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Projective Surfaces and Pre-Normalized Blaschke Immersions of Codimension Two

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ABSTRACT

We prove that any non-degenerate surface in the projective 3-space has a local lift as a minimal pre-normalized Blaschke immersion into the equicentroaffine 4-space. Furthermore, an indefinite surface in the projective 3-space has a local lift as a pre-normalized Blaschke immersion into the equicentroaffine 4-space satisfying the Einstein condition if and only if the surface is projectively applicable to an affine sphere.

Keywords: *projective surface; affine sphere; pre-normalized Blaschke immersion; centroaffine minimality; Einstein condition.*

AMS Subject Classification (2010): *Primary: 53A15; Secondary: 53A20, 53A07*

1. Introduction

Differential geometry of surfaces in the real projective space $\mathbf{P}^3$ has a long history from the early twentieth century. We can find a lot of papers and books concerning about this topic in references in [1, 8, 9].

In the previous paper [2], the authors studied centroaffine surfaces in the affine space $\mathbf{R}^3$ from the viewpoint of projective differential geometry by regarding centroaffine surfaces in $\mathbf{R}^3$ as surfaces in $\mathbf{P}^3$. In contrast to [2], in this article, we shall study projective surfaces in $\mathbf{P}^3$ from the viewpoint of equicentroaffine differential geometry of codimension two by regarding projective surfaces in $\mathbf{P}^3$ as surfaces in $\mathbf{R}^4$.

In 1993, Nomizu and Sasaki [6] gave a new approach by using the equicentroaffine geometry of surfaces in $\mathbf{R}^4$. A point of such geometry of codimension two is how to take transversal vector fields for a surface. One transversal vector field is the radial vector field, that is, the position vector field of a surface, and the other is chosen to be a pre-normalized Blaschke normal vector field, which was defined in [6]. See [4, 5, 11, 12] for other choices of transversal vector fields. Following [6], Furuhata [3] studied surfaces in $\mathbf{R}^4$ with vanishing shape operator, which can be considered from a viewpoint of a certain variation problem. We call them minimal pre-normalized Blaschke surfaces in this article. It is a natural question to determine surfaces in $\mathbf{P}^3$ admitting local lifts as minimal pre-normalized Blaschke surfaces in $\mathbf{R}^4$. In this article, we give an answer to this question, which claims any surface has such a lift (Theorem 5.1).

It is an interesting problem to characterize a surface in $\mathbf{P}^3$ in terms of the property whether it has a certain special lift in $\mathbf{R}^4$ or not. Affine spheres are important objects not only in equiaffine differential geometry, but also in projective differential geometry. We show that a surface projectively applicable to an affine sphere is characterized to have a local lift such that the Ricci tensor field of the induced connection is constant multiple of the pre-normalized Blaschke metric (Theorem 6.1).

2. Projective surfaces

We denote by $\mathbf{P}^n$ the real projective space of dimension n. In this section, we shall review the surface theory in $\mathbf{P}^3$. See [1, 8, 9] and references therein for more detail.

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A *projective surface* is an immersion from a 2-dimensional manifold M into $\mathbf{P}^3$. If we take local coordinates (x, y) on M, a projective surface $\underline{z}$ is given by a lift z into $\mathbf{R}^4 \setminus \{0\}$:

$$z(x,y) = (z^1(x,y), z^2(x,y), z^3(x,y), z^4(x,y)),$$

where

$$\underline{z}(x,y) = [z^1(x,y) : z^2(x,y) : z^3(x,y) : z^4(x,y)]$$

via homogeneous coordinates on $\mathbf{P}^3$. In the following, we assume that the vectors z_{xy}, z_x, z_y, z are linearly independent at each point (x, y), which is independent of the choice of the lift. Then z_{xx} and z_{yy} can be written as

$$z_{xx} = lz_{xy} + az_x + bz_y + pz,\ z_{yy} = mz_{xy} + cz_x + dz_y + qz$$

for some functions l, m, a, b, c, d, p, q. Given a projective surface $\underline{z}$, we associate a surface in $\mathbf{R}^3$ by fixing inhomogeneous coordinates: for example, for the mapping $\underline{z}$ above and the inhomogeneous coordinates $[1 : u : v : w]$, we have a surface in $\mathbf{R}^3$ as

$$(z^2(x,y)/z^1(x,y), z^3(x,y)/z^1(x,y), z^4(x,y)/z^1(x,y)).$$

It is easy to see that the symmetric $(0, 2)$-tensor ψ defined by

$$\psi = ldx^2 + 2dxdy + mdy^2$$

is conformal to the second fundamental form of the surface in $\mathbf{R}^3$. We call $\underline{z}$ an *indefinite* projective surface if ψ is indefinite.

In the following, we assume that $\underline{z}$ is an indefinite projective surface. Then taking asymptotic line coordinates for the corresponding surface in $\mathbf{R}^3$, that is, taking coordinates so that $\ell = m = 0$, and rescaling the lift, we may assume that a lift z satisfies a system of the form

$$z_{xx} = bz_y + pz,\ z_{yy} = cz_x + qz, \tag{2.1}$$

which is called a *canonical system*.

