X and $A_W X$ have no component in L_1 and L_3 , for any $X \in \Gamma(L)$.
- (iii) $A_{FY}X$ has no component L_2 and L_3 , for any $X \in \Gamma(L)$ and $Y \in \Gamma(L')$.

Proof. M is mixed geodesic if and only if

$$g(h(X, Y), \xi) = 0 \text{ and } g(h(X, Y), W) = 0,$$

for any $N \in \Gamma(ltrTM), W \in \Gamma(s(TM^\perp)), X \in \Gamma(L)$ and $Y \in \Gamma(L')$. From (2.4), (2.7), (2.8) and (2.12) we obtain

$$g(h(X, Y), \xi) = g(Y, A_\xi^*X)$$

and

$$g(h(X, Y), W) = g(Y, A_W X).$$

Thus we get (i) \Leftrightarrow (ii). Similarly from (2.4), (2.5), (3.3) and (3.4) we obtain

$$g(h(X, Y), \xi) = -g(A_{FY}X, F\xi)$$

and

$$g(h(X, Y), W) = -g(A_{FY}X, FW).$$

Therefore we conclude (i) \Leftrightarrow (iii). Thus we have our assertion □

If M is coisotropic submanifold, that is $S(TM^\perp) = 0$, then from (2.11) we have the following theorem.

Theorem 4.2. Let $(M, g, S(TM))$ be a screen semi-invariant coisotropic submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g}, F)$. Then the following assertions are equivalent:

- (i) M is mixed geodesic.
- (ii) A_{ξ}^*X have no component in L_1 , for any $X \in \Gamma(L)$.
- (iii) $h^*(X, F\xi) = 0$, for any $X \in \Gamma(L)$.

Theorem 4.3. Let $(M, g, S(TM), S(TM^{\perp}))$ be a screen semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g}, F)$. Then the following assertions are equivalent:

- (i) M is L -geodesic.
- (ii) $A_{\xi}^*X \in \Gamma(L_1 \perp L_3)$ and $g(Y, A_W X) = g(Y, D^l(X, W))$, for any $X, Y \in \Gamma(L)$
- (iii) $\nabla_X FY$ have no component in L_2 and L_3 , for any $X, Y \in \Gamma(L)$.

Proof. It is known that M is L -geodesic if and only if

$$g(h(X, Y), \xi) = 0 \text{ and } g(h(X, Y), W) = 0,$$

for any $X, Y \in \Gamma(L)$, $N \in \Gamma(\text{ltr}TM)$, $W \in \Gamma(s(TM^{\perp}))$. Accordingly from (2.4), (2.7), (2.8) and (2.12) we obtain

$$g(h(X, Y), \xi) = g(Y, A_{\xi}^*X)$$

and

$$g(h(X, Y), W) = g(Y, A_W X - D^l(X, W)).$$

Thus we get (i) \Leftrightarrow (ii). Similarly from (2.4), (2.5), (3.3) and (3.4) we have

$$g(h(X, Y), \xi) = -g(\nabla_X FY, F\xi)$$

and

$$g(h(X, Y), W) = -g(\nabla_X FY, FW).$$

This is (i) \Leftrightarrow (iii). □

From above the theorem, we have the following theorem.

Theorem 4.4. Let $(M, g, S(TM))$ be a screen semi-invariant coisotropic submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g}, F)$. Then the following assertions are equivalent:

- (i) M is L -geodesic.
- (ii) $A_{\xi}^*X \in \Gamma(L_1)$, for any $X \in \Gamma(L)$.
- (iii) $\nabla_X^* F\xi \in \Gamma(L_1)$, for any $X \in \Gamma(L)$.

Theorem 4.5. Let $(M, g, S(TM), S(TM^{\perp}))$ be a screen semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g}, F)$. Then the following assertions are equivalent:

- (i) M, L' - geodeziktir.
- (ii) $A_W X$ and A_{ξ}^*X have no component in L_1 and L_3 , for any $X \in \Gamma(L')$.
- (iii) $A_{FY} X$ have no component in L_2 and L_3 , for any $X, Y \in \Gamma(L')$.

