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BERTRAND MATE OF A BIHARMONIC CURVE IN CARTAN-VRANCEANU 3-DIMENSIONAL SPACE

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Abstract. Non-geodesic biharmonic curves in Cartan-Vranceanu 3-dimensional spaces are studied in [3]. In this paper, we characterize parametric equations of Bertrand mate of a biharmonic curve in the Cartan-Vranceanu 3-space.

1. Introduction

Harmonic maps $f : (M, g) \to (N, h)$ between a compact Riemannian manifold (M, g), and a Riemannian manifold (N, h), are the critical points of the energy functional

$$E(f) = \frac{1}{2}\int_M |df|^2 \, v_g$$

defined by J. Eells and J.H. Sampson [7]. From the first variation formula it follows that f is harmonic if and only if its tension field

$$\tau(f) = trace\nabla df$$

vanishes.

We can define the bienergy of a map f by

$$E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 \, v_g,$$

and say that is biharmonic if it is a critical point of the bienergy (as suggested by the same authors [7]). Jiang derived the first and the second variation formula for the bienergy [10], showing that the Euler-Lagrange equation for E_2 is

$$\tau_2(f) = -J^f(\tau(f)) = -\Delta\tau(f) - traceR^N(df, \tau(f))df = 0,$$

where J^f denotes the Jacobi operator of f. The equation $\tau_2(f) = 0$ is called the biharmonic equation. Note that the harmonic maps are also biharmonic. Therefore, we are interested in non-harmonic biharmonic maps, which are called proper biharmonic maps.

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In this paper, we restrain our attention to curves $\gamma : I \to (N, h)$ parametrized by arc length, from an open interval $I \subset \mathbb{R}$ to a Riemannian manifold. In this case, putting $T = \gamma\prime$, the tension field becomes $\tau(\gamma) = \nabla_T T$ and the biharmonic equation reduces to the fourth order differential equation

$$\nabla_T^3 T + R(T, \nabla_T T)T = 0.$$

The homogeneous Riemannian spaces with a large isometry group play a special role among the 3-dimensional manifolds of non-constant sectional curvature For these spaces, except for those with constant negative curvature, there is a nice local representation given by the following two-parameter family of Riemannian metrics which are called the Cartan-Vranceanu metric

$$ds^2_{l,m} = \frac{dx^2 + dy^2}{[1 + m(x^2 + y^2)]^2} + \left(dz + \frac{l}{2}\frac{ydx - xdy}{[1 + m(x^2 + y^2)]}\right)^2, l, m \in \mathbb{R} \tag{1.1}$$

defined on $3-$dimensional manifold M, where $M = \mathbb{R}^3$ if $m \geq 0$, and $M = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \langle \frac{-1}{m}\right\}$ otherwise. The family of metrics (1.1) includes all 3-dimensional homogeneous metrics whose group of isometries have dimension 4 or 6, except for those of constant negative sectional curvature. Biharmonic curves on $(M, ds^2_{l,m})$ have been already studied for particular values of ℓ and m. In particular, if $l = m = 0$, $(M, ds^2_{l,m})$ is the Euclidean space and γ is biharmonic if and only if it is a line [6]; if $\ell^2 = 4m$ and $l \neq 0$, $(M, ds^2_{l,m})$ is locally the $3-$dimensional sphere and the proper biharmonic curves were classified in [1], where it was proved that they are helices; if $\ell = 0$ and $m < 0$, $(M, ds^2_{l,m})$ is isometric to $H^2 \times \mathbb{R}$ with the product metric and it can be shown that all biharmonic curves are geodesics; if $m = 0$ and $\ell \neq 0$, $(M, ds^2_{l,m})$ is the Heisenberg space H_3 endowed with a left invariant metric and the explicit solutions of the biharmonic curves were obtained in [2] the generalized ones are obtained in [8]; if $\ell = 1$ a study of the biharmonic curves was given also in [4]; if $l^2 \neq 4m$ and $m \neq 0$ explicit formulas for non-geodesic biharmonic curves of the $3-$dimensional Cartan-Vranceanu space and the generalized ones are obtained in [3, 9].

