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ON SOME GEOMETRIC PROPERTIES OF FINITE BLASCHKE PRODUCTS

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Abstract. In this paper we consider some geometric properties of finite Blaschke products for the unit disc and for the upper half plane.

1. Introduction

The rational function

$$B(z) = \beta \prod_{i=1}^{n} \frac{z - a_i}{1 - \overline{a_i} z}$$

is called a finite Blaschke product of degree n for the unit disc where $|\beta| = 1$ and $|a_i| < 1$, $1 \leq i \leq n$. The following finite Blaschke products are called canonical:

$$B(z) = z \prod_{j=1}^{n-1} \frac{z - a_j}{1 - \overline{a_j} z}, |a_j| < 1 \text{ for } 1 \leq j \leq n-1. \tag{1.1}$$

Some geometric properties of canonical finite Blaschke products were studied in [3], [4], [6], [8], [9], [10] and [11].

From [2], it is known that the most general transformation which maps the upper half plane $\mathbb{U} = \{z \in \mathbb{C} : \mathtt{Im} z > 0\}$ onto the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is of the form

$$f(z) = e^{i\theta} \frac{z - \alpha}{z - \overline{\alpha}}, \ \alpha \in \mathbb{U}. \tag{1.2}$$

Let the points α and θ be fixed. Then clearly, the transformation

$$f^{-1}(z) = \frac{z\overline{\alpha} - e^{i\theta}\alpha}{z - e^{i\theta}} \tag{1.3}$$

maps the unit disc $\mathbb{D}$ onto the upper half plane $\mathbb{U}$.

The rational function

$$\widetilde{B}(z) = e^{i\theta} \prod_{k=1}^{n} (\frac{z - z_k}{z - \overline{z_k}})^{m_k}$$

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is called a finite Blaschke product for the upper half plane where z_k, $1 \leq k \leq n$ is a complex number in the upper half plane. For each $x \in \mathbb{R}$, we have $\left|\widetilde{B}(x)\right| = 1$ [7].

In Section 2 we give the description of our problem. In Section 3 and Section 4 we consider some geometric properties of finite Blaschke products for the upper half plane of the following form

$$\widetilde{B}(z) = (B \circ f)(z), \tag{1.4}$$

where f is defined in (1.2), B is defined in (1.1) and of degree 2 or 3, respectively. In section 5, we deal with finite Blaschke products of the forms (1.1) and (1.4) of degree 4 and their geometric properties.

2. Description of the Problem

We begin the following theorem about finite Blaschke products for the upper half plane.

Theorem 2.1. *Let z_k, $1 \leq k \leq n$ be the distinct points in the upper half plane and $\widetilde{B}$ be a Blaschke product for the upper half plane of the following form:*

$$\widetilde{B}(z) = e^{i\theta} \prod_{k=1}^{n} \frac{z - z_k}{z - \overline{z_k}}.$$

Then for every point $\lambda \in \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$, there exist the n distinct points $x_1, x_2, ..., x_n \in \mathbb{R}$ such that

$$\widetilde{B}(x_1) = \widetilde{B}(x_2) = ... = \widetilde{B}(x_n) = \lambda.$$

Proof. Let $\widetilde{B}(z) = e^{i\theta} \prod\limits_{k=1}^{n} \frac{z-z_k}{z-\overline{z_k}}$ be a finite Blaschke product for the upper half plane with n distinct zeros and λ be a fixed point on the unit circle. Because $\widetilde{B}$ is a rational function of degree n, the equation $\widetilde{B}(x_j) = \lambda$ has exactly n solutions with multiplicities. We must show if $\widetilde{B}(x_j) = \widetilde{B}(x_k)$ then $x_j \neq x_k$ for $j, k = 1, 2, ..., n$ and $x_j \in \mathbb{R}$. We know that $\left|\widetilde{B}(z_j)\right| = 1$ if and only if $z_j \in \mathbb{R}$. Then we must show $x_j \neq x_k$. If we take the logarithmic derivative of $\widetilde{B}$, we have

$$\frac{\widetilde{B}'}{\widetilde{B}} = \sum_{j=1}^{n} \frac{z_j - \overline{z}_j}{(z - z_j)(z - \overline{z}_j)}.$$

It can be easily seen that $\frac{\widetilde{B}'(x)}{\widetilde{B}(x)} \neq 0$ and so $\widetilde{B}'(x) \neq 0$ for every $x \in \mathbb{R}$. Hence, λ has exactly n different preimages of real numbers. □

