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INTRINSIC PROOFS OF THE EXISTENCE OF GENERALIZED FINSLER CONNECTIONS

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Abstract. Let $(M, F, \mathcal{H})$ be a Finsler-Ehresmann manifold, where (M, F) is a Finsler manifold endowed with an arbitrary horizontal structure $\mathcal{H}$. In the present paper, we give an intrinsic proof of the existence of generalized Chern connection on $(M, F, \mathcal{H})$. Note that in [3], the author gave a local coordinates proof of this existence which is very long and quite laborious. Subsequently, we give an axiomatic formalism of generalized Cartan, Berwald and Hashiguchi connections on $(M, F, \mathcal{H})$ and establish some relations between these connections and generalized Chern connections.

1. Introduction

The theory of connections is an important field of differential geometry. It was initially developed to solve pure geometrical problems. Going back to the construction of Cartan, Berwald, Chern and Hashiguchi connections on the Finsler manifold (M, F), it easily follows that these connections are not related to the choice of a particular Ehresmann connection i.e. a horizontal distribution $\mathcal{H}$ complementary to $\mathcal{V} = \ker d\pi$ where, $\pi : TM_0 \to M$ and $TM_0 := \{(x, y) \in TM : y \neq 0\}$ in the sense that it defines a direct sum $TTM_0 = \mathcal{H} \oplus \mathcal{V}$. So by providing (M, F) of any choice of subbundle $\mathcal{H} \subset TTM_0$, we obtain the Finsler-Ehresmann manifold, on which we extended the mentioned connections. In [3], the author introduced and studied the case of generalized chern connection but using the local coordinate system on an open subset of M, the existence proof of this connection which he proposes was very long and quite tedious. We therefore propose an improvement of this proof and we treat the case of the other important finslerian connections, namely Cartan, Berwald and Hashiguchi connections. So the purpose of the paper is to give an axiomatic formulation of generalized Cartan, Berwald and Hashiguchi connections. We also provide an intrinsic (coordinate free) proof of the existence and uniqueness theorem of these connections. Our proof have the advantages of being simple, systematic and guided by the Riemannian case.

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If we restrict to the case of classical connections induced by the Finslerian structure, we obtain by using only the pull-back formalism, an intrinsic formulation of these connections. This simplifies what is done in [5], where the study is made by combining the pull-back approach and the one of Klein-Grifone. The paper consists of four parts :
The first part is an introductory section (§2), which provides a brief account of the basic definitions and concepts necessary for this work. For more details, we refer to [1] and [3].

In the second part (§3), we propose an outcome (lemma 3.1) that we used to provide an intrinsic proof of the existence of generalized Chern connection and other connections that we treat. The third part (§4) is devoted to an intrinsic proof of the existence and uniqueness theorem of generalized Cartan connection. Moreover, an intrinsic relationship between this connection and the generalized Chern connection is obtained. The fifth and fourth part (§5, §6), provides an intrinsic proof of the existence and uniqueness theorems of generalized Berwald and Hashiguchi connections on $(M, F, \mathcal{H})$. In the same manner as the others connections, we give an intrinsic relationship between Berwald and chern connection, and Hashiguchi and chern connections on $(M, F, \mathcal{H})$. It is worth mentioning that, the local expressions relating all these connections (Cartan, Berwald, Hashiguchi) to the Chern connection in the classical (M, F) Finsler manifold, coincide with the existing classical local results.

2. Preliminaries

2.1. **Finsler-Ehresmann manifold.** In this section we recall briefly the concept of Finsler-Ehresmann manifold. For a detailed description, the reader may also refer to [3].

Let (M, F) be a Finsler manifold and G^i the spray coefficients of F.
By $N^i_j := \frac{\partial G^i}{\partial y^j}$ we define a vector form

$$\theta_c = \frac{\partial}{\partial x^i} \otimes \frac{1}{F}(dy^i + N^i_j dx^j) \tag{2.1}$$

Then from θ_c and π_*, we can obtain the horizontal distribution $\mathcal{H}_c$ and the vertical distribution $\mathcal{V}$ define by :

$$\mathcal{H}_c := \ker \theta_c \tag{2.2}$$

$$\mathcal{V} := \ker \pi_* \tag{2.3}$$

With the Riemannian metric of Sasaki type induced by F, $\mathcal{H}_c$ is orthogonal to $\mathcal{V}$, and we have the decomposition,

$$TTM_0 = \mathcal{H}_c \oplus \mathcal{V}. \tag{2.4}$$

Therefore the manifold TM_0 admits an Ehresmann connection directly related to objects N^i_j.

Definition 2.1. An Ehresmann connection associated with $\pi : TM_0 \longrightarrow M$ is a smooth distribution $\mathcal{H} \subset TTM_0$ called horizontal subbundle of the connection which is complementary to $\mathcal{V}$ in the sense that it defines the direct sum $TTM_0 = \mathcal{H} \oplus \mathcal{V}$.

The objects N^i_j, play an important role in the construction of the connection on π^*TM. However this construction does not depend either on a particular choice of

N^i_j. Which justifies the generalization made here.
Indeed the projection π being natural, the vertical subbundle $\mathcal{V}$ is determined in the unique way. In contrary, the choice of a complementary of $\mathcal{V}$ namely the horizontal subbundle $\mathcal{H}$ is not determined in such a way as canonical. Thanks to the chosen of an Ehresmann connection, we selecte the subbundle $\mathcal{H}$.

Let $P_{\mathcal{H}}$ a projection on vertical bundle $\mathcal{V}$ along of $\mathcal{H}$. So

$$P_{\mathcal{H}} : TTM_0 \longrightarrow \mathcal{V} \quad \textit{such that} \quad P_{\mathcal{H}} \circ P_{\mathcal{H}} = P_{\mathcal{H}} \text{ and } Im\,(P_{\mathcal{H}}) = \mathcal{V}$$

$\mathcal{H} = \ker P_{\mathcal{H}}$ is the subbundle of the Erhesmann connection.

Recall that locally, once the projector $P_{\mathcal{H}}$ is chooses, it acts on the basis $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ of TTM_0, in the following way :

$$P_{\mathcal{H}}(\frac{\partial}{\partial x^i}) = \mathbf{N}^j_i \frac{\partial}{\partial y^j}, \tag{2.5}$$

$$P_{\mathcal{H}}(\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial y^i} \tag{2.6}$$

Where the coefficients $\mathbf{N}^i_j$ correspond to the choice of the connection $\mathcal{H}$.

