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ON KILLING VECTOR FIELDS ON A TANGENT BUNDLE WITH $g-$ NATURAL METRIC. PART II

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Abstract. The tangent bundle of a Riemannian manifold (M, g) with non-degenerate $g-$ natural metric G that admits a Killing vector field decomposes into four classes. Properties of these classes are investigated. A complete structure of the Lie algebra of Killing vector fields for some subclasses is given.

1. Introduction

In the first part of the paper ([9], see also [10]) we have developed the method by Tanno ([18]) to investigate Killing vector fields on TM with an arbitrary, non-degenerate $g-$ natural metric. The method applied Taylor's formula to components of the vector field that was supposed to be an infinitesimal isometry. It is known that an infinitesimal affine transformation, in particular an infinitesimal isometry, is determined by the values of its components and their first partial derivatives at a point ([14], p. 232). It appears by applying the Taylor's formula there are at most four generators of the infinitesimal isometry: two vectors and two tensors of type $(1, 1)$.

We have proved the following

Theorem 1.1. *([9], [10]) Let (TM, G) be a tangent bundle of a Riemannian manifold (M, g), $dimM > 2$, with $g-$ natural non-degenerate metric G. Let Z be a Killing vector field on TM with its Taylor series expansion around a point $(x, 0) \in TM$ given by (3.2) and (3.3). Then for each such a point there exists a neighbourhood $U \subset M$, $x \in U$, that one of the following cases occurs:*

(1) $2ba_2 - a_1 b_2 \neq 0$. *Then*

$$\nabla_k X_l + \nabla_l X_k = 0, \quad \nabla_k Y_l + \nabla_l Y_k = 0, \tag{1.1}$$

$$P_{kl} + P_{lk} = 0, \quad K_{kl} + K_{lk} = 0. \tag{1.2}$$

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(2) $2ba_2 - a_1b_2 = 0$ *and either* $a_1a_2b_2 \neq 0$ *or* $a_2 \neq 0$ *and* $b_2 = 0$. *Then*

$$P_{kl} + P_{lk} + 2\left(\nabla_k X_l + \nabla_l X_k\right) = 0, \tag{1.3}$$

$$a_2\left(\nabla_k Y_l + \nabla_l Y_k\right) + A\left(\nabla_k X_l + \nabla_l X_k\right) = 0, \tag{1.4}$$

$$a_2\left(K_{kl} + K_{lk}\right) - a_1\left(\nabla_k X_l + \nabla_l X_k\right) = 0. \tag{1.5}$$

(3) $a_2b_2 \neq 0$ *and* $a_1 = b_1 - a_1' = 0$. *Then*

$$P_{kl} + P_{lk} + 2\left(\nabla_k X_l + \nabla_l X_k\right) = 0, \tag{1.6}$$

$$a_2\left(\nabla_k Y_l + \nabla_l Y_k\right) + A\left(\nabla_k X_l + \nabla_l X_k\right) = 0, \tag{1.7}$$

$$K_{kl} + K_{lk} = 0. \tag{1.8}$$

(4) $a_2 = b_2 = 0$. *Then*

$$\nabla_k X_l + \nabla_l X_k = 0, \quad P_{kl} + P_{lk} = 0, \quad AK_{lk} + a_1\nabla_l Y_k = 0. \tag{1.9}$$

In the above theorem we have put $a_j = a_j(r^2)_{|(x,0)\in TM}$, $b_j = b_j(r^2)_{|(x,0)\in TM}$, $a_j' = a_j'(r^2)_{|(x,0)\in TM}$, $A = a_1 + a_3$ and $b = b_1 - a_1'$.

Above theorem splits (TM, G) into four classes. In section 4 of the paper for each such class further properties are proved separately. Some restrictions on a number of generators are found (cf. for example 3.5 and Corollary after it). Moreover, a complete structure of Killing vector fields on TM for some subclasses is given (Theorems 4.3 and 4.7). In the next section some classical lifts of some tensor fields from (M, g) to (TM, G) are discussed.

Finally, in the Appendix we collect some known facts and theorems that we use throughout the paper.

Throughout the paper all manifolds under consideration are smooth and Hausdorff ones. The metric g of the base manifold M is always assumed to be Riemannian one.

The computations in local coordinates were partially carried out and checked using MathTensor$^{\text{TM}}$ and Mathematica software.