An increasing interest of the theory of curves makes a development of special curves, which can be characterized by the relationship between the Frenet vectors of the curves. Bertrand curves, discovered by J. Bertrand in 1850, are one of the most important and interesting topics of classical special curve theory. A Bertrand curve is defined as a special curve which shares its principal normals with another special curve (called Bertrand mate). Note that Bertrand mates are particular examples of offset curves used in computer-aided design [12]. The concept of Bertrand mate of a biharmonic curve in the Heisenberg group $Heis^3$, which is a special case of Cartan-Vranceanu metrics, are studied in [11]. Choi and his colleagues characterized Bertrand curves and their mate curves in 3- dimensional space forms [5].

In this paper, the Bertrand mate curves of the proper biharmonic curves in Cartan-Vranceanu 3-dimensional spaces $(M, ds^2_{l,m})$, with $l^2 \neq 4m$ and $m \neq 0$ are studied. We give the explicit parametric equations of that Bertrand mate curve in Cartan-Vranceanu 3-dimensional space.

2. Prelimary

2.1. **Riemannian Structure of Cartan-Vranceanu 3-Spaces.** We consider the following two-parameter family of Riemannian metrics, called the Cartan-Vranceanu metrics

$$ds^2_{l,m} = \frac{dx^2 + dy^2}{[1 + m(x^2 + y^2)]^2} + \left(dz + \frac{l}{2}\frac{ydx - xdy}{[1 + m(x^2 + y^2)]}\right)^2, l, m \in \mathbb{R} \tag{2.1}$$

defined on 3−dimensional manifold M, where $M = \mathbb{R}^3$ if $m \geq 0$, and $M = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \langle \frac{-1}{m}\right\}$ otherwise.

The Cartan-Vranceanu metric (2.1) can be written as:

$$ds^2_{l,m} = \sum_{i=1}^{3} \omega^i \otimes \omega^i$$

where , putting $F = 1+ m(x^2 + y^2)$, in

$$\omega^1 = \frac{dx}{F},\ \omega^2 = \frac{dy}{F},\ \omega^3 = dz + \frac{l}{2}\frac{ydx - xdy}{F}, \tag{2.2}$$

and the orthonormal basis of dual vector fields to the 1−forms (2.2) is

$$E_1 = F\frac{\partial}{\partial x} - \frac{ly}{2}\frac{\partial}{\partial z},\ E_2 = F\frac{\partial}{\partial y} + \frac{lx}{2}\frac{\partial}{\partial z},\ E_3 = \frac{\partial}{\partial z}. \tag{2.3}$$

The Levi-Civita connection with respect to the orthonormal basis (2.3) is given by

$$\begin{aligned}
\nabla_{E_1} E_1 &= 2myE_2,\ \nabla_{E_1} E_2 = -2myE_1 + \frac{l}{2}E_3 \\
\nabla_{E_2} E_2 &= 2mxE_1,\ \nabla_{E_2} E_1 = -2mxE_2 - \frac{l}{2}E_3 \\
\nabla_{E_3} E_3 &= 0, \qquad \nabla_{E_1} E_3 = \nabla_{E_3} E_1 = -\frac{l}{2}E_2 \\
\nabla_{E_2} E_3 &= \nabla_{E_3} E_2 = \frac{l}{2}E_1.
\end{aligned}$$

Also, we obtain the bracket relations

$$\begin{aligned}
[E_1, E_2] &= (-2myF)\frac{\partial}{\partial x} + (2mxF)\frac{\partial}{\partial y} + (Fl)\frac{\partial}{\partial z},\ [E_1, E_3] = 0,\ [E_2, E_3] = 0 \\
[E_1, E_1] &= 0, [E_2, E_2] = 0, [E_3, E_3] = 0.
\end{aligned}$$

We shall adopt the following notation and sign convention. The curvature operator is given by

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$

while the Riemann-Christoffel tensor field and the Ricci tensor field are given by

$$R(X, Y, Z, W) =< R(X, Y)Z, W >,\ \rho(X, Y) = trace(Z \to R(X, Z)Y),$$

where X, Y, Z, W are smooth vector fields on $(M, ds^2_{l,m})$. The non vanishing components of the Riemann-Christoffel and the Ricci tensor fields are

$$R_{1212} = 4m - \frac{3}{4}l^2,\ R_{1313} = \frac{l^2}{4},\ R_{2323} = \frac{l^2}{4}$$