It is easy to verify that the vector fields

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - \mathbf{N}^j_i \frac{\partial}{\partial y^j}, \tag{2.7}$$

form a basis of $\mathcal{H} = \ker P_{\mathcal{H}}$. Similary we have the covector

$$\delta y^i = dy^i + \mathbf{N}^i_j dx^j. \tag{2.8}$$

For the sake of computations let us introduce the morphism $\bar{\theta} : TTM_0 \longrightarrow \pi^*TM$ define by : $\bar{\theta} = \nu \circ P_{\mathcal{H}}$ where ν is the canonical map $\nu : \mathcal{V} \longrightarrow \pi^*TM$.

Therefore each subbundle is defined as

$$\mathcal{H} := \ker \bar{\theta}. \tag{2.9}$$

Definition 2.2. A Finsler-Ehresmann manifold which we denote by $(M, F, \mathcal{H})$ is a finslerian structure equipped with the Ehresmann connection $\mathcal{H}$.

Remark 2.1. In order to be coherent with the conventions of working with objects invariant under the transformation $(y \longrightarrow \lambda y)$ we will used $\theta := \frac{\bar{\theta}}{F}$ instead of $\bar{\theta}$. And locally we have

$$\theta\left(F\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial x^i},$$
$$\theta\left(\frac{\delta}{\delta x^i}\right) = 0,$$

3. Generalized Chern connection.

In this section we improve the result in [3] by providing an intrinsic proof of the existence theorem of generalized Chern connection.
But before set out this proof, recall that a finslerian structure F on the manifold M defines a fundamental tensor g. This tensor is defined on TM_0 and is a metric on the pulled-back bundle π^*TM [1].

As in the riemannian case we have the following result:

Lemma 3.1. *Let (M, F) be a Finsler manifold, g the fundamental tensor of F and π^*TM the pulled-back bundle on TM_0 . Let $\xi \in \Gamma(\pi^*TM)$ there exists precisely one section $\xi^\flat \in \Gamma(\pi^*T^*M)$ such that for all $\eta \in \Gamma(\pi^*TM)$, $\xi^\flat(\eta) = g(\xi, \eta)$.*
And conversely, let $\alpha \in \Gamma(\pi^*T^*M)$ there exists precisely one section $\alpha^\sharp \in \Gamma(\pi^*TM)$ such that for all $\eta \in \Gamma(\pi^*TM)$, $g(\alpha^\sharp, \eta) = \alpha(\eta)$.

Proof. Note that, the space $\Gamma(\pi^*TM)$ is the module on $C^\infty(TM_0)$. So the metric g defined a $C^\infty(TM_0)$-linear product on $\Gamma(\pi^*TM)$ with values in $C^\infty(TM_0)$:

$$g : \Gamma(\pi^*TM) \times \Gamma(\pi^*TM) \longrightarrow C^\infty(TM_0). \tag{3.1}$$

By posing, for all $(x, y) \in TM_0$,

$$g(\xi, \eta)(x, y) = g(x, y)(\xi_{(x,y)}, \eta_{(x,y)}). \tag{3.2}$$

As g is nondegenerate, the $C^\infty(TM_0)$-linear map

$$\begin{array}{rccc} g^\flat : & \Gamma(\pi^*TM) & \longrightarrow & \Gamma(\pi^*T^*M) \\ & \xi & \longmapsto & g^\flat(\xi) \end{array} \tag{3.3}$$

such that $g^\flat(\xi)(x, y) = g(x, y)(\xi_{(x,y)}, \bullet)$, for all $(x, y) \in TM_0$, is an isomorphism and, the $C^\infty(TM_0)$-linear map:

$$\begin{array}{rccc} g^\sharp : & \Gamma(\pi^*T^*M) & \longrightarrow & \Gamma(\pi^*TM) \\ & \alpha & \longmapsto & g^\sharp(\alpha) \end{array} \tag{3.4}$$

such that $\alpha_{(x,y)} = g(x, y)((g^\sharp(\alpha))_{(x,y)}, \bullet)$ for all $(x, y) \in TM_0$ is the reciprocal isomorphism. The result follows from these isomorphisms. □

Note that the pullback bundle π^*TM, is the quotient of the tangent bundle TTM_0 and we have the following short sequence of vectors bundle, relating the tangent bundle TTM_0 and the pullback bundle π^*TM:

$$0 \longrightarrow \mathcal{V} \xrightarrow{i} TTM_0 \xrightarrow{\pi_*} \pi^*TM \longrightarrow 0$$

Where $\pi_* : TTM_0 \longrightarrow \pi^*TM$ is the derivative of the map $\pi : TM_0 \longrightarrow M$.

The purpose is to build on our Finsler-Ehresmann manifold $(M, F, \mathcal{H})$, a Koszulian formulation of Finsler connection ∇:

$$\begin{array}{rcl} \nabla : \Gamma(TTM_0) \times & \Gamma(\pi^*TM) & \longrightarrow \Gamma(\pi^*TM) \\ & (X, \xi) & \longmapsto \nabla_X \xi \end{array} \tag{3.5}$$

This is a connection on π^*TM associated to an arbitrary choice of $\mathcal{H} = \ker \theta$. It is called a generalized Chern connection.

Proposition 3.1. [3] *Let $(M, F, \mathcal{H})$ be a Finsler-Ehresmann manifold and g, a fundamental tensor of F. There exist a unique linear connection ∇ on π^*TM such that for all $X, Y \in \Gamma(TTM_0)$ and $\xi, \eta \in \Gamma(\pi^*TM)$ we have the following :*

(a) *Symmetry*

$$\nabla_X \pi_* Y - \nabla_Y \pi_* X = \pi_*[X, Y], \tag{3.6}$$

(b) *Almost g-compatibility*

$$(\nabla_X g)(\xi, \eta) = 2A(\theta(X), \xi, \eta), \tag{3.7}$$

where A is the Cartan tensor.

A proof of the above proposition is given in [3] but the author has done in a local approach through a local coordinates system on an open set of M. What makes the proof very long and laborious. Thus hiding the intrinsic aspect which we would expect.

So we propose an intrinsic proof of the existence of this connection, with the advantage that it is short, simple and similar to what is done in the Riemannian case for the Levi-Civita connection.