Let B be any given Blaschke product of the form (1.1) of degree n. It is well known that for any specified point $\lambda \in \partial\mathbb{D}$, there exist n distinct points of $\partial\mathbb{D}$ that B maps to λ. There are several studies on the determination problem of these points (see [3], [4], [6], [8], [9] and [10]). In the next chapters we consider any Blashke product $\widetilde{B}$ of the form (1.4) and try to determine the points x_k and x_l on the real axis such that $\widetilde{B}(x_1) = \widetilde{B}(x_2) = \lambda$ for any $\lambda \in \partial\mathbb{D}$. Also in some special cases we consider the finite Blaschke products of the form (1.1) of degree 4.

3. Blaschke Products of Degree Two

In [3], it was given the following theorem for the finite Blaschke products of the form (1.1) of degree 2.

Theorem 3.1. *(See [3] Theorem 2) Let $B(z) = z(z-a)/(1-\overline{a}z)$ be a Blaschke product with $a \neq 0$. For $\lambda \in \partial\mathbb{D}$, let z_1 and z_2 be the two distinct points satisfying $B(z_1) = B(z_2) = \lambda$. Then the line joining z_1 and z_2 passes through the point a. Conversely, if we consider any line L through the point a, then for the points z_1 and z_2 at which L intersects $\partial\mathbb{D}$ it is the case that $B(z_1) = B(z_2)$.*

Now we give the following theorem about finite Blaschke products of the form (1.4) of degree 2.

Theorem 3.2. *Let the points α and θ be fixed points, f be the transformation of the form (1.2) and $\widetilde{B}(z) = (B \circ f)(z)$ where*

$$B(z) = z\frac{z - f(a)}{1 - \overline{f(a)}z} \text{ and } \mathrm{Im}(a) > 0. \tag{3.1}$$

For $\lambda \in \partial\mathbb{D}$, let x_1 and x_2 be the two distinct points satisfying

$$\widetilde{B}(x_1) = \widetilde{B}(x_2) = \lambda.$$

Then the points x_1, x_2, a and $\overline{\alpha}$ lie on the same circle.

Conversely, if we consider any circle C through the points a and $\overline{\alpha}$, then for the points x_1 and x_2, at which C intersects the real axis, we have $\widetilde{B}(x_1) = \widetilde{B}(x_2)$.

Proof. For any $\lambda \in \partial\mathbb{D}$, let x_1 and x_2 be the two distinct points satisfying

$$\widetilde{B}(x_1) = \widetilde{B}(x_2) = \lambda.$$

By the definition of $\widetilde{B}$, we have

$$B(f(x_1)) = B(f(x_2)).$$

Then by Theorem 3.1, the line L joining the points $f(x_1)$ and $f(x_2)$ passes through the point $f(a)$. Then the image of L under the transformation f^{-1} is a circle passing through the points x_1, x_2 and a. Since the line joining $f(x_1)$, $f(x_2)$ and $f(a)$ through the point ∞, then the circle C through the points x_1, x_2 and a should contains the point $\overline{\alpha}$.

Conversely, let C be any circle passing through the points $a, \overline{\alpha}$. Clearly, C cuts the real axis at two points x_1 and x_2 since $\mathrm{Im}(a) > 0$ and $\mathrm{Im}(\overline{\alpha}) < 0$. Let $z_1 = f(x_1)$ and $z_2 = f(x_2)$ (note that $f(x_1)$ and $f(x_2)$ cannot equal to $e^{i\theta}$). Since $\overline{\alpha} \in C$, the image of $f(C)$ is a line passing through the point $f(a)$ an so by Theorem 3.1 we have $B(z_1) = B(z_2)$, that is

$$B(f(x_1)) = B(f(x_2))$$

and then we obtain

$$\widetilde{B}(x_1) = \widetilde{B}(x_2).$$

□

Example 3.1. Let $\alpha = \frac{i\sqrt{3}}{2}$, $a = \frac{1}{2} + i\frac{1}{2}$, $\theta = \frac{\pi}{3}$ and $\widetilde{B}(z) = (B \circ f)(z)$ where $B(z)$ is the Blaschke product of the form (3.1). Then by Theorem 3.2, any circle passing through the points a and $\overline{\alpha}$ cuts the real axis at two points x_1 and x_2 at which we have $\widetilde{B}(x_1) = \widetilde{B}(x_2)$ (See Figure 1).