$$\rho_{11} = \rho_{22} = 4m - \frac{l^2}{2},\ \rho_{33} = \frac{l^2}{2}.$$

2.2. **Biharmonic Curves in Cartan-Vranceanu 3-Spaces.** To study the biharmonic curves in $3-$dimensional Cartan-Vranceanu space, we shall use their Frenet vector fields and equations. Let $\gamma : I \to (M, ds^2_{l,m})$ be a differentiable curve parametrized by arc length and let $\{T = T_i E_i, N = N_i E_i, B = B_i E_i\}$ be the orthonormal frame field tangent to M along γ decomposed with respect to the orthonormal basis (2.3). Then we have the following Frenet equations

$$\begin{aligned} \nabla_T T &= \kappa N \\ \nabla_T N &= -\kappa T + \tau B \\ \nabla_T B &= -\tau N \end{aligned}$$

where κ and τ are curvatures of γ.

From now on we shall assume that $l^2 \neq 4m$ and $m \neq 0$.

From [3] the following system for the proper biharmonic curves, that is for biharmonic curves with $\kappa \neq 0$, we have

$$\begin{aligned} \kappa &= constant \neq 0 \\ \kappa^2 + \tau^2 &= \frac{l^2}{4} - (l^2 - 4m)B_3^2 \\ \tau' &= (4m - l^2)N_3 B_{3.} \end{aligned}$$

Theorem 2.1. *If $\gamma : I \to (M, ds^2_{l,m})$ is a proper biharmonic curve parametrized by arc length, then it is a helix* [3].

Corollary 2.1. *Let $\gamma : I \to (M, ds^2_{l,m})$ be a curve parametrized by arc length. Then γ is a proper biharmonic curve if and only if*

$$\begin{aligned} k &= cnst \neq 0 \\ \tau &= cnst \\ N_3 &= 0 \\ k^2 + \tau^2 &= \frac{l^2}{4} - \left(l^2 - 4m\right) B_3^2 \end{aligned}$$

[3].

Lemma 2.1. *Let $\gamma : I \to (M, ds^2_{l,m})$ a non-geodesic curve parametrized by arc length. If $N_3 = 0$, then*

$$T = \sin\alpha_0 \cos\beta(s) E_1 + \sin\alpha_0 \sin\beta(s) E_2 + \cos\alpha_0 E_3, \tag{2.4}$$

where $\alpha_0 \in (0, \pi)$ [3].

Theorem 2.2. *Let $(M, ds^2_{l,m})$ be the Cartan-Vranceanu space with $l^2 \neq 4m$ and $m \neq 0$. Assume that $\delta = l^2 + (16m - 5l^2)\sin^2\alpha_0 \geq 0$, $\alpha_0 \in (0, \pi)$, and denote by $2\omega_{1,2} = -l\cos\alpha_0 \pm \sqrt{\delta}$. Then, the parametric equations of all proper biharmonic curves of $(M, ds^2_{l,m})$ are of the following three types.*

Type I

$$
\begin{aligned}
x(s) &= b\sin\alpha_0\sin\beta(s)+c;\ \ b,c\in\mathbb{R}, b\rangle 0\\
y(s) &= -b\sin\alpha_0\cos\beta(s)+d;\ \ d\in\mathbb{R}\\
z(s) &= \frac{l}{4m}\beta(s)+\frac{1}{4m}\left[\left(4m-l^2\right)\cos\alpha_0-l\omega_{1,2}\right]s
\end{aligned}
\tag{2.5}
$$

where β is a non-constant solution of the following ODE:

$$\beta'+2md\sin\alpha_0\cos\beta(s)-2mc\sin\alpha_0\sin\beta(s)=l\cos\alpha_0+2mb\sin^2\alpha_0+\omega_{1,2}$$

and the constants satisfy

$$c^2+d^2=\frac{b}{m}\left\{(l\cos\alpha_0+\omega_{1,2}-\frac{1}{b})+mb\sin^2\alpha_0\right\}.$$