Proof. For any $N \in \Gamma(\text{ltr}TM)$, $W \in \Gamma(s(TM^{\perp}))$ and $X, Y \in \Gamma(L)$, by using (2.12) and Gauss- Weingarten formulas we obtain

$$g(h(X, Y), \xi) = g(Y, A_{\xi}^*X),$$

and

$$g(h(X, Y), W) = g(Y, A_W X).$$

Thus we get (i) \Leftrightarrow (ii). Moreover from (3.2), (3.3) and Gauss-Weingarten formulas we get

$$g(h(X, Y), \xi) = -g(A_{FY} X, F\xi)$$

and

$$g(h(X, Y), W) = -g(A_{FY} X, FW).$$

This is (i) \Leftrightarrow (iii). □

Theorem 4.6. Let $(M, g, S(TM), S(TM^\perp))$ be a screen semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g}, F)$. Then the following assertions are equivalent:

- (i) L is parallel distribution.
- (ii) $h(X, FY) = 0, X, Y \in \Gamma(L)$.
- (iii) $(\nabla_X f)Y = 0$, for any $X, Y \in \Gamma(L)$.

Proof. For any $N \in \Gamma(\text{ltr}TM), W \in \Gamma(s(TM^\perp))$ and $X, Y \in \Gamma(L)$ by using Gauss-Weingarten formulas we have

$$g(\nabla_X Y, F\xi) = -g(h^l(X, FY), \xi),$$

and

$$g(\nabla_X Y, FW) = -g(h^s(X, FY), W).$$

Thus we get (i) \Leftrightarrow (ii). Since F is parallel, from (2.4), (4.1) we obtain

$$h(X, FY) = -(\nabla_X f)Y + w\nabla_X Y + Fh(X, Y)$$

If we take tangential and transversal parts of this last equation we have

$$(\nabla_X f)Y = Fh(X, Y).$$

Thus we get (ii) \Leftrightarrow (iii). □

Proposition 4.3. Let $(M, g, S(TM), S(TM^\perp))$ be a screen semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g}, F)$. Then the distribution L' is parallel on M if and only if $A_{FY}X \in \Gamma(L')$.

Proof. For any $X \in \Gamma(TM), Y \in \Gamma(L')$ and $Z \in \Gamma(L_0)$, from (3.2), (3.3) and Gauss-Weingarten formulas we obtain

$$g(\nabla_X Y, N) = g(\widetilde{\nabla}_X Y, N) = g(\widetilde{\nabla}_X FY, FN) = -g(A_{FY}X, FN), \tag{4.17}$$

$$g(\nabla_X Y, FN) = g(\widetilde{\nabla}_X FY, N) = -g(A_{FY}X, N), \tag{4.18}$$

$$g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X Y, Z) = g(\widetilde{\nabla}_X FY, FZ) = g(A_{FY}X, FZ), \tag{4.19}$$

From (4.17), (4.18) and (4.19), proof is completed. □

Theorem 4.7. Let $(M, g, S(TM), S(TM^\perp))$ be a screen semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g}, F)$. Then the following assertions are equivalent:

- (i) L is integrable.
- (ii) $h(X, FY) = h(Y, FX), X, Y \in \Gamma(L)$.
- (iii) $(\nabla_X f)Y = (\nabla_Y f)X, X, Y \in \Gamma(L)$.

Proof. For any $X, Y \in \Gamma(L)$, from (2.4) we obtain

$$g([X, Y], F\xi) = g(h(X, FY) - h(Y, FX), \xi).$$

Thus we get (i) \Leftrightarrow (ii). From (2.4) and (4.1) we obtain

$$h(X, FY) = F\nabla_X Y + Fh(X, Y) - \nabla_X FY. \tag{4.20}$$

Interchanging role of X and Y we have

$$h(Y, FX) = F\nabla_Y X + Fh(Y, X) - \nabla_Y FX. \tag{4.21}$$

From (4.20) and (4.21) we get

$$h(X, FY) - h(Y, FX) = -(\nabla_X f)Y + (\nabla_Y f)X + w([X, Y])$$

If we take tangential and transversal parts of this last equation we have

(ii) \Leftrightarrow (iii). □

Proposition 4.4. *Let $(M, g, S(TM), S(TM^\perp))$ be a screen semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g}, F)$. Then the distribution L' is integrable if and only if*