2. Preliminaries

2.1. **Conventions and basic formulas.** Let (M, g) be a pseudo-Riemannian manifold of dimension n with metric g. The Riemann curvature tensor R is defined by

$$R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

In a local coordinate neighbourhood $(U, (x^1, ..., x^n))$ its components are given by

$$R(\partial_i, \partial_j)\partial_k = R(\partial_i, \partial_j, \partial_k) = R^r_{kji}\partial_r = \left(\partial_i\Gamma^r_{jk} - \partial_j\Gamma^r_{ik} + \Gamma^r_{is}\Gamma^s_{jk} - \Gamma^r_{js}\Gamma^s_{ik}\right)\partial_r,$$

where $\partial_k = \frac{\partial}{\partial x^k}$ and Γ^r_{jk} are the Christoffel symbols of the Levi-Civita connection ∇. We have

$$\partial_l g_{hk} = g_{hk;l} = \Gamma^r_{hl}g_{rk} + \Gamma^r_{kl}g_{rh}.$$

The Ricci identity is

$$\nabla_i\nabla_j X_k - \nabla_j\nabla_i X_k = X_{k,ji} - X_{k,ij} = -X^s R_{skji}. \tag{2.1}$$

The Lie derivative of a metric tensor g is given by

$$\left(L_X g\right)(Y, Z) = g\left(\nabla_Y X, Z\right) + g\left(Y, \nabla_Z X\right)$$

for all vector fields X, Y, Z on M. In local coordinates $(U, (x^1, ..., x^n))$ we get

$$(L_{X^r\partial_r} g)_{ij} = \nabla_i X_j + \nabla_j X_i,$$

where $X_k = g_{kr} X^r$.

We shall need the following properties of the Lie derivative

$$(2.2) \quad L_X \Gamma^h_{ji} = \nabla_j \nabla_i X^h + X^r R_{rjis} g^{sh} = \frac{1}{2} g^{hr} \left[\nabla_j \left(L_X g_{ir} \right) + \nabla_i \left(L_X g_{jr} \right) - \nabla_r \left(L_X g_{ji} \right) \right].$$

If $L_X \Gamma^h_{ji} = 0$, then X is said to be an infinitesimal affine transformation.

The vector field X is said to be the Killing vector field or infinitesimal isometry if $L_X g = 0$, ([20], p. 23 and 24).

2.2. **Tangent bundle.** Let x be a point of a Riemannian manifold (M, g), $\dim M = n$, covered by coordinate neighbourhoods $(U, (x^j))$, $j = 1, ..., n$. Let TM be tangent bundle of M and $\pi : TM \longrightarrow M$ be a natural projection on M. If $x \in U$ and $u = u^r \frac{\partial}{\partial x^r}_{|x} \in T_x M$, then $(\pi^{-1}(U), ((x^r), (u^r))$, $r = 1, ..., n$, is a coordinate neighbourhood on TM.

For all $(x, u) \in TM$ we denote by $V_{(x,u)}TM$ the kernel of the differential at (x, u) of the projection $\pi : TM \longrightarrow M$, i.e.,

$$V_{(x,u)}TM = Ker \left(d\pi_{|(x,u)} \right),$$

which is called the vertical subspace of $T_{(x,u)}TM$ at (x, u).

To define the horizontal subspace of $T_{(x,u)}TM$ at (x, u), let $V \subset M$ and $W \subset T_x M$ be open neighbourhoods of x and 0 respectively, diffeomorphic under exponential mapping $exp_x : T_x M \longrightarrow M$. Furthermore, let $S : \pi^{-1}(V) \longrightarrow T_x M$ be a smooth mapping that translates every vector $Z \in \pi^{-1}(V)$ from the point y to the point x in a parallel manner along the unique geodesic connecting y and x. Finally, for a given $u \in T_x M$, let $R_{-u} : T_x M \longrightarrow T_x M$ be a translation by u, i.e. $R_{-u}(X_x) = X_x - u$. The connection map

$$K_{(x,u)} : T_{(x,u)}TM \longrightarrow T_x M$$

of the Levi-Civita connection ∇ is given by

$$K_{(x,u)}(Z) = d(exp_p \circ R_{-u} \circ S)(Z)$$

for any $Z \in T_{(x,u)}TM$.