It is straightforward to see that the integrability condition for (2.1) is given by

$$\begin{cases} L_y = -2bc_x - cb_x,\ M_x = -2cb_y - bc_y, \\ bM_y + 2Mb_y + b_{yyy} = cL_x + 2Lc_x + c_{xxx}, \end{cases} \tag{2.2}$$

where

$$L = -b_y - 2p,\ M = -c_x - 2q. \tag{2.3}$$

Definition 2.1. Let $\underline{z}$ and $\underline{w}$ be indefinite projective surfaces and choose a lift z of $\underline{z}$ satisfying (2.1). We say that $\underline{w}$ is *projectively applicable* to $\underline{z}$ if $\underline{w}$ has a lift w satisfying a canonical system with the same b and c in (2.1).

By use of transformation formulas for b and c, it is easy to see that the above definition is independent of choice of z. In other references, it might be assumed in addition that $\underline{w}$ is not projectively equivalent to $\underline{z}$.

3. Affine spheres

We first recall a formulation of the theory of affine hypersurfaces and then we define a notion of affine spheres in $\mathbf{P}^3$. Such surfaces are found to form an important class of projective surfaces. See [7, 10] for more detail.

For an immersion F from an n-dimensional manifold M into the affine space $\mathbf{R}^{n+1}$ with the standard flat connection D, we choose a transversal vector field ξ along F. Then, the Gauss-Weingarten formulas for the immersion F are given by

$$\begin{cases} D_X F_* Y = F_* \nabla_X Y + h(X,Y)\xi, \\ D_X \xi = -F_* SX + \tau(X)\xi \end{cases} \quad (X, Y \in \mathfrak{X}(M)),$$

where $\mathfrak{X}(M)$ is the set of all vector fields on M. Then ∇, h, S and τ define a torsion-free affine connection, a symmetric $(0, 2)$-tensor, a $(1, 1)$-tensor and a 1-form on M, respectively, which we call *the induced connection, the* *affine fundamental form*, *the affine shape operator* and *the transversal connection form*, respectively. We fix a volume form ω on $\mathbf{R}^{n+1}$ which is parallel with respect to D and define a volume form θ on M by

$$\theta(X_1, \dots, X_n) = \omega(F_* X_1, \dots, F_* X_n, \xi)$$

for $X_1, \dots, X_n \in \mathfrak{X}(M)$, called the volume form induced by F and ξ. Then we have

$$\nabla_X \theta = \tau(X)\theta \quad (X \in \mathfrak{X}(M)).$$

We call F to be *non-degenerate* if h is non-degenerate; this property is independent of the choice of ξ. Then we can find a transversal vector field ξ such that $\tau = 0$. We call the immersion F with such a vector field ξ an *equiaffine immersion*. Moreover, we can find a unique transversal vector field ξ up to sign such that $\tau = 0$ and θ is equal to the volume form with respect to h. Then, we call F with ξ a *Blaschke immersion*, and ξ the *Blaschke normal vector field* of F.

A Blaschke immersion is called an *affine sphere* if the affine shape operator S is a scalar operator. An affine sphere is said to be *proper* or *improper* if S is nonzero or zero, respectively.

Definition 3.1. A projective surface in $\mathbf{P}^3$ is called an *affine sphere* if it is locally an affine sphere in some affine chart $\mathbf{R}^3$ in $\mathbf{P}^3$.

If we use the canonical system (2.1), the integrability condition for indefinite affine spheres in $\mathbf{P}^3$ can be stated as follows.

Lemma 3.1. *Let z be a lift of an indefinite projective surface satisfying* (2.1) *and*

$$z = (e^{\frac{1}{2}\varphi}, e^{\frac{1}{2}\varphi} F) \tag{3.1}$$

for an $\mathbf{R}$-valued function φ and a surface F in $\mathbf{R}^3$. Then F is an affine sphere if and only if there exists some $k \in \mathbf{R}$ such that

$$b_y = b\varphi_y, \; c_x = c\varphi_x, \tag{3.2}$$

$$\varphi_{xy} = bc + ke^{-\varphi}. \tag{3.3}$$

Moreover, F is a proper affine sphere if $k \neq 0$ and an improper affine sphere if $k = 0$.

Proof. From (2.1) and (3.1), we have

$$p = \frac{1}{2}\varphi_{xx} + \frac{1}{4}\varphi_x^2 - \frac{1}{2}b\varphi_y, \; q = \frac{1}{2}\varphi_{yy} + \frac{1}{4}\varphi_y^2 - \frac{1}{2}c\varphi_x, \tag{3.4}$$

$$F_{xx} = -\varphi_x F_x + bF_y, \; F_{yy} = cF_x - \varphi_y F_y. \tag{3.5}$$

Note that $\omega(F_x, F_y, F_{xy}) \neq 0$, since $\underline{z}$ is indefinite. If we put

$$\xi = \lambda F_{xy}, \; \lambda^2 = \pm \frac{1}{\omega(F_x, F_y, F_{xy})},$$

then, from (3.5), it is straightforward to see that

$$(\log \lambda)_x = \varphi_x, \; (\log \lambda)_y = \varphi_y$$

and hence,

$$\begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} = \lambda \begin{pmatrix} -\varphi_{xy} + bc & -b\varphi_y + b_y \\ -c\varphi_x + c_x & -\varphi_{xy} + bc \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}, \tag{3.6}$$

which shows that F is an equiaffine immersion with ξ. Moreover, it is easy to see that ξ is a Blaschke normal vector field.