$$A_{FX}Y = A_{FY}X, \quad X, Y \in \Gamma(L'),$$

Proof. For any $N \in \Gamma(\text{ltr}TM)$, $Z \in \Gamma(L_0)$ and $X, Y \in \Gamma(L')$, from (2.5) we obtain

$$g([X, Y], FN) = g(A_{FX}Y - A_{FY}X, N), \tag{4.22}$$

$$g([X, Y], Z) = g(A_{FX}Y - A_{FY}X, FZ). \tag{4.23}$$

nnd

$$g([X, Y], Z) = g(A_{FX}Y - A_{FY}X, FZ). \tag{4.24}$$

From (4.22) and (4.24) we get the proof. □

Theorem 4.8. *Let $(M, g, S(TM), S(TM^\perp))$ be a screen semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g}, F)$. Then M is a locally product manifold according to the decomposition (4.8) if and only if f is parallel with respect to induced connection ∇ , that is $\nabla f = 0$.*

Proof. Let M be a locally product manifold. Then the leaves of distributions L and L' are both totally geodesic in M . Since the distribution L is invariant with respect to F then, for any $Y \in \Gamma(L)$, $FY \in \Gamma(L)$. Thus $\nabla_X Y$ and $\nabla_X fY$ belong to $\Gamma(L)$. We know that $\widetilde{\nabla}F = 0$. Then from the Gauss formula we obtain

$$\nabla_X fY + h(X, fY) = f\nabla_X Y + w\nabla_X Y + Fh(X, Y). \tag{4.25}$$

Comparing the tangential and normal parts with respect to L of (4.25), we have

$$(\nabla_X f)Y = 0. \tag{4.26}$$

For any $X \in \Gamma(TM)$ and $Z \in \Gamma(L')$, since $fZ = 0$, we get $\nabla_X fZ = 0$ and $f\nabla_X Z = 0$, that is $(\nabla_X f)Z = 0$. Thus we have $\nabla f = 0$ on M .

Conversely, we assume that $\nabla f = 0$ on M . Then we have $\nabla_X fY = f\nabla_X Y$, for any $X, Y \in \Gamma(L)$ and $\nabla_W fU = f\nabla_W U = 0$, for any $U, W \in \Gamma(L')$. Hence, the leaves of the distributions L and L' are totally geodesic in M . □

5. Lightlike Submanifolds of Semi-Riemannian Product Manifolds with Quarter Symmetric Non-Metric Connection

In this section we investigated screen semi-invariant lightlike submanifolds of a semi-Riemannian product manifold with quarter symmetric non-metric connection and we study their geometric properties.

Let (M, g, F) be a semi-Riemannian product manifold and $\widetilde{\nabla}$ be the Levi-Civita connection on M . If we set

$$\widetilde{D}_X Y = \widetilde{\nabla}_X Y + \pi(Y)FX \tag{5.1}$$

for any $X, Y \in \Gamma(T\widetilde{M})$, then \widetilde{D} is a linear connection on \widetilde{M} , where u is a 1-form on \widetilde{M} with U as associated vector field, that is

$$\pi(X) = \widetilde{g}(X, U).$$

The torsion tensor of \widetilde{D} on \widetilde{M} denoted by \widetilde{T} . Then we obtain

$$\widetilde{T}(X, Y) = \pi(Y)FX - \pi(X)FY, \tag{5.2}$$

and

$$(\widetilde{D}_X \widetilde{g})(Y, Z) = -\pi(Y)\widetilde{g}(FX, Z) - \pi(Z)\widetilde{g}(FX, Y), \tag{5.3}$$

for any $X, Y \in \Gamma(T\widetilde{M})$. Thus \widetilde{D} is a quarter-symmetric non-metric connection on \widetilde{M} . From (5.1) we have