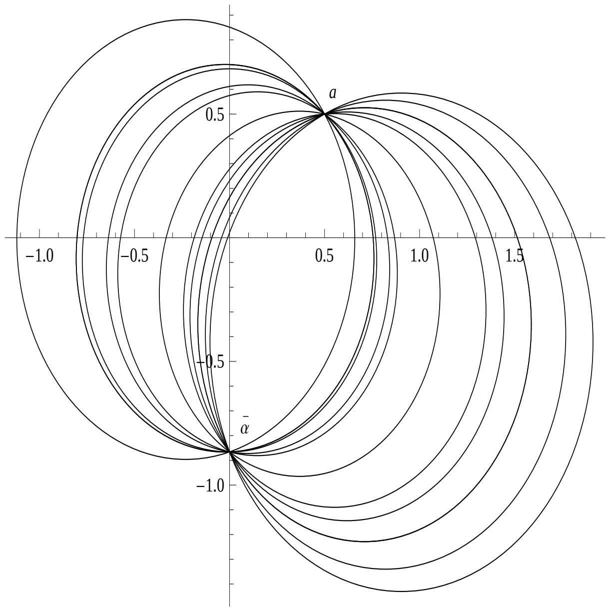

FIGURE 1.

4. BLASCHKE PRODUCTS OF DEGREE THREE

We recall the following theorem.

Theorem 4.1. *(See [3] Theorem 1) Let B be a Blaschke product of degree three of the form (1.1) with distinct zeros at the points 0, a_1 and a_2. For λ on the unit circle, let z_1, z_2 and z_3 denote the points mapped to λ under B. Then the lines joining z_j and z_k for $j \neq k$ are tangent to the ellipse E with equation*

$$|z - a_1| + |z - a_2| = |1 - \overline{a_1} a_2| .$$

Conversely, every point on E is the point of tangency of a line segment joining two distinct points z_1 and z_2 on the unit circle for which $B(z_1) = B(z_2)$.

We give the following theorem about finite Blaschke products for the upper half plane.

Theorem 4.2. *Let $\widetilde{B}(z) = (B \circ f)(z)$ be a finite Blaschke product of degree three where $B(z) = z\frac{z-a_1}{1-\overline{a_1}z}\frac{z-a_2}{1-\overline{a_2}z}$, $|a_1|, |a_2| < 1$ and f be any transformation of the form (1.2). For $\lambda \in \partial\mathbb{D}$, let x_1 and x_2 be the two distinct real points satisfying*

$$\widetilde{B}(x_1) = \widetilde{B}(x_2) = \lambda.$$

Then the circle C passing through the points x_1, x_2 and $\overline{\alpha}$ is tangent to the curve

$$f^{-1}(E) : \left|w(e^{i\theta} - a_1) + a_1\overline{\alpha} - e^{i\theta}\alpha\right| + \left|w(e^{i\theta} - a_2) + a_2\overline{\alpha} - e^{i\theta}\alpha\right| - \left|w(1 - \overline{a_1}a_2) + \overline{\alpha}\overline{a_1}a_2 - \overline{\alpha}\right| = 0. \tag{4.1}$$

Conversely each point of $f^{-1}(E)$ is the point of tangency with the curve $f^{-1}(E)$ and the circle passing through the point $\overline{\alpha}$ and the points x_1, x_2 on the real axis for which $\widetilde{B}(x_1) = \widetilde{B}(x_2)$.

Proof. For any $\lambda \in \partial\mathbb{D}$, let x_1 and x_2 be the two distinct points satisfying

$$\widetilde{B}(x_1) = \widetilde{B}(x_2) = \lambda.$$

By the definition of $\widetilde{B}$, we have

$$B(f(x_1)) = B(f(x_2)).$$

Then by Theorem 4.1, the line L joining the points $f(x_1)$ and $f(x_2)$ is tangent to the ellipse

$$E : |z - a_1| + |z - a_2| = |1 - \overline{a_1}a_2|\,.$$

Then the image of L under the transformation f^{-1} is a circle passing through the points x_1, x_2 and $\overline{\alpha}$. This circle is tangent to the curve with the equation (4.1).