Now we assume that F is an affine sphere. Then from (3.6), we have (3.2) which can be solved as

$$b = f(x)e^{\varphi}, \; c = g(y)e^{\varphi} \tag{3.7}$$

for some functions f and g of one variable. Then from (2.2), (2.3), (3.4) and (3.7), we have

$$\varphi_{xxy} + \varphi_x\varphi_{xy} = (3fg\varphi_x + f'g)e^{2\varphi}, \; \varphi_{xyy} + \varphi_y\varphi_{xy} = (3fg\varphi_y + fg')e^{2\varphi},$$

which are equivalent to

$$(\varphi_{xy}e^{\varphi} - fge^{3\varphi})_x = 0, \; (\varphi_{xy}e^{\varphi} - fge^{3\varphi})_y = 0.$$

Hence there exists some $k \in \mathbf{R}$ such that

$$\varphi_{xy}e^{\varphi} - fge^{3\varphi} = k,$$

which is equivalent to (3.3) from (3.7). From (3.3) and (3.6), F is proper if and only if $k \neq 0$. □

4. Centroaffine immersions of codimension two

Following [6], we shall explain the notion of centroaffine immersions of codimension two.

We denote by η the radial vector field on $\mathbf{R}^{n+2}$. An immersion F from an n-dimensional manifold M into $\mathbf{R}^{n+2}$ is called a *centroaffine immersion of codimension two* if $\eta \circ F$ is a transversal vector field along F and there exists a vector field ξ along F which is transversal to the direct sum of the vector space spanned by $(\eta \circ F)(x)$ and $F_* T_x M$ at each point $x \in M$. We denote by D the standard flat connection on $\mathbf{R}^{n+2}$. If f is a centroaffine immersion of codimension two, we have the following equations:

$$\begin{cases} D_X \eta = F_* X, \\ D_X F_* Y = F_* \nabla_X Y + h(X,Y)\xi + T(X,Y)\eta, \\ D_X \xi = -F_* SX + \tau(X)\xi + \rho(X)\eta \end{cases} \tag{4.1}$$

for $X, Y \in \mathfrak{X}(M)$. Then ∇ defines a torsion-free affine connection, and both h and T symmetric $(0,2)$-tensors on M. Moreover, S defines a $(1,1)$-tensor, and both τ and ρ are 1-forms on M. We fix a volume form ω on $\mathbf{R}^{n+2}$ which is parallel with respect to D and define a volume form θ on M by

$$\theta(X_1, \dots, X_n) = \omega(F_* X_1, \dots, F_* X_n, \xi, \eta) \tag{4.2}$$

for $X_1, \dots, X_n \in \mathfrak{X}(M)$. Then we have

$$\nabla_X \theta = \tau(X)\theta \quad (X \in \mathfrak{X}(M)).$$

The following is the fundamental result concerning about reduction of codimension of centroaffine immersions of codimension two.

Proposition 4.1 ([6])**.** *Let F be a centroaffine immersion of codimension two. In the case that* $\operatorname{rank} h \geq 2$*, the image of F is contained in some affine hyperplane if and only if $T = \lambda h$ for some function λ. If $h = 0$ and $n \geq 2$, then the image of F is contained in some affine hyperplane which goes through 0.*

If F is a centroaffine immersion of codimension two such that h is non-degenerate, we call F to be *non-degenerate*; this is independent of the choice of ξ. Let F be a non-degenerate centroaffine immersion of codimension two. Then we can find a transversal vector field ξ such that $\tau = 0$. We call F with such ξ an *equiaffine immersion*. We can also find a transversal vector field ξ determined $\operatorname{mod} \eta$ up to sign such that $\tau = 0$ and θ is equal to the volume form with respect to h. We call F with such ξ a *Blaschke immersion*. Moreover, we can find a unique transversal vector field ξ up to sign such that it satisfies all the above conditions with the equation

$$\operatorname{tr}_h((X,Y) \mapsto T(X,Y) + h(SX,Y)) = 0.$$

We call F with such ξ a *pre-normalized Blaschke immersion*, and ξ the *pre-normalized Blaschke normal vector field* of F.

Definition 4.1 ([3])**.** A pre-normalized Blaschke immersion F is called to be *centroaffine minimal*, or *minimal* for short, if it is extremal for the integral of the volume form θ among any variation in the pre-normalized Blaschke normal direction, which is equivalent to the condition $\operatorname{tr} S = 0$.