$$(\widetilde{D}_X F)Y = \pi(FY)FX - \pi(Y)X. \tag{5.4}$$

Replacing X by FX and Y by FY in (5.4) we obtain

$$(\tilde{D}_{FX}F)FY = \pi(Y)X - \pi(FY)FX. \quad (5.5)$$

Thus we have

$$(\tilde{D}_XF)Y + (\overline{D}_{FX}F)FY = 0. \quad (5.6)$$

If we set

$$'F(X, Y) = \tilde{g}(FX, Y) \quad (5.7)$$

for any $X, Y \in \Gamma(T\overline{M})$, from (5.1) we get

$$(\tilde{D}_X 'F)(Y, Z) = (\tilde{\nabla}_X 'F)(Y, Z) - \pi(Y)\tilde{g}(X, Z) - \pi(Z)\tilde{g}(X, Y). \quad (5.8)$$

From (5.1) the curvature tensor \tilde{R}^D of the quarter-symmetric non-metric connection \tilde{D} is given by

$$\tilde{R}^D(X, Y)Z = \tilde{R}(X, Y)Z + \tilde{\lambda}(X, Z)FY - \tilde{\lambda}(Y, Z)FX, \quad (5.9)$$

for any $X, Y, Z \in \Gamma(T\overline{M})$, where $\tilde{\lambda}$ is a $(0, 2)$ -tensor given by $\tilde{\lambda}(X, Z) = (\tilde{\nabla}_X \pi)(Z) - \pi(Z)\pi(FX)$. If we set $\tilde{R}^D(X, Y, Z, W) = \tilde{g}(\tilde{R}^D(X, Y)Z, W)$, then, from (5.9), we obtain

$$\tilde{R}^D(X, Y, Z, W) = -\tilde{R}^D(Y, X, Z, W).$$

We note that the Riemannian curvature tensor \tilde{R}^D of \tilde{D} does not satisfy the other curvature-like properties. But, from (5.9), we have

$$\begin{aligned} \tilde{R}^D(X, Y)Z + \tilde{R}^D(Y, Z)X + \tilde{R}^D(Z, X)Y &= (\tilde{\lambda}(Z, Y) - \tilde{\lambda}(Y, Z))FX \\ &+ (\tilde{\lambda}(X, Z) - \tilde{\lambda}(Z, X))FY \\ &+ (\tilde{\lambda}(Y, X) - \tilde{\lambda}(X, Y))FZ. \end{aligned}$$

Thus we have the following proposition.

Proposition 5.1. *Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of a semi-Riemannian product manifold $(\overline{M}, \tilde{g})$. Then the first Bianchi identity of the quarter symmetric non-metric connection \tilde{D} on M is provided if and only if $\tilde{\lambda}$ is symmetric.*

Let M be a lightlike submanifold of a semi-Riemannian product manifold $(\overline{M}, \tilde{g})$ with quarter symmetric non-metric connection \tilde{D} . Then the Gauss and Weingarten formulas with respect to \tilde{D} are given by, respectively,

$$\tilde{D}_X Y = D_X Y + \tilde{h}(X, Y), \quad X, Y \in \Gamma(TM) \quad (5.10)$$

$$\tilde{D}_X N = -\tilde{A}_N X + \tilde{\nabla}_X^l N + \tilde{D}^s(X, N), \quad N \in \Gamma(ltrTM), \quad (5.11)$$

and

$$\tilde{D}_X W = -\tilde{A}_W X + \tilde{\nabla}_X^s W + \tilde{D}^l(X, W), \quad W \in \Gamma(s(TM^\perp)) \quad (5.12)$$

for any $X, Y \in \Gamma(TM)$, where $D_X Y$, $\tilde{A}_N X$, $\tilde{A}_u X \in \Gamma(TM)$, and $\tilde{\nabla}^l$ and $\tilde{\nabla}^s$ are linear connections on $ltrTM$ and $s(TM^\perp)$, respectively. Both \tilde{A}_N and \tilde{A}_u are linear operators on $\Gamma(TM)$. From (5.1), (5.10), (5.11) and (5.12) we obtain