Conversely, let C be any circle passing through the point $\overline{\alpha}$ and tangent to the curve $f^{-1}(E)$. Then clearly C cuts the real axis at two points x_1 and x_2. Let $z_1 = f(x_1)$ and $z_2 = f(x_2)$ (note that $f(x_1)$ and $f(x_2)$ cannot be equal to $e^{i\theta}$). Since $\overline{\alpha} \in C$, the image $f(C)$ is a line passing through the points $f(x_1)$, $f(x_2)$ and tangent to the curve E. By Theorem 4.1, we have $B(z_1) = B(z_2)$, that is, $B(f(x_1)) = B(f(x_2))$ and then we have

$$\widetilde{B}(x_1) = \widetilde{B}(x_2).$$

□

From [1] and [5], we know that the image of ellipses under the transformation

$$T(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$ and $c \neq 0$ cannot be an ellipse. Hence the image of the ellipse $|z - a_1| + |z - a_2| = |1 - \overline{a_1}a_2|$ cannot be an ellipse under the transformation $f^{-1}(z) = \frac{z\overline{\alpha} - e^{i\theta}\alpha}{z - e^{i\theta}}$.

Example 4.1. Let $\alpha = \frac{i\sqrt{3}}{2}$, $\theta = \frac{\pi}{3}$, $a_1 = \frac{1}{2}$, $a_2 = \frac{i}{2}$, and $\widetilde{B}(z) = (B \circ f)(z)$ where $f(z) = e^{i\theta}\frac{z-\alpha}{z-\overline{\alpha}}$ and $B(z) = z\frac{z-a_1}{1-\overline{a_1}z}\frac{z-a_2}{1-\overline{a}_2 z}$. It is clear that for a fixed $\lambda \in \partial\mathbb{D}$, we get three distinct real numbers x_1, x_2, x_3 satisfying $\widetilde{B}(x_1) = \widetilde{B}(x_2) = \widetilde{B}(x_3) = \lambda$. Then the circles passing through x_1, x_2 and $\overline{\alpha}$ are tangent to the curve

$$f^{-1}(E) : \left|w(\tfrac{1+i\sqrt{3}}{2} - \tfrac{1}{2}) - \tfrac{i\sqrt{3}}{4} - (\tfrac{1+i\sqrt{3}}{2})\tfrac{i\sqrt{3}}{2}\right|$$
$$+ \left|w(\tfrac{1+i\sqrt{3}}{2} - \tfrac{i}{2}) + \tfrac{\sqrt{3}}{4} - (\tfrac{1+i\sqrt{3}}{2})\tfrac{i\sqrt{3}}{2}\right| - \left|w(1 - \tfrac{i}{4}) + \tfrac{\sqrt{3}}{8} + \tfrac{i\sqrt{3}}{2}\right| = 0$$

(See Figure 2).

5. Blaschke Products of Degree Four

Let B be any finite Blaschke product of the form (1.1) and of degree 4. Assume that B is a composition of two Blaschke products of degree 2 and E be the Poncelet curve of B. Using Lemma 4, Lemma 5 and Theorem 2 in [6], we see that E is an ellipse if and only if E has the equation

$$E : |z - a_1| + |z - a_2| = |1 - \overline{a_1}a_2| \sqrt{\frac{|a_1|^2 + |a_2|^2 - 2}{|a_1|^2 |a_2|^2 - 1}}. \tag{5.1}$$

Let B be a finite Blaschke product whose Poncelet curve is an ellipse and of degree 4. Let $\widetilde{B}(z) = (B \circ f)(z)$. Similar arguments used in Section 4 can be

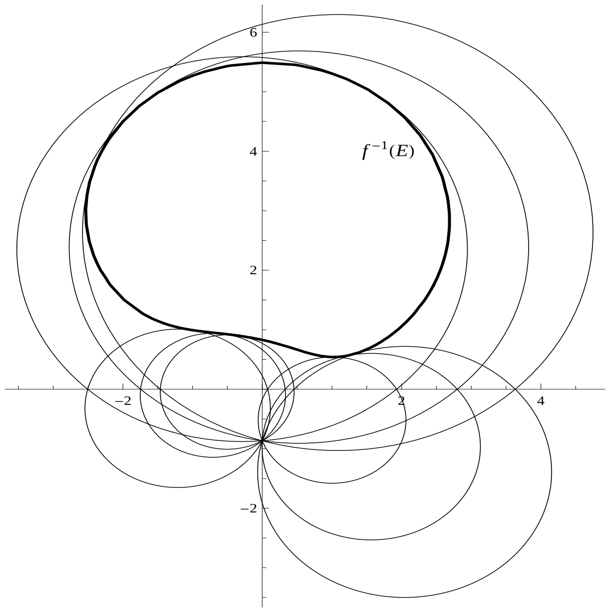

FIGURE 2.