Example 4.1. A curve in $\mathbf{P}^2$ is a map from an interval into $\mathbf{P}^2$. If we use homogeneous coordinates on $\mathbf{P}^2$, a curve in $\mathbf{P}^2$ is given by a lift z into $\mathbf{R}^3 \setminus \{0\}$:

$$z(t) = (z_1(t), z_2(t), z_3(t)).$$

In the following, we assume that the vectors z'', z', z are linearly independent at each point t. Then z_1, z_2, z_3 can be given by linearly independent solutions of a third-order linear differential equation:

$$z''' + p_1 z'' + p_2 z' + p_3 z = 0. \tag{4.3}$$

By the assumption, z is a centroaffine immersion of codimension two with a transversal vector field $\xi = z''$. From (4.1), we have

$$\nabla_{\frac{d}{dt}} \frac{d}{dt} = 0, \; h\left(\frac{d}{dt}, \frac{d}{dt}\right) = 1, \; T\left(\frac{d}{dt}, \frac{d}{dt}\right) = 0. \tag{4.4}$$

In particular, if we denote by ω_h the volume form with respect to h, we have

$$\omega_h\left(\frac{d}{dt}\right) = 1. \tag{4.5}$$

From (4.1) and (4.3), we have

$$S\left(\frac{d}{dt}\right) = p_2,\ \tau\left(\frac{d}{dt}\right) = -p_1,\ \rho\left(\frac{d}{dt}\right) = -p_3. \tag{4.6}$$

In particular, z is an equiaffine immersion with ξ if and only if $p_1 = 0$. The volume form θ is given by

$$\theta\left(\frac{d}{dt}\right) = \omega(z', \xi, \eta) = \omega(z, z', z''). \tag{4.7}$$

From (4.5) and (4.7), z is a Blaschke immersion with ξ if and only if

$$p_1 = 0,\ \omega(z, z', z'') = 1.$$

Moreover, from (4.4) and (4.6), we have

$$T\left(\frac{d}{dt}, \frac{d}{dt}\right) + h\left(S\left(\frac{d}{dt}\right), \frac{d}{dt}\right) = p_2$$

so that

$$\mathrm{tr}_h\,\{(X, Y) \mapsto T(X, Y) + h(SX, Y)\} = p_2.$$

Hence z is a pre-normalized Blaschke immersion with ξ if and only if z is given by a Laguerre-Forsyth canonical form:

$$z''' + p_3 z = 0$$

with normalization

$$\omega(z, z', z'') = 1.$$

5. Pre-normalized lifts and centroaffine minimality

In this section, we shall consider lifts of indefinite projective surfaces in $\mathbf{P}^3$ as pre-normalized Blaschke immersions into $\mathbf{R}^4$. For an arbitrary chosen $\mathbf{R}$-valued function φ, we define a lift w of $\underline{z}$ by $w = e^{-\frac{1}{2}\varphi} z$, where z is a lift of $\underline{z}$ satisfying the canonical system (2.1). Then we have

$$\begin{aligned}&(w, w_x, w_y, w_{xy}) \\ &= (z, z_x, z_y, z_{xy}) e^{-\frac{1}{2}\varphi} \begin{pmatrix} 1 & -\frac{1}{2}\varphi_x & -\frac{1}{2}\varphi_y & -\frac{1}{2}\varphi_{xy} + \frac{1}{4}\varphi_x\varphi_y \\ 0 & 1 & 0 & -\frac{1}{2}\varphi_y \\ 0 & 0 & 1 & -\frac{1}{2}\varphi_x \\ 0 & 0 & 0 & 1 \end{pmatrix}.\end{aligned} \tag{5.1}$$

It follows that the vectors w_{xy}, w_x, w_y, w are linearly independent at each point (x, y) since we assume that the vectors z_{xy}, z_x, z_y, z are linearly independent at each point (x, y). For some scalar functions λ and μ, where λ is assumed never to vanish, we define a transversal vector field ξ given by

$$\xi = \lambda w_{xy} + \mu w. \tag{5.2}$$

Then, w is a centroaffine immersion of codimension two with the transversal vector field ξ. We determine these functions λ and μ so that the immersion w with ξ turns out to be a pre-normalized Blaschke immersion in the following. A direct computation shows that

$$w_{xx} = -\varphi_x w_x + b w_y + \tilde{p} w,\ w_{yy} = c w_x - \varphi_y w_y + \tilde{q} w, \tag{5.3}$$

where

$$\tilde{p} = -\frac{1}{2}\varphi_{xx} - \frac{1}{4}\varphi_x^2 + \frac{1}{2}b\varphi_y + p,\ \tilde{q} = -\frac{1}{2}\varphi_{yy} - \frac{1}{4}\varphi_y^2 + \frac{1}{2}c\varphi_x + q.$$

Moreover, we have

$$\begin{cases} \xi_x = \{\lambda(-\varphi_{xy} + bc) + \mu\}w_x + \lambda(-b\varphi_y + b_y + \tilde{p})w_y \\ \quad + \left(-\varphi_x + \dfrac{\lambda_x}{\lambda}\right)\xi + \left\{\mu\varphi_x + \lambda(\tilde{p}_y + b\tilde{q}) - \dfrac{\lambda_x}{\lambda}\mu + \mu_x\right\}w, \\ \xi_y = \lambda(-c\varphi_x + c_x + \tilde{q})w_x + \{\lambda(-\varphi_{xy} + bc) + \mu\}w_y \\ \quad + \left(-\varphi_y + \dfrac{\lambda_y}{\lambda}\right)\xi + \left\{\mu\varphi_y + \lambda(\tilde{q}_x + c\tilde{p}) - \dfrac{\lambda_y}{\lambda}\mu + \mu_y\right\}w. \end{cases} \tag{5.4}$$