$$D_X Y = \nabla_X Y + \pi(Y)fX, \quad (5.13)$$

$$\tilde{h}(X, Y) = h(X, Y) + \pi(Y)wX, \quad (5.14)$$

$$\tilde{A}_N X = A_N X - \pi(N)fX, \quad (5.15)$$

$$\tilde{\nabla}_X^l N = \nabla_X^l N + \pi(N)w_l X, \quad (5.16)$$

$$\tilde{D}^s(X, N) = D^s(X, N) + \pi(N)w_s X, \quad (5.17)$$

and

$$\tilde{A}_W X = A_W X - \pi(W)fX, \tag{5.18}$$

$$\tilde{\nabla}_X^s W = \nabla_X^s W + \pi(W)w_s X, \tag{5.19}$$

$$\tilde{D}^l(X, W) = D^l(X, W) + \pi(W)w_l X, \tag{5.20}$$

for any $X, Y \in \Gamma(TM)$. In where w_s and w_l are projections on $ltrTM$ and $s(TM^\perp)$, respectively. From (2.18) and (5.13) we get

$$(D_X g)(Y, Z) = g(h^l(X, Y), Z) + g(h^l(X, Z), Y) - \pi(Y)g(fX, Z) - \pi(Z)g(fX, Y), \tag{5.21}$$

On the other hand, the torsion tensor of the induced connection D is

$$T^D(X, Y) = \pi(Y)fX - \pi(X)fY. \tag{5.22}$$

From last two equations we have the following proposition.

Proposition 5.2. *Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of a semi-Riemannian product manifold (\tilde{M}, \tilde{g}) with quarter symmetric non-metric connection \tilde{D} . Then the induced connection D on the lightlike submanifold M is a quarter-symmetric non-metric connection.*

With analysis of (2), we can write

$$D_X PY = D_X^* PY + \tilde{h}^*(X, PY), \quad X, Y \in \Gamma(TM) \tag{5.23}$$

$$D_X \xi = -\tilde{A}_\xi^* X + \tilde{\nabla}_X^{*t} \xi \tag{5.24}$$

for any $X, Y \in \Gamma(TM)$, where $D_X^* PY, \tilde{A}_\xi^* X \in \Gamma(s(TM))$ and $\tilde{h}^*(X, PY) \in \Gamma(RadTM)$. From (2.11), (2.12), (5.13), (5.23) and (5.24) we obtain

$$D_X^* PY = \nabla_X^* PY + \pi(PY)PfX, \tag{5.25}$$

$$\tilde{h}^*(X, PY) = h^*(X, PY) + \pi(PY) \sum_i^r \eta_i(fX)\xi_i, \tag{5.26}$$

$$\tilde{A}_\xi^* X = A_\xi^* X - \pi(\xi)PfX, \tag{5.27}$$

$$\tilde{\nabla}_X^{*t} \xi = \nabla_X^{*t} \xi + \pi(\xi) \sum_i^r \eta_i(fX)\xi, \tag{5.28}$$

where $D_X^* PY, \tilde{A}_\xi^* X \in \Gamma(S(TM)), \eta_i(X) = g(X, N_i)$, and $\{\xi_1, \dots, \xi_r\}$ is basis of $\Gamma(RadTM)$ for $i \in \{1, \dots, r\}$.

From (5.14), (5.27) and (5.15), (5.26) we have

$$\begin{aligned} g(\tilde{h}^l(X, PY), \xi) &= g(h^l(X, PY), \xi) + \pi(Y)g(w_l X, \xi) \\ &= g(A_\xi^* X, PY) + \pi(Y)g(w_l X, \xi) \\ &= g(\tilde{A}_\xi^* X, PY) + \pi(\xi)g(PfX, PY) + \pi(Y)g(w_l X, \xi), \end{aligned} \tag{5.29}$$

and

$$\begin{aligned} g(\tilde{h}^*(X, PY), N) &= g(h^*(X, PY), N) + \pi(PY)\eta(fX) \\ &= g(A_N X, PY) + \pi(PY)\eta(fX) \\ &= g(\tilde{A}_N X, PY) + \pi(N)g(fX, PY) + \pi(PY)\eta(fX), \end{aligned} \tag{5.30}$$