Type II *If $\beta=\beta_0=cnst$ and $\cos\beta_0\sin\beta_0\neq 0$, the parametric equations are*

$$
\begin{aligned}
x(s) &= x(s)\\
y(s) &= x(s)\tan\beta_0+a\\
z(s) &= \frac{1}{4m}\left[\left(4m-l^2\right)\cos\alpha_0-l\omega_{1,2}\right]s+b,\ \ b\in\mathbb{R}
\end{aligned}
$$

where

$$a=\frac{\omega_{1,2}+l\cos\alpha_0}{2m\sin\alpha_0\cos\beta_0}$$

and $x(s)$ is a solution of the following ODE:

$$x'=(1+m[x^2+(x\tan\beta_0+a)^2])\sin\alpha_0\cos\beta_0.$$

Type III *If $\cos\beta_0\sin\beta_0=0$, up to interchange of x with y, $\cos\beta_0=0$ and the parametric equations are*

$$
\begin{aligned}
x(s) &= x_0=\pm\frac{\omega_{1,2}+l\cos\alpha_0}{2m\sin\alpha_0}\\
y(s) &= y(s)\\
z(s) &= \frac{1}{4m}\left[\left(4m-l^2\right)\cos\alpha_0-l\omega_{1,2}\right]s+b,\ \ b\in\mathbb{R}
\end{aligned}
$$

where $y(s)$ is a solution of the following ODE:

$$y'=\pm(1+m[x_0^2+y^2])\sin\alpha_0$$

[3].

Proof. The proof of this theorem can be found in [3]. □

3. Bertrand Mate of Biharmonic Curve in Cartan-Vranceanu 3-Spaces

A curve $\alpha: I\to(M,ds^2_{l,m})$ with $\kappa\neq 0$ is called a Bertrand curve if there exists a curve $\alpha^*: I\to(M,ds^2_{l,m})$ such that the principal normal lines of α and α^* at $s\in I$ are equal. In this case α^* is called Bertrand mate of α.

Let $\alpha: I\to(M,ds^2_{l,m})$ be a Bertrand curve parametrized by arc length . A Bertrand mate of is as follows:

$$\alpha^*(s)=\alpha(s)+\lambda N(s),\ \forall s\in I,\tag{3}$$

where λ is constant.

Let the Frenet frames of the curves α and α^* be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively.

In the following theorem, we obtain the explicit parametric equations of the Bertrand mate of the proper biharmonic curve in Cartan-Vranceanu 3- dimensional spaces.

Theorem 3.1. *Let $(M, ds^2_{l,m})$ be the Cartan-Vranceanu space with $l^2 \neq 4m$ and $m \neq 0$. Assume that $\delta = l^2 + (16m - 5l^2)\sin^2\alpha_0 \geq 0$, $\alpha_0 \in (0, \pi)$, and denote by $2\omega_{1,2} = -l\cos\alpha_0 \pm \sqrt{\delta}$. Let $\alpha : I \to E^3$ be a unit curve and α^* its evolute curve on $(M, ds^2_{l,m})$. Then, the parametric equations of α^* are*

Type I

$$\begin{aligned}
x^*(s) &= b\sin\alpha_0\sin\beta(s) + c + \lambda(-F\sin\beta(s));\ b, c \in \mathbb{R}, b\rangle 0 \\
y^*(s) &= -b\sin\alpha_0\cos\beta(s) + d + \lambda(F\cos\beta(s));\ d \in \mathbb{R} \\
z^*(s) &= \frac{l}{4m}\beta(s) + \frac{1}{4m}\left[\left(4m - l^2\right)\cos\alpha_0 - lw_{1,2}\right]s \\
&\quad +\lambda\left(\frac{ld}{2}\sin\beta(s) + \frac{lc}{2}\cos\beta(s)\right)
\end{aligned} \tag{3.1}$$

where $b, c, d \in \mathbb{R}$, $b > 0$,

$$F = 1 + m(b^2\sin^2\alpha_0 + c^2 + d^2 + 2b\sin\alpha_0(c\sin\beta(s) - d\cos\beta(s))),$$

β is a non-constant solution of the following ODE:

$$\beta' + 2md\sin\alpha_0\cos\beta(s) - 2mc\sin\alpha_0\sin\beta(s) = l\cos\alpha_0 + 2mb\sin^2\alpha_0 + \omega_{1,2}$$

and the constants satisfy

$$c^2 + d^2 = \frac{b}{m}\left\{(l\cos\alpha_0 + \omega_{1,2} - \frac{1}{b}) + mb\sin^2\alpha_0\right\}.$$