For any smooth vector field $Z : M \longrightarrow TM$ and $X_x \in T_x M$ we have

$$K(dZ_x(X_x)) = (\nabla_X Z)_x.$$

Then $H_{(x,u)}TM = Ker(K_{(x,u)})$ is called the horizontal subspace of $T_{(x,u)}TM$ at (x, u).

The space $T_{(x,u)}TM$ tangent to TM at (x, u) splits into direct sum

$$T_{(x,u)}TM = H_{(x,u)}TM \oplus V_{(x,u)}TM.$$

We have isomorphisms

$$H_{(x,u)}TM \sim T_x M \sim V_{(x,u)}TM.$$

For any vector $X \in T_x M$ there exist the unique vectors: X^h given by $d\pi(X^h) = X$ and X^v given for any function f on M by $X^v(df) = Xf$. The vectors X^h and

X^v are called respectively the horizontal and the vertical lifts of X to the point $(x,u) \in TM$.

The vertical lift of a vector field X on M is a unique vector field X^v on TM such that at each point $(x,u) \in TM$ its value is a vertical lift of X_x to the point (x,u). The horizontal lift of a vector field is defined similarly.

If $((x^j), (u^j))$, $i = 1, ..., n$, is a local coordinate system around the point $(x,u) \in TM$ where $u \in T_xM$ and $X = X^j \frac{\partial}{\partial x^j}$, then

$$X^h = X^j \frac{\partial}{\partial x^j} - u^r X^s \Gamma^j_{rs} \frac{\partial}{\partial u^j}, \quad X^v = X^j \frac{\partial}{\partial u^j},$$

where Γ^j_{rs} are Christoffel symbols of the Levi-Civita connection ∇ on (M,g). We shall write $\partial_k = \frac{\partial}{\partial x^k}$ and $\delta_k = \frac{\partial}{\partial u^k}$. Cf. [8] or [13]. See also [21].

In the paper we shall frequently use the frame $(\partial^h_k, \partial^v_l) = \left(\left(\frac{\partial}{\partial x^k}\right)^h, \left(\frac{\partial}{\partial x^l}\right)^v \right)$ known as the adapted frame.

Every metric g on M defines a family of metrics on TM. Between them a class of so called $g-$ natural metrics is of special interest. The well-known Cheeger-Gromoll and Sasaki metrics are special cases of the $g-$ natural metrics ([15]).

Lemma 2.1. *([4], [5]) Let (M,g) be a Riemannian manifold and G be a $g-$ natural metric on TM. There exist functions a_j, b_j $:< 0, \infty) \longrightarrow R$, $j = 1, 2, 3$, such that for every X, Y, $u \in T_xM$*

$$\begin{aligned} G_{(x,u)}(X^h, Y^h) &= (a_1 + a_3)(r^2) g_x(X,Y) + (b_1 + b_3)(r^2) g_x(X,u) g_x(Y,u), \\ G_{(x,u)}(X^h, Y^v) &= a_2(r^2) g_x(X,Y) + b_2(r^2) g_x(X,u) g_x(Y,u), \\ G_{(x,u)}(X^v, Y^h) &= a_2(r^2) g_x(X,Y) + b_2(r^2) g_x(X,u) g_x(Y,u), \\ G_{(x,u)}(X^v, Y^v) &= a_1(r^2) g_x(X,Y) + b_1(r^2) g_x(X,u) g_x(Y,u), \end{aligned} \tag{2.3}$$

where $r^2 = g_x(u,u)$. For $\dim M = 1$ the same holds for $b_j = 0$, $j = 1, 2, 3$.

Following ([4]) we put

(1) $a(t) = a_1(t)\,(a_1(t) + a_3(t)) - a_2^2(t)$,
(2) $F_j(t) = a_j(t) + t b_j(t)$,
(3) $F(t) = F_1(t)\,[F_1(t) + F_3(t)] - F_2^2(t)$
for all $t \in < 0, \infty)$.

We shall often abbreviate: $A = a_1 + a_3$, $B = b_1 + b_3$.