Proof. **Existence**. Consider the equation of almost g-compatibility (3.7) for the two sections $\pi_*Y, \pi_*Z \in \Gamma(\pi^*TM)$, then :

$$X.g(\pi_*Y, \pi_*Z) = g(\nabla_X \pi_*Y, \pi_*Z) + g(\pi_*Y, \nabla_X \pi_*Z) + 2A(\theta(X), \pi_*Y, \pi_*Z). \tag{3.8}$$

Cyclic permutation on (X, Y, Z) in the above formula yields three equations. Adding two of these equations and subtracting the third, gives by the condition (a) the following equation :

$$\begin{aligned} 2g(\nabla_X \pi_*Y, \pi_*Z) &= X.g(\pi_*Y, \pi_*Z) + Y.g(\pi_*Z, \pi_*X) - Z.g(\pi_*X, \pi_*Y) \\ &\quad + g(\pi_*[X,Y], \pi_*Z) - g(\pi_*[Y,Z], \pi_*X) + g(\pi_*[Z,X], \pi_*Y) \\ &\quad - 2\mathcal{A}(X,Y,Z), \end{aligned} \tag{3.9}$$

where

$$\mathcal{A}(X,Y,Z) = A(\theta(X), \pi_*Y, \pi_*Z)) + A(\theta(Y), \pi_*Z, \pi_*X)) - A(\theta(Z), \pi_*X, \pi_*Y)). \tag{3.10}$$

For fixed $X, Y \in \Gamma(TTM_0)$, we consider the 1-form $\frac{1}{2}\omega \in \Gamma(\pi^*T^*M)$ assigning the right hand side of (3.9) to each $\pi_*Z \in \Gamma(\pi^*TM)$ i.e

$$\begin{aligned} \frac{1}{2}\omega(\pi_*Z) &= X.g(\pi_*Y, \pi_*Z) + Y.g(\pi_*Z, \pi_*X) - Z.g(\pi_*X, \pi_*Y) \\ &\quad + g(\pi_*[X,Y], \pi_*Z) - g(\pi_*[Y,Z], \pi_*X) + g(\pi_*[Z,X], \pi_*Y) \\ &\quad - 2\mathcal{A}(X,Y,Z), \quad \forall Z \in \Gamma(TTM_0). \end{aligned} \tag{3.11}$$

Then $\frac{1}{2}\omega$ is tensorial in π_*Z. In fact, for $f \in C^\infty(TM_0)$

$$\begin{aligned} \frac{1}{2}\omega(f\pi_*Z) &= Xg(\pi_*Y, f\pi_*Z) + Yg(f\pi_*Z, \pi_*X) - fZg(\pi_*X, \pi_*Y) \\ &\quad + g(\pi_*[X,Y], f\pi_*Z) - g(\pi_*[Y,fZ], \pi_*X) + g(\pi_*[fZ,X], \pi_*Y) \\ &\quad - 2\mathcal{A}(X,Y,fZ) \\ &= (Xf)g(\pi_*Y, \pi_*Z) + fXg(\pi_*Y, \pi_*Z) + (Yf)g(\pi_*Z, \pi_*X) \\ &\quad + fYg(\pi_*Z, \pi_*X) - fZg(\pi_*X, \pi_*Y) + fg(\pi_*[X,Y], \pi_*Z) \\ &\quad - (Yf)g(\pi_*Z, \pi_*X) - fg(\pi_*[Y,Z], \pi_*X)) - (Xf)g(\pi_*Z, \pi_*Y)) \\ &\quad + fg(\pi_*[X,Z], \pi_*Y)) - 2f\mathcal{A}(X,Y,Z) \\ &= \frac{1}{2}f\omega(\pi_*Z), \end{aligned} \tag{3.12}$$

and the additivity in π_*Z is obvious.
Therefore, by lemma 2.1, there exists precisely one section $\xi \in \Gamma(\pi^*TM)$ such that

$$\omega(\pi_*Z) = 2g(\xi, \pi_*Z)$$

We thus put $\nabla_X \pi_*Y := \xi$. It remains to show that this defines a connection verifying the conditions (a) and (b) of the theorem : The additivity with respect to X and Y is clear, the tensorial behavior with respect to X follows as in (3.12) and the derivative property

$$\nabla_X \pi_* fY = f\nabla_X \pi_* Y + ((\pi_* X)f)\pi_* Y$$

in the same manner. For the condition (a) , we have by (3.9) ,

$$\begin{aligned}
2g(\nabla_X \pi_* Y - \nabla_Y \pi_* X, \pi_* Z) &= 2g(\nabla_X \pi_* Y, \pi_* Z) - 2g(\nabla_Y \pi_* X, \pi_* Z) \\
&= X.g(\pi_* Y, \pi_* Z) + Y.g(\pi_* Z, \pi_* X) - Z.g(\pi_* X, \pi_* Y) \\
&+ g(\pi_*[X, Y], \pi_* Z) - g(\pi_*[Y, Z], \pi_* X) + g(\pi_*[Z, X], \pi_* Y) \\
&- 2\mathcal{A}(X, Y, Z) - Yg(\pi_* X, \pi_* Z) - Xg(\pi_* Z, \pi_* Y) \\
&+ Z.g(\pi_* X, \pi_* Y) - g(\pi_*[Y, X], \pi_* Z) + g(\pi_*[X, Z], \pi_* Y) \\
&- g(\pi_*[Z, Y], \pi_* X) + 2\mathcal{A}(X, Y, Z) \\
&= 2g(\pi_*[X, Y], \pi_* Z) \qquad (3.13)
\end{aligned}$$

Therefore

$$\nabla_X \pi_* Y - \nabla_Y \pi_* X = \pi_*[X, Y]$$

Since g is nondegenerate. Likewise we deduce (b) from (3.9), in the same manner. For the uniqueness, the proof is the same as in [3]. □

4. Generalized Cartan connection

The aim of the present section is to build firstly an axiomatic formulation of generalized Cartan connection $\bar{\nabla}$ and secondly to provide an intrinsic proof of the existence and uniqueness theorem for this connection. More over we give an intrinsic relationship between Cartan and chern connection and, we show that the local expressions of this relation in the canonical case coincide with the existing classical local results.