From (4.1), (5.2), (5.3) and (5.4), we have the following:

$$\begin{cases} \nabla_{\frac{\partial}{\partial x}}\dfrac{\partial}{\partial x} = -\varphi_x\dfrac{\partial}{\partial x} + b\dfrac{\partial}{\partial y},\ \nabla_{\frac{\partial}{\partial y}}\dfrac{\partial}{\partial y} = c\dfrac{\partial}{\partial x} - \varphi_y\dfrac{\partial}{\partial y}, \\ \nabla_{\frac{\partial}{\partial x}}\dfrac{\partial}{\partial y} = \nabla_{\frac{\partial}{\partial y}}\dfrac{\partial}{\partial x} = 0, \end{cases} \tag{5.5}$$

$$h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = h\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = 0,\ h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \frac{1}{\lambda}, \tag{5.6}$$

$$T\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \tilde{p},\ T\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = -\frac{\mu}{\lambda},\ T\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \tilde{q}, \tag{5.7}$$

$$\begin{cases} S\left(\dfrac{\partial}{\partial x}\right) = \{\lambda(\varphi_{xy} - bc) - \mu\}\dfrac{\partial}{\partial x} + \lambda(b\varphi_y - b_y - \tilde{p})\dfrac{\partial}{\partial y}, \\ S\left(\dfrac{\partial}{\partial y}\right) = \lambda(c\varphi_x - c_x - \tilde{q})\dfrac{\partial}{\partial x} + \{\lambda(\varphi_{xy} - bc) - \mu\}\dfrac{\partial}{\partial y}, \end{cases} \tag{5.8}$$

$$\tau\left(\frac{\partial}{\partial x}\right) = -\varphi_x + \frac{\lambda_x}{\lambda},\ \tau\left(\frac{\partial}{\partial y}\right) = -\varphi_y + \frac{\lambda_y}{\lambda}. \tag{5.9}$$

Since z satisfies (2.1), we have $\omega(z_x, z_y, z_{xy}, z)$ is a nonzero constant, and denote it by C_0. By (5.1), we have

$$\begin{aligned} \theta\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) &= \omega(w_x, w_y, \xi, w) \\ &= \lambda\omega(w_x, w_y, w_{xy}, w) \\ &= \lambda e^{-2\varphi}\omega(z_x, z_y, z_{xy}, z) \\ &= C_0\lambda e^{-2\varphi}. \end{aligned} \tag{5.10}$$

Lemma 5.1. *Let z be a lift of an indefinite projective surface in $\mathbf{P}^3$ satisfying (2.1). Set a centroaffine immersion $w = e^{-\frac{1}{2}\varphi}z$ of codimension two, and ξ as in (5.2). Then ξ is a pre-normalized Blaschke normal vector field of w if and only if*

$$(\lambda, \mu) = C\left(e^{\varphi}, \frac{1}{2}(\varphi_{xy} - bc)e^{\varphi}\right), \tag{5.11}$$

where C is a nonzero constant $\pm|C_0|^{-1/2}$ as in (5.10).

Proof. From (5.9), w with ξ is an equiaffine immersion if and only if

$$\lambda = Ce^{\varphi} \tag{5.12}$$

for some $C \in \mathbf{R} \setminus \{0\}$. From (5.6) and (5.10), w with ξ is a Blaschke immersion if and only if $C = \pm|C_0|^{-1/2}$ in (5.12).

From (5.6), (5.7) and (5.8), we have

$$T\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) + h\left(S\left(\frac{\partial}{\partial x}\right), \frac{\partial}{\partial y}\right) = \varphi_{xy} - bc - 2\frac{\mu}{\lambda},$$

which shows that it is a pre-normalized Blaschke immersion if and only if (5.11) holds. □

Let $\underline{z}$ be an indefinite projective surface in $\mathbf{P}^3$, z a lift satisfying (2.1) and $w = e^{-\frac{1}{2}\varphi}z$ a lift as a pre-normalized Blaschke immersion with ξ given by (5.2) with (5.11). Then, from (5.8) and (5.11), we have

$$\operatorname{tr} S = C(\varphi_{xy} - bc)e^{\varphi}. \tag{5.13}$$

Theorem 5.1. *Any indefinite projective surface in $\mathbf{P}^3$ has a local lift as a minimal pre-normalized Blaschke immersion of codimension two.*

Proof. From (5.13) it is enough to consider the equation

$$\varphi_{xy} = bc,$$

which can be solved locally. □

The same holds for a definite projective surface; refer to Corollary A.1.

Remark 5.1. In Theorem 5.1, let w be a local lift as a minimal pre-normalized Blaschke immersion of codimension two. Then a local lift $f(x)g(y)w$ for functions f and g of one variable has also the same property.

6. Pre-normalized lifts and Einstein condition

In this section, we shall study properties of a lift of a surface projectively applicable to an affine sphere.