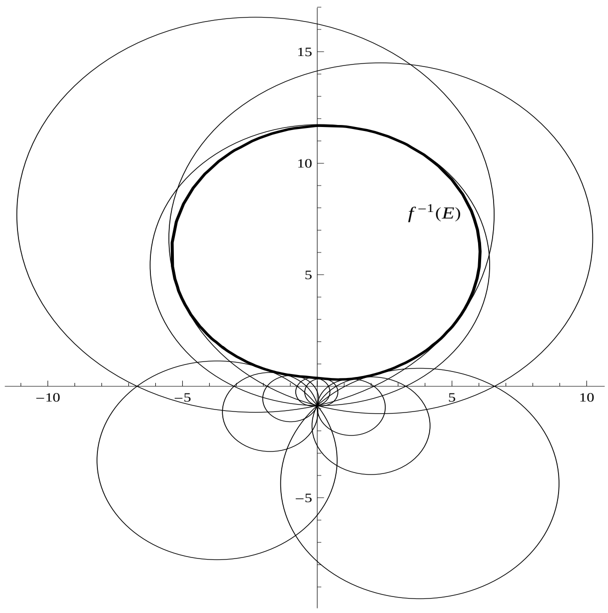

FIGURE 3.

applied to $\widetilde{B}(z)$. As an example let us take $a_1 = \frac{1}{3}, a_2 = \frac{i}{3}, \alpha = \frac{i\sqrt{3}}{2}, \theta = \frac{\pi}{3}$, then we get the following figure (See Figure 3).

Now we recall the following theorem for the finite Blaschke products of degree 4 and of the form (1.1).

Theorem 5.1. (*See* [9] *Theorem* 4.1) *Let* a_1, a_2, a_3 *be three distinct nonzero complex numbers with* $|a_i| < 1$ *for* $1 \leq i \leq 3$ *and* $B(z) = z\prod_{j=1}^{3} \frac{z-a_i}{1-\overline{a_i}z}$ *be a Blaschke product of degree* 4 *with the condition that one of its zeros, say* a_1, *satisfies the following equation:*

$$a_1 + \overline{a_1}a_2a_3 = a_2 + a_3.$$

i) If L *is any line through the point* a_1, *then for the points* z_1 *and* z_2 *at which* L *intersects* $\partial\mathbb{D}$, *we have* $B(z_1) = B(z_2)$.

ii) The unit circle $\partial\mathbb{D}$ *and any circle through the points* 0 *and* $\frac{1}{\overline{a_1}}$ *have exactly two distinct intersection points* z_1 *and* z_2. *Then we have* $B(z_1) = B(z_2)$ *for these intersection points.*

We give the following theorem which has a nice geometric interpretation.

Theorem 5.2. *Let* a_1, a_2 *and* a_3 *be three distinct nonzero complex numbers with* $|a_i| < 1$ *for* $1 \leq i \leq 3$ *and* $B(z) = z\prod_{i=1}^{3} \frac{z-a_i}{1-\overline{a_i}z}$ *be a Blaschke product of degree* 4 *with the condition that one of its zeros, say* a_1, *satisfies the following equation:*

$$a_1 + \overline{a_1}a_2a_3 = a_2 + a_3.$$

Then the Poncelet curve associated with B *is the ellipse* E *with the equation*

$$E : |z - a_2| + |z - a_3| = |1 - \overline{a_2}a_3| \sqrt{\frac{|a_2|^2 + |a_3|^2 - 2}{|a_2|^2 |a_3|^2 - 1}}.$$

Proof. In the proof of Theorem 5.1, it was shown that $B(z)$ can be written as the composition of two Blaschke products of degree 2 as

$$B(z) = B_2 \circ B_1(z)$$

where

$$B_1(z) = z\frac{z - a_1}{1 - \overline{a_1}z} \text{ and } B_2(z) = z\frac{z + a_2a_3}{1 + \overline{a_2a_3}z}.$$

From Lemma 4 in [6], we know that the foci of the ellipse E are the roots of the equation

$$t^2 - (a_1 + \overline{a_1}a_2a_3)t + a_2a_3 = 0. \tag{5.2}$$

By the hypothesis we have $a_1 + \overline{a_1}a_2a_3 = a_2 + a_3$ and so we get

$$t^2 - (a_2 + a_3)t + a_2a_3 = 0.$$

Then the roots of the equation (5.2) are a_2, a_3. So the equation of the ellipse E as the following:

$$E : |z - a_2| + |z - a_3| = |1 - \overline{a_2}a_3| \sqrt{\frac{|a_2|^2 + |a_3|^2 - 2}{|a_2|^2 |a_3|^2 - 1}}.$$