Theorem 4.1. *Let $(M, F, \mathcal{H})$ be a Finsler-Ehresmann manifold and g be a fundamental tensor of F. There exist a unique linear connection $\bar{\nabla}$ on π^*TM such that for all $X, Y \in \Gamma(TTM_0)$:*

(a) *Symmetry*

$$\bar{\nabla}_X \pi_* Y - \bar{\nabla}_Y \pi_* X = \pi_*[X, Y] + \left(A^\sharp(\theta(X), \pi_* Y, \bullet) - A^\sharp(\theta(Y), \pi_* X, \bullet)\right), \qquad (4.1)$$

(b) *Metric-compatibility*

$$\bar{\nabla} g = 0, \qquad (4.2)$$

where, $A^\sharp(\xi, \eta, \bullet)$ is the section of π^*TM define by :

$$g(A^\sharp(\xi, \eta, \bullet), \mu) = A(\xi, \eta, \mu) \quad \forall \xi, \eta, \mu \in \Gamma(\pi^*TM) \qquad (4.3)$$

Proof. Uniqueness . Since $\bar{\nabla}$ is metric with respect to fundamental tensor g, we have for all $X, Y, Z \in \Gamma(TTM_0)$:

$$X.g(\pi_* Y, \pi_* Z) = g(\bar{\nabla}_X \pi_* Y, \pi_* Z) + g(\pi_* Y, \bar{\nabla}_X \pi_* Z). \qquad (4.4)$$

Cyclic permutation on (X, Y, Z) in the above formula yields three equations. Adding two of these equations and subtracting the third, it follows from condition ***a)*** :

$$\begin{aligned} 2g(\bar\nabla_X \pi_*Y, \pi_*Z) &= X.g(\pi_*Y, \pi_*Z) + Y.g(\pi_*Z, \pi_*X) - Z.g(\pi_*X, \pi_*Y) \\ &+ g(\pi_*[X, Y], \pi_*Z) - g(\pi_*[Y, Z], \pi_*X) + g(\pi_*[Z, X], \pi_*Y) \\ &+ g\left(A^\sharp(\theta(X), \pi_*Y, \bullet) - A^\sharp(\theta(Y), \pi_*X, \bullet), \pi_*Z)\right) \\ &- g\left(A^\sharp(\theta(Y), \pi_*Z, \bullet) - A^\sharp(\theta(Z), \pi_*Y, \bullet), \pi_*X)\right) \\ &+ g\left(A^\sharp(\theta(Z), \pi_*X, \bullet) - A^\sharp(\theta(X), \pi_*Z, \bullet), \pi_*Y)\right) \\ &\overset{(3.3)}{=} X.g(\pi_*Y, \pi_*Z) + Y.g(\pi_*Z, \pi_*X) - Z.g(\pi_*X, \pi_*Y) \\ &+ g(\pi_*[X, Y], \pi_*Z) - g(\pi_*[Y, Z], \pi_*X) + g(\pi_*[Z, X], \pi_*Y) \\ &+ 2A(\theta(Z), \pi_*X, \pi_*Y) - 2A(\theta(Y), \pi_*X, \pi_*Z). \end{aligned}$$

Then,

$$\begin{aligned} 2g(\bar\nabla_X \pi_*Y, \pi_*Z) &= X.g(\pi_*Y, \pi_*Z) + Y.g(\pi_*Z, \pi_*X) - Z.g(\pi_*X, \pi_*Y) \\ &+ g(\pi_*[X, Y], \pi_*Z) - g(\pi_*[Y, Z], \pi_*X) + g(\pi_*[Z, X], \pi_*Y) \\ &+ 2A(\theta(Z), \pi_*X, \pi_*Y) - 2A(\theta(Y), \pi_*X, \pi_*Z). \end{aligned} \tag{4.5}$$

Now if there are two connections $\bar\nabla^1$ and $\bar\nabla^2$ satisfying the conditions (a) and (b) then the relation (4.5) implies that

$$g(\bar\nabla^1_X \xi - \bar\nabla^2_X \xi, \eta) = 0, \quad \forall X \in \Gamma(TTM_0) \; and \; \xi, \eta \in \Gamma(\pi^*TM) \tag{4.6}$$

And the uniqueness follows.

Existence. For fixed $X, Y \in \Gamma(TTM_0)$, we consider the 1-form $\frac{1}{2}\bar\omega \in \Gamma(\pi^*T^*M)$ assigning the right hand side of (4.5) to each $\pi_*Z \in \Gamma(\pi^*TM)$. Then $\frac{1}{2}\bar\omega$ is tensorial in π_*Z. In fact, for $f \in C^\infty(TM_0)$

$$\begin{aligned} &\frac{1}{2}\bar\omega(f\pi_*Z) = X.g(\pi_*Y, f\pi_*Z) + Y.g(f\pi_*Z, \pi_*X) - fZ.g(\pi_*X, \pi_*Y) \\ &+ g(\pi_*[X, Y], f\pi_*Z) - g(\pi_*[Y, fZ], \pi_*X) + g(\pi_*[fZ, X], \pi_*Y) \\ &+ 2A(f\theta(Z), \pi_*X, \pi_*Y) - 2A(\theta(Y), \pi_*X, f\pi_*Z) \\ &= (Xf)g(\pi_*Y, f\pi_*Z) + fX.g(\pi_*Y, f\pi_*Z) + (Yf).g(\pi_*Z, \pi_*X) + fY.g(\pi_*Z, \pi_*X) \\ &- fZ.g(\pi_*X, \pi_*Y) + fg(\pi_*[X, Y], \pi_*Z)) - (Yf).g(\pi_*Z, \pi_*X) - fg(\pi_*[Y, Z], \pi_*X) \\ &+ fg(\pi_*[Z, X], \pi_*Y) - (Xf)g(\pi_*Y, f\pi_*Z) + 2f\,(A(\theta(Z), \pi_*X, \pi_*Y) \\ &- A(\theta(Y), \pi_*X, \pi_*Z)) \\ &= \frac{1}{2}f\bar\omega(\pi_*Z) + (Xf)g(\pi_*Y, \pi_*Z) + (Yf)g(\pi_*Z, \pi_*X) \\ &- (Xf)g(\pi_*Y, \pi_*Z) - (Yf)g(\pi_*Z, \pi_*X) \\ &= \frac{1}{2}f\bar\omega(\pi_*Z), \end{aligned} \tag{4.7}$$

and the additivity in π_*Z is obvious. Therefore, by lemma 2.1 there exists precisely one section $\xi \in \Gamma(\pi^*T^*M)$ such that