Type II *If $\beta = \beta_0 = cnst$ and $\cos\beta_0\sin\beta_0 \neq 0$, the parametric equations are*

$$\begin{aligned}
x^*(s) &= x(s) + \lambda(-F\sin\beta_0) \\
y^*(s) &= x(s)\tan\beta_0 + a + \lambda(F\cos\beta_0) \\
z^*(s) &= \frac{1}{4m}\left[\left(4m - l^2\right)\cos\alpha_0 - lw_{1,2}\right]s + b \\
&\quad +\lambda\left(\frac{lx(s)}{2\cos\beta_0} + \frac{la}{2}\sin\beta_0\right)
\end{aligned}$$

where $b \in \mathbb{R}$,

$$a = \frac{\omega_{1,2} + l\cos\alpha_0}{2m\sin\beta_0\cos\beta_0},\ \ F = 1 + m\left(\frac{x^2(s)}{\cos^2\beta_0} + 2ax(s)\tan\beta_0 + a^2\right)$$

and $x(s)$ is a solution of the following ODE:

$$x' = (1 + m[x^2 + (x\tan\beta_0 + a)^2])\sin\alpha_0\cos\beta_0.$$

Type III *If $\cos\beta_0\sin\beta_0 = 0$, up to interchange of x with y, $\cos\beta_0 = 0$ and the parametric equations are*

$$\begin{aligned}
x^*(s) &= x_0 + \lambda(-F\sin\beta_0) \\
y^*(s) &= y(s) \\
z^*(s) &= \frac{1}{4m}\left[\left(4m - l^2\right)\cos\alpha_0 - lw_{1,2}\right]s + b \\
&\quad +\lambda\left(\frac{ly(s)}{2}\sin\beta_0\right)
\end{aligned}$$

where $b \in \mathbb{R}$,

$$F = 1 + m\left(x_0^2 + y^2(s)\right)$$

and $y(s)$ *is a solution of the following ODE:*

$$y' = \pm(1 + m[x_0^2 + y^2])\sin\alpha_0.$$

Proof. The covariant derivative of the vector field T given by (2.4) is

$$\begin{aligned}
\nabla_T T &= [-\beta'\sin\alpha_0\sin\beta - 2my\sin^2\alpha_0\cos\beta\sin\beta \\
&\quad +2mx\sin^2\alpha_0\sin^2\beta + l\cos\alpha_0\sin\alpha_0\sin\beta]E_1 \\
&\quad +[\beta'\sin\alpha_0\cos\beta + 2my\sin^2\alpha_0\cos^2\beta \\
&\quad -2mx\sin^2\alpha_0\cos\beta\sin\beta - l\cos\alpha_0\sin\alpha_0\cos\beta]E_2 \\
&= \kappa N,
\end{aligned}$$

where

$$\kappa = |\beta' + 2my\sin\alpha_0\cos\beta - 2mx\sin\alpha_0\sin\beta - l\cos\alpha_0|\sin\alpha_0.$$

We assume that

$$\omega = \beta' + 2my\sin\alpha_0\cos\beta - 2mx\sin\alpha_0\sin\beta - l\cos\alpha_0 > 0.$$

Then we have

$$\kappa = \omega\sin\alpha_0$$

and

$$N = -\sin\beta E_1 + \cos\beta E_2. \tag{3.2}$$

Using the orthonormal basis (4) in (14), we obtain

$$N = (-F\sin\beta(s))\frac{\partial}{\partial x} + (F\cos\beta(s))\frac{\partial}{\partial y} + \left(\frac{ld}{2}\sin\beta(s) + \frac{lc}{2}\cos\beta(s)\right)\frac{\partial}{\partial z}$$

If we replace (2.5) and (3.2) in (3), we get (3.1). The case of Type II and Type III can be derived in a similar way. □

Remark 3.1. We know that if $m < 0$, Cartan-Vranceanu metric is defined on $M = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \langle \frac{-1}{m}\right\}$. Therefore , the evolute α^* of all three types must also satisfy the below conditions and the conditions of the Theorem 6, respectively:

For Type I,

$$\frac{1}{m} + \lambda^2 F - 2b\lambda\sin\alpha_0 - 2c\lambda\sin\beta(s) + 2d\lambda\cos\beta(s) < 0.$$

For Type II,

$$\frac{1}{m} + \lambda^2 F + 2a\lambda\cos\beta_0 < 0.$$

For Type III,

$$\frac{1}{m} - 2\lambda x_0 \sin\beta_0 + \lambda^2 F \sin^2\beta_0 < 0.$$

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