Also, from (5.27) we obtain

$$\begin{aligned} g(\tilde{A}_\xi^* PX, PY) &= g(A_\xi^* PX, PY) - \pi(\xi)g(PfX, PY) \\ &= g(A_\xi^* PY, PX) - \pi(\xi)g(PfX, PY), \end{aligned}$$

and

$$g(\tilde{A}_\xi^*PY, PX) = g(A_\xi^*PY, PX) - \pi(\xi)g(PfY, PX),$$

Then from the last two equation and (5.27), we obtain

$$g(\tilde{A}_\xi^*PX, PY) - g(\tilde{A}_\xi^*PY, PX) = \pi(\xi)g(PfY, PX) - \pi(\xi)g(PfX, PY), \tag{5.31}$$

and

$$\tilde{A}_\xi^*\xi = -\pi(\xi)f\xi, \tag{5.32}$$

Lemma 5.1. *Let $(M, g, S(TM), S(TM^\perp))$ be a semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold (\tilde{M}, \tilde{g}) . Then*

$$g(\tilde{h}^l(X, Y), \xi) = g(A_\xi^*X, Y) = g(\tilde{A}_\xi^*X, Y) + \pi(\xi)g(PfX, Y), \quad \xi \in \Gamma(RadTM), \tag{5.33}$$

for any $X \in \Gamma(L)$ and $Y \in \Gamma(L')$

Proof. For any $X \in \Gamma(L)$ and $Y \in \Gamma(L')$ from (5.14) we get $\tilde{h}^l(X, Y) = h^l(X, Y)$. Thus from (5.27) we have

$$g(\tilde{h}^l(X, Y), \xi) = g(h^l(X, Y), \xi) = g(A_\xi^*X, Y) = g(\tilde{A}_\xi^*X, Y) + \pi(\xi)g(PfX, Y).$$

□

From Lemma(5.1) and Teorem(4.1), we have the following Corollary

Corollary 5.1. *Let $(M, g, S(TM), S(TM^\perp))$ be a semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold (\tilde{M}, \tilde{g}) . The following assertions are equivalent:*

- (i) M is mixed geodesic.
- (ii) M is mixed geodesic with respect to quarter symmetric non-metric connection.
- (iii) A_ξ^*X and A_WX have no component in L_1 and L_3 , for any $X \in \Gamma(L)$ and $Y \in \Gamma(L')$.
- (iv) $A_{FY}X$ have no component L_2 and L_3 , for any $X \in \Gamma(L)$ and $Y \in \Gamma(L')$.

Theorem 5.1. *Let $(M, g, S(TM), S(TM^\perp))$ be a semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold (\tilde{M}, \tilde{g}) . For any $X, Y \in \Gamma(L)$ the following assertions are equivalent:*

- (i) M is L - geodesic with respect to quarter symmetric non-metric connection.
- (ii) $A_\xi^*X \in \Gamma(L_1 \perp L_3)$ and $g(Y, A_WX) = g(Y, D^l(X, W))$.
- (iii) $\nabla_X FY$ have no component L_2 and L_3 .
- (iv) M is L - geodesic.

Proof. For any $X, Y \in \Gamma(L)$, we know that

$$\tilde{h}(X, Y) = h(X, Y).$$

Thus from Teorem(4.3) we have the proof. □

Theorem 5.2. *Let $(M, g, S(TM), S(TM^\perp))$ be a semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold (\tilde{M}, \tilde{g}) . M is L' - geodesic with respect to quarter symmetric non-metric connection if and only if*

$$A_{FY}X = -\pi(Y)X, \quad X, Y \in \Gamma(L')$$

Proof. For any $X, Y \in \Gamma(L')$ and $W \in \Gamma(s(TM^\perp))$ we obtain

$$g(\tilde{h}(X, Y), W) = -g(A_{FY}X + \pi(Y)X, FW),$$

and

$$g(\tilde{h}(X, Y), \xi) = -g(A_{FY}X + \pi(Y)X, F\xi),$$

From the last two equations we get the proof. □

Theorem 5.3. Let $(M, g, S(TM), S(TM^\perp))$ be a semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. The distribution L is parallel with respect to quarter symmetric non-metric connection D if and only if L is parallel with respect to ∇ .