$$\bar\omega(\pi_*Z) = 2g(\xi, \pi_*Z)$$

We thus put $\bar{\nabla}_X \pi_* Y := \xi$. It remains to show that $\bar{\nabla}$ define a connection verifying the conditions (a) and (b) of theorem : The additivity with respect to X and Y is clear, the tensorial behavior with respect X follows as in (4.7) and the derivative property

$$\bar{\nabla}_X \pi_* f Y = f \bar{\nabla}_X \pi_* Y + (Xf)\pi_* Y$$

in the same manner. Indeed, we have :

$$\begin{aligned}
&2g(\bar{\nabla}_X f\pi_* Y, \pi_* Z) = X.g(f\pi_* Y, \pi_* Z) + fY.g(\pi_* Z, \pi_* X) - Z.g(\pi_* X, f\pi_* Y) \\
&+ g(\pi_*[X, fY], \pi_* Z) - g(\pi_*[fY, Z], \pi_* X) + g(\pi_*[Z, X], f\pi_* Y) \\
&+ 2A(\theta(Z), \pi_* X, f\pi_* Y) - 2A(\theta(fY), \pi_* X, \pi_* Z) \\
&= (Xf)g(\pi_* Y, f\pi_* Z) + fX.g(\pi_* Y, f\pi_* Z) + fY.g(\pi_* Z, \pi_* X) - (Zf).g(\pi_* Y, \pi_* X) \\
&- fZ.g(\pi_* Y, \pi_* X) + (Xf)g(\pi_* Y, \pi_* Z) + fg(\pi_*[X, Y], \pi_* Z) + (Zf).g(\pi_* Y, \pi_* X) \\
&- fg(\pi_*[Y, Z], \pi_* X) + fg(\pi_*[Z, X], \pi_* Y) + 2fA(\theta(Z), \pi_* X, \pi_* Y) \\
&- 2fA(\theta(Y), \pi_* X, \pi_* Z) \\
&\overset{(3.5)}{=} 2fg(\bar{\nabla}_X \pi_* Y, \pi_* Z) + 2(Xf)g(\pi_* Y, \pi_* Z)
\end{aligned}$$

$$= 2g(f\bar{\nabla}_X \pi_* Y + (Xf)\pi_* Y, \pi_* Z), \tag{4.8}$$

where

$$\bar{\nabla}_X \pi_* f Y = f \bar{\nabla}_X \pi_* Y + (Xf)\pi_* Y$$

For the condition **a)**, setting

$$T(X, Y) = \bar{\nabla}_X \pi_* Y - \bar{\nabla}_Y \pi_* X - \pi_*[X, Y]. \tag{4.9}$$

We obtain :

$$\begin{aligned}
2g(T(X, Y), \pi_* Z) &= 2g(\bar{\nabla}_X \pi_* Y, \pi_* Z) - 2g(\bar{\nabla}_Y \pi_* X, \pi_* Z) - 2g(\pi_*[X, Y], \pi_* Z) \\
&\overset{(3.5)}{=} A(\theta(X), \pi_* Y, \pi_* Z)) - A(\theta(Y), \pi_* X, \pi_* Z)) \\
&\overset{(3.3)}{=} 2g\left(A^\sharp(\theta(X), \pi_* Y, \bullet) - A^\sharp(\theta(Y), \pi_* X, \bullet), \pi_* Z\right),
\end{aligned}$$

for any $\pi_* Z \in \Gamma(\pi^* TM)$. It follows that,

$$T(X, Y) = A^\sharp(\theta(X), \pi_* Y, \bullet) - A^\sharp(\theta(Y), \pi_* X, \bullet). \tag{4.10}$$

Since g is nondegenerate.

In the same manner, the condition **b)** follows from (4.2). In fact

$$\begin{aligned}
&2(\bar{\nabla}_X g)(\pi_* Y, \pi_* Z) = 2X.g(\pi_* Y, \pi_* Z) - 2g(\bar{\nabla}_X \pi_* Y, \pi_* Z) - 2g(\bar{\nabla}_X \pi_* Z, \pi_* Y) \\
&= 2X.g(\pi_* Y, \pi_* Z) - X.g(\pi_* Y, \pi_* Z) + Y.g(\pi_* Z, \pi_* X) - Z.g(\pi_* X, \pi_* Y) \\
&- g(\pi_*[X, Y], \pi_* Z) + g(\pi_*[Y, Z], \pi_* X) - g(\pi_*[Z, X], \pi_* Y) \\
&- 2A(\theta(Z), \pi_* X, \pi_* Y) - 2A(\theta(Y), \pi_* X, \pi_* Z) - X.g(\pi_* Y, \pi_* Z) - Y.g(\pi_* Z, \pi_* X) \\
&+ Z.g(\pi_* X, \pi_* Y) - g(\pi_*[X, Y], \pi_* Z) + g(\pi_*[Y, Z], \pi_* X) - g(\pi_*[Z, X], \pi_* Y)
\end{aligned}$$

$$+ 2A(\theta(Z), \pi_* X, \pi_* Y) + 2A(\theta(Y), \pi_* X, \pi_* Z) = 0 \tag{4.11}$$

for any $X, Y, Z \in \Gamma(TTM_0)$, where the existence and unicity of $\bar{\nabla}$. □

Corollary 4.1. *The generalized Cartan connection $\bar{\nabla}$ is explicitly expressed in terms of the generalized Chern connection ∇ in the form :*

$$\bar{\nabla}_X \pi_* Y = \nabla_X \pi_* Y + A^\sharp(\theta(X), \pi_* Y, \bullet). \tag{4.12}$$

Proof. Replacing $X, Y, Z \in \Gamma(TTM_0)$ by X^H, Y^H, Z^H in (3.9) and (4.5), we get

$$2g(\bar{\nabla}_{X^H} \pi_* Y^H, \pi_* Z^H) = 2g(\nabla_{X^H} \pi_* Y^H, \pi_* Z^H)$$

It follow that $\bar{\nabla}_{X^H} \pi_* Y^H = \nabla_{X^H} \pi_* Y^H$. Similary, replacing X, Y by X^V, Y^H in (4.1) we obtain

$$\bar{\nabla}_{X^V} \pi_* Y^H = A^\sharp(\theta(X^V), \pi_* Y^H, \bullet)$$

From wich $\bar{\nabla}_X \pi_* Y = \nabla_X \pi_* Y + A^\sharp(\theta(X), \pi_* Y, \bullet)$. □

Remark 4.1. In the classical Finsler manifold ie for $\mathcal{H} = \mathcal{H}_c$ and $\theta = \theta_c$, we find locally the well-known relation between Cartan and Chern connection [1].

$$\bar{\alpha}^i_j = \alpha^i_j + A^i_{jk} \frac{\delta y^k}{F}, \tag{4.13}$$

where $\bar{\alpha}$ and α are respectively the Cartan and Chern connection 1-forms.