Let z be a lift of an indefinite projective surface satisfying (2.1) and $w = e^{-\frac{1}{2}\varphi}z$ a lift as a pre-normalized Blaschke immersion with ξ given by (5.2) with (5.11). If we denote by R and Ric the curvature tensor and the Ricci tensor of ∇ for w, respectively, from (5.5) we have

$$\begin{cases} R\left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right)\dfrac{\partial}{\partial x} = (\varphi_{xy} - bc)\dfrac{\partial}{\partial x} + (b\varphi_y - b_y)\dfrac{\partial}{\partial y}, \\ R\left(\dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial x}\right)\dfrac{\partial}{\partial y} = (c\varphi_x - c_x)\dfrac{\partial}{\partial x} + (\varphi_{xy} - bc)\dfrac{\partial}{\partial y}, \end{cases}$$

so that

$$\begin{cases} \operatorname{Ric}\left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial x}\right) = -b\varphi_y + b_y, \ \operatorname{Ric}\left(\dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial y}\right) = -c\varphi_x + c_x, \\ \operatorname{Ric}\left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right) = \varphi_{xy} - bc. \end{cases} \tag{6.1}$$

We call the condition

$$\operatorname{Ric} = \alpha h \tag{6.2}$$

for a constant $\alpha \in \mathbf{R}$ the *Einstein condition for a pre-normalized Blaschke immersion*.

Theorem 6.1. *An indefinite projective surface in $\mathbf{P}^3$ has a lift as a pre-normalized Blaschke immersion into $\mathbf{R}^4$ satisfying the Einstein condition if and only if the surface is projectively applicable to an affine sphere.*

Proof. Let z be a lift of an indefinite projective surface $\underline{z}$ satisfying (2.1) and $w = e^{-\frac{1}{2}\varphi}z$ a lift as a pre-normalized Blaschke immersion with ξ given by (5.2) with (5.11). If we put $\alpha = Ck$, from (5.6), (5.11) and (6.1), the condition (6.2) is equivalent to (3.3) and (3.2), which implies that $\underline{z}$ is projectively applicable to an affine sphere given by $\tilde{z} = (e^{\frac{1}{2}\varphi}, e^{\frac{1}{2}\varphi}F)$. □

A. Equicentroaffine geometry of immersions of codimension two

We now give equicentroaffine geometric properties of pre-normalized Blaschke immersions of an n-dimensional manifold into $\mathbf{R}^{n+2}$, though the basic is already stated in Section 4. It is not necessary to assume h is indefinite in this section.

At first, we note the formulas below in a general setting.

Remark A.1. Let ∇ be an affine connection of torsion free, and h a pseudo-Riemannian metric on an n-dimensional manifold M. Let ∇^* be the dual connection of ∇ with respect to h, which is by definition given as

$$Xh(Y,Z) = h(\nabla_X Y, Z) + h(Y, \nabla^*_X Z)$$

for any vector fields $X, Y, Z \in \mathfrak{X}(M)$. Then,

$$\mathrm{div}^{\nabla}\mathrm{grad}_h\psi = \mathrm{tr}_h\mathrm{Hess}^{\nabla^*}\psi \tag{A.1}$$

holds for any function ψ on M, where div^{∇} denotes the divergence relative to ∇, grad_h the gradient relative to h, and Hess^{∇^*} the Hessian relative to ∇^*. In fact, taking temporarily an orthonormal frame $X_i \in \mathfrak{X}(M)$ with respect to h such that $h(X_i, X_j) = \varepsilon_i\delta_{ij}$, $\varepsilon_i = \pm 1$, we have

$$\begin{aligned} \text{LHS of (A.1)} &= \mathrm{tr}\{X \mapsto \nabla_X \mathrm{grad}_h\psi\} \\ &= \sum_j \varepsilon_j h(X_j, \nabla_{X_j}\mathrm{grad}_h\psi) \\ &= \sum_j \varepsilon_j\{X_j h(X_j, \mathrm{grad}_h\psi) - h(\nabla^*_{X_j}X_j, \mathrm{grad}_h\psi)\} \\ &= \sum_j \varepsilon_j\{X_j(X_j\psi) - \nabla^*_{X_j}X_j\psi\} = \text{RHS of (A.1)}. \end{aligned}$$

If the volume form of h is parallel with respect to ∇, then

$$\triangle_h\psi = \mathrm{div}^{\nabla^h}\mathrm{grad}_h\psi = \mathrm{div}^{\nabla}\mathrm{grad}_h\psi, \tag{A.2}$$

where ∇^h is the Levi-Civita connection of h, and $\triangle_h$ is the Laplacian with respect to h.

Let D be the standard flat connection of $\mathbf{R}^{n+2}$, and ω a D-parallel volume form on $\mathbf{R}^{n+2}$. Let $F : M \to \mathbf{R}^{n+2}$ be a pre-normalized Blaschke immersion of an n-dimensional manifold M with normal ξ, and ∇, h, T, S, ρ as in the formulas (4.1). For a function ψ on M, we set $\widetilde{F} = e^{\psi}F$ and take its pre-normalized Blaschke normal vector field $\widetilde{\xi}$. Define $\widetilde{\nabla}$, $\widetilde{h}$, $\widetilde{T}$, $\widetilde{S}$ and $\widetilde{\rho}$ by

$$\begin{aligned} &D_X\widetilde{F}_*Y = \widetilde{F}_*\widetilde{\nabla}_XY + \widetilde{h}(X,Y)\widetilde{\xi} + \widetilde{T}(X,Y)\widetilde{\eta}, \\ &D_X\widetilde{\xi} = -\widetilde{F}_*\widetilde{S}X + \widetilde{\rho}(X)\widetilde{\eta} \end{aligned}$$

for $X, Y \in \mathfrak{X}(M)$, where $\widetilde{\eta}$ is the radial vector field for $\widetilde{F}$.