□

Let $B(z)$ be given as in the statement of Theorem 5.2. For any $\lambda \in \partial\mathbb{D}$, let z_1, z_2, z_3 and z_4 be the four distinct points satisfying $B(z_1) = B(z_2) = B(z_3) = B(z_4) = \lambda$. Then the Poncelet curve associated with B is an ellipse E with foci a_2 and a_3 and by Theorem 5.1, the lines joining z_1, z_3 and z_2, z_4 passes through the point a_1.

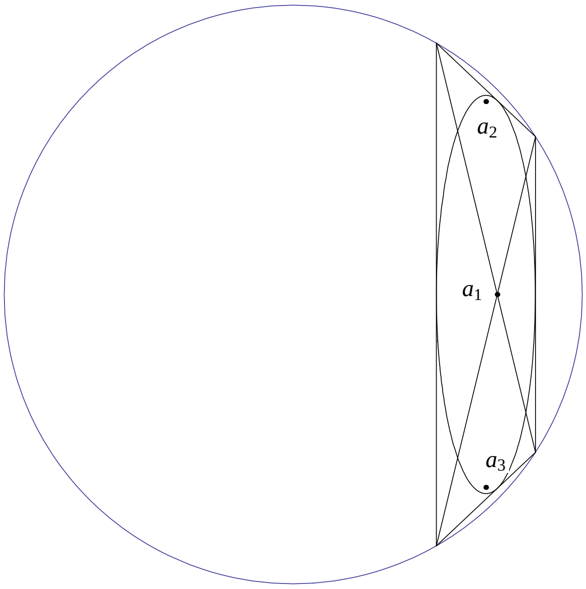

FIGURE 4.

Example 5.1. Let $a_1 = \frac{12}{17}, a_2 = \frac{2}{3} - i\frac{2}{3}, a_3 = \frac{2}{3} + i\frac{2}{3}$ and $B(z) = z\prod_{i=1}^{3} \frac{z-a_i}{1-\overline{a_i}z}$. The Poncelet curve associated with B is an ellipse with foci a_2 and a_3 (See Figure 4).

Using Theorem 5.1, we give another theorem related to finite Blaschke products for the upper half plane.

Theorem 5.3. *Let a_1, a_2, a_3 be three distinct nonzero complex numbers with $|a_i| < 1$ for $1 \leq i \leq 3$ and $B(z) = z\prod_{i=1}^{3} \frac{z-a_i}{1-\overline{a_i}z}$ be a Blaschke product of degree 4 with the condition that one of its zeros, say a_1,satisfies the following equation:*

$$a_1 + \overline{a_1}a_2a_3 = a_2 + a_3.$$

Let f be any transformation of the form (1.2) and $\widetilde{B}(z) = (B \circ f)(z)$, where B is a finite Blaschke product of degree 4 of the above form. If K is any circle through the points $f^{-1}(a_1)$ and $\overline{\alpha}$, then for the points x_1 and x_2 where K intersects the real axis, we have $\widetilde{B}(x_1) = \widetilde{B}(x_2)$.

Proof. In the proof of Theorem 5.1, it was shown that $B(z)$ can be written as the composition of two Blaschke products of degree 2 as

$$B(z) = B_2 \circ B_1(z)$$

where

$$B_1(z) = z\frac{z - a_1}{1 - \overline{a_1}z} \text{ and } B_2(z) = z\frac{z + a_2a_3}{1 + \overline{a_2a_3}z}.$$

Let K be any circle passing through $f^{-1}(a_1)$, $\overline{\alpha}$ and x_1, x_2 be the points at which K intersects the real axis. Let $z_1 = f(x_1)$ and $z_2 = f(x_2)$. Since $\overline{\alpha} \in K$, the image of $f(K)$ is a line passing through the point a_1 and so by Theorem 3.1 we have $B_1(z_1) = B_1(z_2)$ and then we get $B_1(f(x_1)) = B_1(f(x_2))$. Hence, we obtain

$$\widetilde{B}(x_1) = B_2(B_1(f(x_1))) = B_2(B_1(f(x_2))) = \widetilde{B}(x_2).$$

□

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