5. Generalized Berwald connection

In this section we provide an intrinsic proof of an intrinsic version of the existence and uniqueness theorem for the generalized Berwald connection $\tilde{\nabla}$. Moreover, we deduce an explicit expression relating this connection and Chern connection.

Theorem 5.1. *Let $(M, F, \mathcal{H})$ be a Finsler-Ehresmann manifold and g be a fundamental tensor of F. There exist a unique linear connection $\tilde{\nabla}$ on π^*TM such that for all $X, Y \in \Gamma(TTM_0)$, $\xi, \eta \in \Gamma(\pi^*TM)$:*

(a) *Symmetry*

$$\tilde{\nabla}_X \pi_* Y - \tilde{\nabla}_Y \pi_* X = \pi_*[X, Y], \tag{5.1}$$

(b) *Almost g-compatibility*

$$(\tilde{\nabla}_X g)(\xi, \eta) = 2A(\theta(X), \xi, \eta) - 2L(\pi_* X, \xi, \eta), \tag{5.2}$$

where L is the Landsberg tensor associated with the Ehresmann connection.

Proof. Uniqueness . As in the proof of theorem 2.1, for all $X, Y, Z \in \Gamma(TTM_0)$:
(5.3)
$$X.g(\pi_* Y, \pi_* Z) = g(\bar{\nabla}_X \pi_* Y, \pi_* Z) + g(\pi_* Y, \bar{\nabla}_X \pi_* Z) + 2A(\theta(X), \xi, \eta) - 2L(\pi_* X, \xi, \eta).$$

Cyclic permutation on (X, Y, Z) in the above formula yields three equations. Adding two of these equations and subtracting the third, gives by the condition ***a)*** the following relation : we have.

$$\begin{aligned} 2g(\tilde{\nabla}_X \pi_* Y, \pi_* Z) &= X.g(\pi_* Y, \pi_* Z) + Y.g(\pi_* Z, \pi_* X) - Z.g(\pi_* X, \pi_* Y) \\ &\quad + g(\pi_*[X, Y], \pi_* Z) - g(\pi_*[Y, Z], \pi_* X) + g(\pi_*[Z, X], \pi_* Y) \\ &\quad - 2\mathcal{A}(X, Y, Z) + 2\mathcal{L}(X, Y, Z), \end{aligned} \tag{5.4}$$

where

$$\begin{aligned}\mathcal{A}(X,Y,Z) &= A(\theta(X),\pi_*Y,\pi_*Z)+A(\theta(Y),\pi_*Z,\pi_*X)-A(\theta(Z),\pi_*X,\pi_*Y)\\ \mathcal{L}(X,Y,Z) &= L(\pi_*X,\pi_*Y,\pi_*Z)+L(\pi_*Y,\pi_*Z,\pi_*X)-L(\pi_*Z,\pi_*X,\pi_*Y)\\ &= L(\pi_*X,\pi_*Y,\pi_*Z)\end{aligned} \tag{5.5}$$

Then, if there are two connections $\tilde{\nabla}^1$ and $\tilde{\nabla}^2$ satisfying the conditions (a) and (b) then the relation (3.4) implies that

$$g(\tilde{\nabla}^1_X\xi-\tilde{\nabla}^2_X\xi,\eta)=0,\quad \forall X\in\Gamma(TTM_0)\ and\ \xi,\eta\in\Gamma(\pi^*TM), \tag{5.6}$$

and the uniqueness follows. Existence. Fixing $X,Y\in\Gamma(TTM_0)$, we consider as for the proof of proposition 2.1 the 1-form $\frac{1}{2}\tilde{\omega}\in\Gamma(\pi^*T^*M)$ assigning the right hand side of (3.4) to each $\pi_*Z\in\Gamma(\pi^*TM)$. Then $\frac{1}{2}\tilde{\omega}$ is tensorial in π_*Z. In fact, for $f\in C^\infty(TM_0)$,

$$\begin{aligned}&\frac{1}{2}\tilde{\omega}(f\pi_*Z)=X.g(\pi_*Y,f\pi_*Z)+Y.g(f\pi_*Z,\pi_*X)-fZ.g(\pi_*X,\pi_*Y)\\ &+g(\pi_*[X,Y],f\pi_*Z)-g(\pi_*[Y,fZ],\pi_*X)+g(\pi_*[fZ,X],\pi_*Y)\\ &-2f\mathcal{A}(X,Y,Z)+2fL(\pi_*X,\pi_*Y,\pi_*Z)\\ &=(Xf)g(\pi_*Y,f\pi_*Z)+fX.g(\pi_*Y,f\pi_*Z)+(Yf).g(\pi_*Z,\pi_*X)+fY.g(\pi_*Z,\pi_*X)\\ &-fZ.g(\pi_*X,\pi_*Y)+fg(\pi_*[X,Y],\pi_*Z))-(Yf).g(\pi_*Z,\pi_*X)-fg(\pi_*[Y,Z],\pi_*X)\\ &+fg(\pi_*[Z,X],\pi_*Y)-(Xf)g(\pi_*Y,f\pi_*Z)-2f\mathcal{A}(X,Y,Z)+2fL(\pi_*X,\pi_*Y,\pi_*Z)\\ &=\frac{1}{2}f\tilde{\omega}(\pi_*Z)+(Xf)g(\pi_*Y,\pi_*Z)+(Yf)g(\pi_*Z,\pi_*X)\\ &-(Xf)g(\pi_*Y,\pi_*Z)-(Yf)g(\pi_*Z,\pi_*X)\\ &=\frac{1}{2}f\tilde{\omega}(\pi_*Z),\end{aligned} \tag{5.7}$$