Lemma A.1. *For pre-normalized Blaschke immersions $F, \widetilde{F} = e^{\psi}F : M \to \mathbf{R}^{n+2}$, the following equations hold.*

$$\widetilde{\nabla}_XY = \nabla_XY + (Y\psi)X + (X\psi)Y - h(X,Y)\mathrm{grad}_h\psi, \tag{A.3}$$

$$\widetilde{h} = e^{2\psi}h, \tag{A.4}$$

$$\widetilde{T} = T + \mathrm{Hess}^{\nabla}\psi - d\psi\otimes d\psi + \frac{1}{2}|\mathrm{grad}_h\psi|^2_h h, \tag{A.5}$$

$$\widetilde{S} = e^{-2\psi}\left\{S - \frac{1}{2}|\mathrm{grad}_h\psi|^2_h\mathrm{Id} + \mathrm{grad}_h\psi\otimes d\psi - \nabla\mathrm{grad}_h\psi\right\}. \tag{A.6}$$

Proof. We express the pre-normalized Blaschke normal vector filed $\widetilde{\xi}$ by

$$u\widetilde{\xi} = F_*U + \xi + a\eta,$$

where U is a vector field, u, a are functions and u is positive. Since $\widetilde{F}_* Y = e^\psi F_* Y + (Ye^\psi)\eta$, we have

$$D_X \widetilde{F}_* Y = F_* \left\{ e^\psi \nabla_X Y + (Ye^\psi)X + (Xe^\psi)Y \right\} + e^\psi h(X,Y)\xi + \left\{ e^\psi T(X,Y) + X(Ye^\psi) \right\} \eta, \tag{A.7}$$

$$\begin{aligned} &\widetilde{F}_* \widetilde{\nabla}_X Y + \widetilde{h}(X,Y)\widetilde{\xi} + \widetilde{T}(X,Y)\widetilde{\eta} \\ &= F_* \left\{ e^\psi \widetilde{\nabla}_X Y + u^{-1}\widetilde{h}(X,Y)U \right\} + u^{-1}\widetilde{h}(X,Y)\xi \\ &\quad + \left\{ \widetilde{\nabla}_X Y e^\psi + u^{-1}a\widetilde{h}(X,Y) + e^\psi \widetilde{T}(X,Y) \right\} \eta, \end{aligned} \tag{A.8}$$

$$\begin{aligned} D_X \widetilde{\xi} = F_* &\left\{ -u^{-1}SX + u^{-1}aX + (Xu^{-1})U + u^{-1}\nabla_X U \right\} \\ &+ \left\{ Xu^{-1} + u^{-1}h(X,U) \right\} \xi \\ &+ \left\{ u^{-1}\rho(X) + X(u^{-1}a) + u^{-1}T(X,U) \right\} \eta, \end{aligned} \tag{A.9}$$

$$\begin{aligned} &-\widetilde{F}_* \widetilde{S}X + \widetilde{\rho}(X)\widetilde{\eta} \\ &= F_* \left\{ -e^\psi \widetilde{S}X \right\} + \left\{ e^\psi \widetilde{\rho}(X) - (\widetilde{S}X)e^\psi \right\} \eta. \end{aligned} \tag{A.10}$$

(Step 1) Comparing the ξ-components of (A.7) and (A.8), we have $\widetilde{h} = ue^\psi h$, and hence

$$\left| \det(\widetilde{h}(X_i, X_j)) \right|^{1/2} = (ue^\psi)^{n/2} \left| \det(h(X_i, X_j)) \right|^{1/2} \tag{A.11}$$

for $X_j \in \mathfrak{X}(M)$. We calculate

$$\begin{aligned} & \omega(\widetilde{F}_* X_1, \ldots, \widetilde{F}_* X_n, \widetilde{\xi}, \widetilde{\eta}) \\ = \;& \omega(e^\psi F_* X_1, \ldots, e^\psi F_* X_n, u^{-1}\xi, e^\psi \eta) \\ = \;& e^{(n+1)\psi} u^{-1} \omega(F_* X_1, \ldots, F_* X_n, \xi, \eta) \\ = \;& e^{(n+1)\psi} u^{-1} \left| \det(h(X_i, X_j)) \right|^{1/2}, \end{aligned}$$

from which (A.11) implies that

$$u = e^\psi, \tag{A.12}$$

and hence (A.4).
(Step 2) Comparing the ξ-components of (A.9) and (A.10), we have

$$U = \mathrm{grad}_h \psi \tag{A.13}$$

by (A.12). Comparing F_*-components of (A.7) and (A.8), we get (A.3) from (A.4), (A.12) and (A.13).
(Step 3) In a similar fashion, comparing the η-components of (A.7) and (A.8) implies