Proof. For any $X, Y \in \Gamma(L)$, since $wX = 0$ we have

$$\widetilde{h}(X, FY) = h(X, FY).$$

Then from teorem(4.6) proof is completed. □

Then from teorem(4.6) we have the following corollary.

Corollary 5.2. Let $(M, g, S(TM), S(TM^\perp))$ be a semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold \widetilde{M} . Then the following assertions are equivalent:

- (i) The distribution L is parallel with respect to quarter symmetric non-metric connection D .
- (ii) The distribution L is parallel with respect to ∇ .
- (iii) $h(X, FY) = 0, X, Y \in \Gamma(L)$.
- (iv) $\widetilde{h}(X, FY) = 0, X, Y \in \Gamma(L)$.
- (v) $(\nabla_X f)Y = 0, X, Y \in \Gamma(L)$.

Proposition 5.3. Let $(M, g, S(TM), S(TM^\perp))$ be a semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. The distribution L' is parallel with respect to quarter symmetric non-metric connection if and only if

$$\widetilde{A}_{FY}X \in \Gamma(L')$$

for any $X, Y \in \Gamma(L')$

Proof. For any $X, Y \in \Gamma(L')$, since $fX = 0$ and $\widetilde{A}_{FY}X = A_{FY}X - \pi(FY)fX$ we have

$$\widetilde{A}_{FY}X = A_{FY}X$$

From Proposition(4.3) and Proposition(5.3) we have the following corollary. □

Theorem 5.4. Let $(M, g, S(TM), S(TM^\perp))$ be a semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. The distribution L is integrable with respect to quarter symmetric non-metric connection if and only if

$$\widetilde{h}(X, FY) = \widetilde{h}(Y, FX), X, Y \in \Gamma(L).$$

Proof. For any $X, Y \in \Gamma(L)$, since $wX = 0$ we get $\widetilde{h}(X, Y) = h(X, Y)$. From theorem(4.7) we have the proof. □

Then from theorem(4.7) we have the following corollary.

Corollary 5.3. Let $(M, g, S(TM), S(TM^\perp))$ be a semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. Then the following assertions are equivalent:

- (i) The distribution L is integrable with respect to quarter symmetric non-metric connection.
- (ii) The distribution L is integrable with respect to ∇ .
- (iii) $\widetilde{h}(X, FY) = \widetilde{h}(Y, FX), X, Y \in \Gamma(L)$.
- (iv) $h(X, FY) = h(Y, FX), X, Y \in \Gamma(L)$.
- (v) $(\nabla_X f)Y = (\nabla_Y f)X, X, Y \in \Gamma(L)$.

Proposition 5.4. Let $(M, g, S(TM), S(TM^\perp))$ be a semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. The distribution L' is integrable with respect to quarter symmetric non-metric connection if and only if

$$\widetilde{A}_{FY}X = \widetilde{A}_{FX}Y, X, Y \in \Gamma(L').$$

Proof. For any $X, Y \in \Gamma(L')$, we know that $fX = 0$. Thus from (5.15) and (5.18) we get $\widetilde{A}_{FY}X = A_{FY}X$. From Proposition(4.4) we have the our assertion. □

Then from proposition(4.4) we give the following corollary.

Corollary 5.4. Let $(M, g, S(TM), S(TM^\perp))$ be a semi-invariant r -lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. The following assertions are equivalent:

- (i) The distribution L' is integrable with respect to quarter symmetric non-metric connection.
- (ii) The distribution L' is integrable with respect to ∇ .
- (iii) $\widetilde{A}_{FY}X = \widetilde{A}_{FX}Y$, $X, Y \in \Gamma(L')$.
- (iv) $A_{FY}X = A_{FX}Y$, $X, Y \in \Gamma(L')$.

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