The additivity in π_*Z is obvious. It follows by lemma 2.1 that there exists precisely one section $\xi\in\Gamma(\pi^*T^*M)$ such that

$$\tilde{\omega}(\pi_*Z)=2g(\xi,\pi_*Z).$$

We thus put $\tilde{\nabla}_X\pi_*Y:=\xi$. It remains to show that $\tilde{\nabla}$ define a connection verifying the conditions (a) and (b). $\tilde{\nabla}$ satisfies condition (a): we have,

$$\begin{aligned}2g(\tilde{\nabla}_X\pi_*Y-\tilde{\nabla}_Y\pi_*X,\pi_*Z)&=2g(\tilde{\nabla}_X\pi_*Y,\pi_*Z)-2g(\tilde{\nabla}_Y\pi_*X,\pi_*Z)\\ &\overset{(4.4)}{=}X.g(\pi_*Y,\pi_*Z)+Y.g(\pi_*Z,\pi_*X)-Z.g(\pi_*X,\pi_*Y)\\ &+g(\pi_*[X,Y],\pi_*Z)-g(\pi_*[Y,Z],\pi_*X)+g(\pi_*[Z,X],\pi_*Y)\\ &-2\mathcal{A}(X,Y,Z)+2\mathcal{L}(X,Y,Z)-Y.g(\pi_*X,\pi_*Z)\\ &-X.g(\pi_*Z,\pi_*Y)+Z.g(\pi_*Y,\pi_*X)-g(\pi_*[Y,X],\pi_*Z)\\ &+g(\pi_*[X,Z],\pi_*Y)-g(\pi_*[Z,Y],\pi_*X)+2\mathcal{A}(X,Y,Z)-2\mathcal{L}(X,Y,Z)\\ &=2g(\pi_*[X,Y],\pi_*Z)\end{aligned} \tag{5.8}$$

From which,

$$\tilde{\nabla}_X\pi_*Y-\tilde{\nabla}_Y\pi_*X=\pi_*[X,Y].$$

$\tilde{\nabla}$ satisfies condition (b): setting $\xi = \pi_* Y$ and $\eta = \pi_* Z$ for $Y, Z \in \Gamma(TTM_0)$, we obtain:

$$\begin{aligned}
2(\tilde{\nabla}_X g)(\xi, \eta) &= 2(\tilde{\nabla}_X g)(\pi_* Y, \pi_* Z) \\
&= 2X.g(\pi_* Y, \pi_* Z) - 2g(\tilde{\nabla}_X \pi_* Y, \pi_* Z) - 2g(\tilde{\nabla}_X \pi_* Z, \pi_* Y) \\
&= 2X.g(\pi_* Y, \pi_* Z) - X.g(\pi_* Y, \pi_* Z) - Y.g(\pi_* Z, \pi_* X) \\
&+ Z.g(\pi_* X, \pi_* Y) - g(\pi_*[X, Y], \pi_* Z) + g(\pi_*[Y, Z], \pi_* X) \\
&- g(\pi_*[Z, X], \pi_* Y) + 2\mathcal{A}(X, Y, Z) - 2\mathcal{L}(X, Y, Z) \\
&- X.g(\pi_* Z, \pi_* Y) - Z.g(\pi_* Y, \pi_* X) \\
&+ Y.g(\pi_* X, \pi_* Z) - g(\pi_*[X, Z], \pi_* Y) + g(\pi_*[Z, Y], \pi_* X) \\
&- g(\pi_*[Y, X], \pi_* Z) + 2\mathcal{A}(X, Z, Y) - 2\mathcal{L}(X, Y, Z) \\
&= 2\mathcal{A}(X, Y, Z) + 2\mathcal{A}(X, Z, Y) - 4\mathcal{L}(X, Y, Z) \\
&= 4A(\theta(X), \pi_* Y, \pi_* Z) - 4L(\pi_* X, \pi_* Y, \pi_* Z) \\
&= 4A(\theta(X), \xi, \eta) - 4L(\pi_* X, \xi, \eta). \qquad (5.9)
\end{aligned}$$

$$\begin{aligned}
2(\tilde{\nabla}_X g)(\xi, \eta) &= 2(\tilde{\nabla}_X g)(\pi_* Y, \pi_* Z) \\
&= 2X.g(\pi_* Y, \pi_* Z) - 2g(\tilde{\nabla}_X \pi_* Y, \pi_* Z) - 2g(\tilde{\nabla}_X \pi_* Z, \pi_* Y) \\
&= 2X.g(\pi_* Y, \pi_* Z) - X.g(\pi_* Y, \pi_* Z) - Y.g(\pi_* Z, \pi_* X) \\
&+ Z.g(\pi_* X, \pi_* Y) - g(\pi_*[X, Y], \pi_* Z) + g(\pi_*[Y, Z], \pi_* X) \\
&- g(\pi_*[Z, X], \pi_* Y) + 2\mathcal{A}(X, Y, Z) - 2\mathcal{L}(X, Y, Z) \\
&- X.g(\pi_* Z, \pi_* Y) - Z.g(\pi_* Y, \pi_* X) \\
&+ Y.g(\pi_* X, \pi_* Z) - g(\pi_*[X, Z], \pi_* Y) + g(\pi_*[Z, Y], \pi_* X) \\
&- g(\pi_*[Y, X], \pi_* Z) + 2\mathcal{A}(X, Z, Y) - 2\mathcal{L}(X, Y, Z) \\
&= 2\mathcal{A}(X, Y, Z) + 2\mathcal{A}(X, Z, Y) - 4\mathcal{L}(X, Y, Z) \\
&= 4A(\theta(X), \pi_* Y, \pi_* Z) - 4L(\pi_* X, \pi_* Y, \pi_* Z) \\
&= 4A(\theta(X), \xi, \eta) - 4L(\pi_* X, \xi, \eta). \qquad (5.10)
\end{aligned}$$

where $(\tilde{\nabla}_X g)(\xi, \eta) = 2A(\theta(X), \xi, \eta) - 2L(\pi_* X, \xi, \eta)$. This completes the proof. □

Corollary 5.1. *The generalized Berwald connection $\tilde{\nabla}$ is explicitly expressed in terms of the generalized Chern connection ∇ in the form :*

$$\tilde{\nabla}_X \pi_* Y = \nabla_X \pi_* Y + L^\sharp(\pi_* X, \pi_* Y, \bullet), \qquad (5.11)$$

where $L^\sharp$ is define by: $g(L^\sharp(\pi_* X, \pi_* Y, \bullet), \xi) = L(\pi_* X, \pi_* Y, \xi)\ \forall \xi \in \Gamma(\pi^* TM)$