$$\begin{aligned} &\widetilde{T}(X,Y) \\ &= T(X,Y) + \mathrm{Hess}^\nabla \psi(X,Y) - (X\psi)(Y\psi) \\ &\quad + h(X,Y) \left\{ |\mathrm{grad}_h \psi|_h^2 - a \right\}, \end{aligned} \tag{A.14}$$

and comparing the F_*-components of (A.9) and (A.10) implies

$$\widetilde{S}X = e^{-2\psi} \left\{ SX - aX + (X\psi)\mathrm{grad}_h \psi - \nabla_X \mathrm{grad}_h \psi \right\}, \tag{A.15}$$

from which

$$\begin{aligned} &\mathrm{tr}_{\widetilde{h}} \widetilde{T} + \mathrm{tr} \widetilde{S} \\ &= e^{-2\psi} \left\{ \mathrm{tr}_h T + \mathrm{tr} S + \mathrm{tr}_h \mathrm{Hess}^\nabla \psi - \mathrm{div}^\nabla \mathrm{grad}_h \psi + n|\mathrm{grad}_h \psi|_h^2 - 2na \right\}. \end{aligned}$$

By the pre-normalized condition and (A.2), we have

$$a = \frac{1}{2} |\mathrm{grad}_h \psi|_h^2, \tag{A.16}$$

and hence (A.5) by (A.14), (A.6) by (A.15). □

As a corollary, we have the following.

Proposition A.1. *For pre-normalized Blaschke immersions* $F, \widetilde{F} = e^{\psi} F : M \to \mathbf{R}^{n+2}$*, the formula*

$$\mathrm{tr}\widetilde{S} = e^{-2\psi} \left(\mathrm{tr}S - \triangle_h \psi - \frac{n-2}{2} |\mathrm{grad}_h \psi|_h^2 \right) \tag{A.17}$$

holds. In particular, $\widetilde{F}$ *is centroaffine minimal if and only if* ψ *satisfies*

$$\triangle_h \psi + \frac{n-2}{2} |\mathrm{grad}_h \psi|_h^2 = \mathrm{tr}S. \tag{A.18}$$

In the case of surfaces, we have the following.

Corollary A.1. *For any pre-normalized Blaschke immersion* $F : M^2 \to \mathbf{R}^4$*, there exists a function* ψ *locally defined on* M *such that* $\widetilde{F} = e^{\psi} F$ *is centroaffine minimal.*

Proposition A.2. *The Ricci tensor fields* $\mathrm{Ric}, \widetilde{\mathrm{Ric}}$ *of the connections* $\nabla, \widetilde{\nabla}$ *induced by pre-normalized Blaschke immersions* $F, \widetilde{F} = e^{\psi} F : M \to \mathbf{R}^{n+2}$ *satisfy*

$$\begin{aligned} \widetilde{\mathrm{Ric}} = \mathrm{Ric} &+ (1-n)\mathrm{Hess}^{\nabla}\psi + \mathrm{Hess}^{\nabla^*}\psi - \triangle_h \psi h \\ &+ (n-2)\left\{ d\psi \otimes d\psi - |\mathrm{grad}_h \psi|_h^2 h \right\}. \end{aligned} \tag{A.19}$$

Proof. By (A.3), we have

$$\begin{aligned} & \widetilde{\nabla}_X \widetilde{\nabla}_Y Z \\ = \; & \nabla_X \nabla_Y Z + \left\{ \nabla_Y Z\psi + (Y\psi)(Z\psi) - h(Y,Z)|\mathrm{grad}_h \psi|_h^2 \right\} X \\ & + \{X(Z\psi)\} Y + \{X(Y\psi)\} Z \\ & + (X\psi)\nabla_Y Z + (Z\psi)\nabla_X Y + (Y\psi)\nabla_X Z \\ & - \{h(X, \nabla_Y Z) - (Y\psi)h(X,Z) + Xh(Y,Z)\} \mathrm{grad}_h \psi \\ & - h(Y,Z)\nabla_X \mathrm{grad}_h \psi \\ & + [(Z\psi)(Y\psi)X + (Z\psi)(X\psi)Y + (X\psi)(Y\psi)Z \\ & - (Z\psi)h(X,Y)\mathrm{grad}_h \psi]. \end{aligned}$$

Since ∇h is symmetric, ∇^* is of torsion free ([7, p.21]). Hence by a direct calculation, we obtain the curvature tensor field as

$$\begin{aligned} & \widetilde{R}(X,Y)Z \\ & = R(X,Y)Z \\ & \quad - \left\{ \mathrm{Hess}^{\nabla}\psi(Y,Z) - (Y\psi)(Z\psi) + h(Y,Z)|\mathrm{grad}_h \psi|_h^2 \right\} X \\ & \quad + \left\{ \mathrm{Hess}^{\nabla}\psi(X,Z) - (X\psi)(Z\psi) + h(X,Z)|\mathrm{grad}_h \psi|_h^2 \right\} Y \\ & \quad + \{h(Y,Z)X\psi - h(X,Z)Y\psi\} \mathrm{grad}_h \psi \\ & \quad + h(X,Z)\nabla_Y \mathrm{grad}_h \psi - h(Y,Z)\nabla_X \mathrm{grad}_h \psi. \end{aligned} \tag{A.20}$$

By taking a trace with respect to X, we have (A.19) from (A.1) and (A.2). □

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