Lemma 2.2. *([4], Proposition 2.7) The necessary and sufficient conditions for a $g-$ natural metric G on the tangent bundle of a Riemannian manifold (M,g) to be non-degenerate are $a(t) \neq 0$ and $F(t) \neq 0$ for all $t \in < 0, \infty)$. If $\dim M = 1$ this is equivalent to $a(t) \neq 0$ for all $t \in < 0, \infty)$.*

For a general overview on $g-$ natural metric we refer the reader to ([1]), ([2]). The components of the Levi-Civita connection of an arbitrary, non-degenerate $g-$ natural metric G are calculated in ([7]). They are the same as in the Riemannian case ([1], p. 112-113).

3. Taylor's formula for Killing vector field and coefficients

Suppose now that

$$Z = Z^a \partial_a + \widetilde{Z}^\alpha \delta_\alpha = Z^a \partial^h_a + (\widetilde{Z}^\alpha + Z^a u^r \Gamma^\alpha_{ar}) \partial^v_\alpha = H^a \partial^h_a + V^\alpha \partial^v_\alpha$$

is a vector field on TM. Throughout the paper the following hypothesis will be used:

(3.1) (M,g) is a Riemannian manifold of dimension n with metric g, H covered by the coordinate system $(U,\ (x^r))$.
(TM,G) is the tangent bundle of M with $g-$natural non-degenerate metric G, covered by a coordinate system $(\pi^{-1}(U),\ (x^r,u^s))$, r,s run through the range $\{1,...,n\}$.
Z is a Killing vector field on TM with local components $(Z^r,\widetilde{Z}^s)$ with respect to the local base (∂_r,δ_s).

Let

$$(3.2)\quad H^a = Z^a = Z^a(x,u) = X^a + K^a_p u^p + \frac{1}{2}E^a_{pq}u^pu^q + \frac{1}{3!}F^a_{pqr}u^pu^qu^r + \frac{1}{4!}G^a_{pqrs}u^pu^qu^ru^s + \cdots,$$

$$(3.3)\quad \widetilde{Z}^a = \widetilde{Z}^a(x,u) = Y^a + \widetilde{P}^a_p u^p + \frac{1}{2}Q^a_{pq}u^pu^q + \frac{1}{3!}S^a_{pqr}u^pu^qu^r + \frac{1}{4!}V^a_{pqrs}u^pu^qu^ru^s + \cdots$$

be expansions of the components Z^a and $\widetilde{Z}^a$ by Taylor's formula in a neighbourhood of a point $(x,0)\in TM$. For each index a the coefficients are values of partial derivatives of Z^a and $\widetilde{Z}^a$ respectively, taken at a point $(x,0)$ and therefore are symmetric in all lower indices. For simplicity we have omitted the remainders.

Lemma 3.1. *(*[18]*) The quantities*

$$X = (X^a(x)) = (Z^a(x,0)), \quad Y = (Y^a(x)) = \left(\widetilde{Z}^a(x,0)\right),$$
$$K = \left(K^a_p(x)\right) = \left(\delta_p Z^a(x,0)\right), \quad E = \left(E^a_{pq}(x)\right) = \left(\delta_p\delta_q Z^a(x,0)\right),$$
$$P = \left(P^a_p(x)\right) = \left(\left(\delta_p\widetilde{Z}^a\right)(x,0) - \partial_p\left(Z^a(x,0)\right)\right)$$

are tensor fields M.

We shall often use the following definitions and abbreviations:

$$S^a_p = P^a_p + \nabla_p X^a, \quad S_{kp} = S^a_p g_{ak}, \quad P_{lk} = P^a_k g_{al},$$

$$K_{lp} = K^a_p g_{al}, \quad E_{kpq} = E_{kqp} = E^a_{pq} g_{ak}, \quad T_{lkp} = T^a_{kp} g_{al},$$

$$M_{pqr} = T_{pqr} + T_{qrp} + T_{rpq}.$$

Moreover, for any $(0,2)$ tensor T we put

$$\overline{T}_{ab} = T_{ab} + T_{ba}, \quad \widehat{T}_{ab} = T_{ab} - T_{ba}.$$

Lemmas 3.2-3.9 were proved in ([9], see also [10]). Hereafter, and unless otherwise specified, all the coefficients $a_j,\ b_j,\ a'_j,\ b'_j,\ A,\ A',\ B,\ B',...$ are considered to be constants, equal to the values at 0 of the corresponding functions.