Proof. Likewise in the proof of Corollary 3.2, after Replacing $X, Y, Z \in \Gamma(TTM_0)$ by X^H, Y^H, Z^H in (5.4), we obtain,

$$2g(\tilde{\nabla}_{X^H} \pi_* Y^H, \pi_* Z^H) = 2g(\nabla_{X^H} \pi_* Y^H, \pi_* Z^H) + 2L(\pi_* X^H, \pi_* Y^H, \pi_* Z^H).$$

Hence, $\tilde{\nabla}_{X^H} \pi_* Y^H = \nabla_{X^H} \pi_* Y^H + L^\sharp(\pi_* X, \pi_* Y^H)$. Moreover, replacing X, Y by X^V, Y^H in (4.1) gives

$$\tilde{\nabla}_{X^V} \pi_* Y^H = 0 = \nabla_{X^V} \pi_* Y^H,$$

which implies $\tilde{\nabla}_X \pi_* Y = \nabla_X \pi_* Y + L^\sharp(\pi_* X, \pi_* Y, \bullet)$. □

Remark 5.1. The same manner as the remark 2.3, in the classical Finsler manifold ie for $\mathcal{H} = \mathcal{H}_c$ and $\theta = \theta_c$, we find locally the well-know relation between Berwald and Chern connection[1].

$$\tilde{\alpha}^i_j = \alpha^i_j + \dot{A}^i_{jk} dx^k, \tag{5.12}$$

where $\tilde{\alpha}$ and α are respectively the Berwald and Chern connection 1-forms. And $\dot{A} = \nabla_l A$ is the horizontal covariant derivative of Cartan tensor A along the distinguished (horizontal) direction[1].

6. Generalized Hashiguchi connection

In this section, we establish an intrinsic proof of the existence and uniqueness theorem of the generalized Hashiguchi connection. Moreover, the relationship between this connection and the Chern connection ∇ is obtained.

Theorem 6.1. *Let $(M, F, \mathcal{H})$ be a Finsler-Ehresmann manifold and g be a fundamental tensor of F. There exist a unique linear connection $\hat{\nabla}$ on π^*TM such that for all $X, Y \in \Gamma(TTM_0)$:*

(a) *Symmetry*

$$\hat{\nabla}_X \pi_* Y - \hat{\nabla}_Y \pi_* X = \pi_*[X, Y] + A^\sharp(\theta(X), \pi_* Y, \bullet) - A^\sharp(\theta(Y), \pi_* X, \bullet) \tag{6.1}$$

(b) *Metric-compatibility*

$$(\hat{\nabla}_X g)(\xi, \eta) = -2L(\pi_* X, \xi, \eta), \tag{6.2}$$

where, $A^\sharp(\xi, \eta, \bullet)$ is the section of π^*TM define by:

$$g(A^\sharp(\xi, \eta, \bullet), \mu) = A(\xi, \eta, \mu) \quad \forall \xi, \eta, \mu \in \Gamma(\pi^*TM) \tag{6.3}$$

Proof. As in the case of Chern, Cartan and Berwald, we have :

$$\begin{aligned} 2g(\hat{\nabla}_X \pi_* Y, \pi_* Z) &= X.g(\pi_* Y, \pi_* Z) + Y.g(\pi_* Z, \pi_* X) - Z.g(\pi_* X, \pi_* Y) \\ &\quad + g(\pi_*[X, Y], \pi_* Z) - g(\pi_*[Y, Z], \pi_* X) + g(\pi_*[Z, X], \pi_* Y) \\ &\quad + 2A(\theta(Z), \pi_* X, \pi_* Y) - 2A(\theta(Y), \pi_* X, \pi_* Z) \\ &\quad + 2L(\pi_* X, \pi_* Y, \pi_* Z), \end{aligned} \tag{6.4}$$

and the proof of the existence an unicity follows in the same manner as the previous case. □

Corollary 6.1. *The generalized Hashiguchi connection $\hat{\nabla}$ is explicitly expressed in terms of the generalized Chern connection ∇ in the form :*

$$\hat{\nabla}_X \pi_* Y = \nabla_X \pi_* Y + A^\sharp(\theta(X), \pi_* Y, \bullet) + L^\sharp(\pi_* X, \pi_* Y, \bullet), \tag{6.5}$$

where $L^\sharp$ is define by: $g(L^\sharp(\pi_* X, \pi_* Y, \bullet), \xi) = L(\pi_* X, \pi_* Y, \xi)$ $\forall \xi \in \Gamma(\pi^*TM)$ and $A^\sharp$ by, $g(A^\sharp(\theta(X), \pi_* Y, \bullet), \mu) = A(\theta(X), \pi_* Y, \mu)$ $\forall \mu \in \Gamma(\pi^*TM)$

Proof. Replacing $X, Y, Z \in \Gamma(TTM_0)$ by X^H, Y^H, Z^H in (6.4), we obtain,

$$2g(\hat{\nabla}_{X^H} \pi_* Y^H, \pi_* Z^H) = 2g(\nabla_{X^H} \pi_* Y^H, \pi_* Z^H) + 2L(\pi_* X^H, \pi_* Y^H, \pi_* Z^H).$$

It follow that $\tilde{\nabla}_{X^H} \pi_* Y^H = \nabla_{X^H} \pi_* Y^H + L^\sharp(\pi_* X, \pi_* Y^H)$.
In addition, replacing X, Y by X^V, Y^H in (6.1) we obtain

$$\hat{\nabla}_{X^V} \pi_* Y^H = A^\sharp(\theta(X), \pi_* Y, \bullet). \tag{6.6}$$

From wich $\hat{\nabla}_X \pi_* Y = \nabla_X \pi_* Y + A^\sharp(\theta(X), \pi_* Y, \bullet) + L^\sharp(\pi_* X, \pi_* Y, \bullet)$ □

Remark 6.1. Similarly to the previous case, we have, in the classical Finsler manifold ie for $\mathcal{H} = \mathcal{H}_c$ and $\theta = \theta_c$, the well-know relation between Hashiguchi and Chern connection[1].

$$\hat{\alpha}^i_j = \alpha^i_j + \dot{A}^i_{jk} dx^k + A^i_{jk} \frac{\delta y^k}{F}. \tag{6.7}$$

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