Lemma 3.2. *Under hypothesis (3.1) at a point* $(x,0)\in TM$ *we have:*

$$a_1T_{lkp}+a_2E_{lkp}=a_1'\left(Y_lg_{kp}-Y_kg_{lp}-Y_pg_{kl}\right)-b_1Y_lg_{kp},$$

$$AE_{lkp}+a_2T_{lkp}+a_2'(g_{kl}Y_p+g_{pl}Y_k)+\frac{1}{2}b_2(2g_{kp}Y_l+g_{lp}Y_k+g_{kl}Y_p)=0. \tag{3.4}$$

If $a\neq 0$, *then*

$$aE_{lkm}=(a_2b_1-a_1b_2-a_2a_1')g_{km}Y_l-\frac{1}{2}(a_1b_2-2a_2a_1'+2a_1a_2')(g_{lm}Y_k+g_{lk}Y_m), \tag{3.5}$$

$$aT_{lkm}=(Aa_1'+a_2b_2-Ab_1)g_{km}Y_l+\frac{1}{2}(a_2b_2-2Aa_1'+2a_2a_2')(g_{lm}Y_k+g_{lk}Y_m), \tag{3.6}$$

$$aM_{lkm}=[2a_2(b_2+a_2')-A(b_1+a_1')](g_{km}Y_l+g_{lk}Y_m+g_{ml}Y_k). \tag{3.7}$$

Moreover,

$$a_2\left[\nabla_k\left(\nabla_lX_p+\nabla_pX_l\right)+\nabla_l\left(\nabla_kX_p+\nabla_pX_k\right)-\nabla_p\left(\nabla_lX_k+\nabla_kX_l\right)\right]+ a_1\left(\nabla_k\nabla_lY_p+\nabla_l\nabla_kY_p\right)=2A'g_{kl}Y_p+B\left(Y_kg_{lp}+Y_lg_{kp}\right), \tag{3.8}$$

$$a\left(\nabla_kK_{lp}+\nabla_lK_{kp}\right)+(a_2b_2+2a_1A'-2a_2a_2')Y_pg_{kl}+\frac{1}{2}(-a_2b_2+2a_1B+2a_2a_2')(Y_kg_{lp}+Y_lg_{kp})=0. \tag{3.9}$$

Lemma 3.3. *Under hypothesis (3.1) we have*

$$2a\nabla_lK_{km}=a_1^2Y^rR_{rmkl}-a_1Bg_{km}Y_l+(-a_1B+a_2b_2-2a_2a_2')g_{lm}Y_k+(-a_2b_2-2a_1A'+2a_2a_2')g_{kl}Y_m, \tag{3.10}$$

$$2a\left(\nabla_lS_{km}-X^rR_{rlkm}\right)+a_1a_2Y^rR_{rmkl}-a_2Bg_{km}Y_l+\left[-a_2B+A\left(b_2-2a_2'\right)\right]g_{lm}Y_k+\left[-2a_2A'-A\left(b_2-2a_2'\right)\right]g_{kl}Y_m=0 \tag{3.11}$$

at the point.

Lemma 3.4. *Under hypothesis (3.1) suppose* $\dim M>2$. *Then on* $M\times\{0\}$

$$T_{kl}=T_{lk}=2\left(b_1-a_1'\right)\overline{S}_{kl}+b_2\overline{K}_{kl}=0,$$

$$a_2F_{labk}+a_1W_{labk}+\frac{1}{2}b_2\left(\widehat{K}_{kl}g_{ab}+\widehat{K}_{bl}g_{ak}+\widehat{K}_{al}g_{bk}+\overline{K}_{ak}g_{bl}\right)+b_1g_{bl}\overline{S}_{ak}+a_1'(g_{kl}\overline{S}_{ab}+g_{al}\overline{S}_{bk})=0.$$

Lemma 3.5. *Under hypothesis (3.1) suppose* $\dim M>1$. *Then*

$$(n-1)\beta Y_l=0$$

on $M\times\{0\}$ *holds, where*

$$\beta=2A(b_1^2-a_1'^2-a_1b_1')+(a_1b_2-2a_2b_1)(3b_2+2a_2')+2a_2\left[2a_1'(b_2+a_2')+a_2b_1'\right].$$

Corollary 3.1. *For the Cheeger-Gromoll metric* g^{CG} *on* TM, *the vector field* Y *vanishes everywhere on* M.

Lemma 3.6. *Under hypothesis (3.1) the identities*

$$3AF_{lkmn} + 3a_2 W_{lkmn} + B\left(g_{kl}\overline{K}_{mn} + g_{lm}\overline{K}_{kn} + g_{ln}\overline{K}_{km}\right) + (b_1 - a_1')\left(Y_{n,l}g_{km} + Y_{m,l}g_{kn} + Y_{k,l}g_{mn}\right) + 2(b_2 + a_2')\left(g_{kl}\overline{S}_{mn} + g_{lm}\overline{S}_{kn} + g_{ln}\overline{S}_{km}\right) + 2b_2\left[g_{km}\left(X_{n,l} + S_{ln}\right) + g_{kn}\left(X_{m,l} + S_{lm}\right) + g_{mn}\left(X_{k,l} + S_{lk}\right)\right] = 0$$

and

$$B\left[g_{kl}\left(K_{mn} - 2K_{nm}\right) + g_{lm}\left(K_{kn} + K_{nk}\right) + g_{ln}\left(K_{mk} - 2K_{km}\right)\right] + 2(b_1 - a_1')\left(2Y_{m,l}g_{kn} - Y_{n,l}g_{km} - Y_{k,l}g_{mn}\right) + 3a_1\left(K^r_n R_{rlmk} + K^r_k R_{rlmn}\right) + b_2\left[2g_{kn}\left(X_{m,l} + S_{lm}\right) - g_{km}\left(X_{n,l} + S_{ln}\right) - g_{mn}\left(X_{k,l} + S_{lk}\right)\right] + (b_2 - 2a_2')\left(2g_{lm}\overline{S}_{kn} - g_{ln}\overline{S}_{km} - g_{kl}\overline{S}_{mn}\right). \tag{3.12}$$

are satisfied at a point $(x, 0) \in TM$.

Lemma 3.7. *Under hypothesis (3.1) relation*

$$3a_2\left[E^p_{bc}\left(R_{pkal} + R^p_{lak}\right) + E^p_{ac}\left(R_{pkbl} + R^p_{lbk}\right) + E^p_{ab}\left(R_{pkcl} + R^p_{lck}\right)\right] + 6A'g_{kl}(T_{abc} + T_{bca} + T_{cab}) + g_{bc}K_{kal} + g_{ca}K_{kbl} + g_{ab}K_{kcl} + g_{cl}L_{abk} + g_{al}L_{bck} + g_{bl}L_{cak} + g_{ck}L_{abl} + g_{ak}L_{bcl} + g_{bk}L_{cal} = 0 \tag{3.13}$$

holds on $M \times \{0\}$, where

$$K_{kal} = K_{lak} = -2b_2\left(S_{ka,l} + S_{la,k} + X_{a,kl} + X_{a,lk}\right) - (b_1 - a_1')(Y_{a,kl} + Y_{a,lk}), \tag{3.14}$$

$$L_{abk} = L_{bak} = 2B\overline{K}_{ab,k} + 3BT_{kab} + (b_2 - 2a_2')\overline{S}_{ab,k} + 3B'(g_{ka}Y_b + g_{kb}Y_a). \tag{3.15}$$

Lemma 3.8. *Under hypothesis (3.1) suppose* $\dim M > 2$. *Then the relation*

$$a_1\left[2E^p_{ab}R_{plck} - E^p_{bk}R_{plac} + E^p_{bc}R_{plak} - E^p_{ak}R_{plbc} + E^p_{ac}R_{plbk}\right] + B\left[\left(E_{ckb} - E_{kcb}\right)g_{al} + \left(E_{cak} - E_{kac}\right)g_{bl} + \left(E_{abk} + E_{bak}\right)g_{cl} - \left(E_{abc} + E_{bac}\right)g_{kl}\right] + (b_1 - a_1')\left[\nabla_l\overline{S}_{bc}g_{ak} - \nabla_l\overline{S}_{bk}g_{ac}\right] + b_2\left[\nabla_l\widehat{K}_{kc}g_{ab} + g_{ak}\left(\frac{3}{2}\nabla_l K_{bc} + \frac{1}{2}\nabla_l K_{cb}\right) - g_{ac}\left(\frac{3}{2}\nabla_l K_{bk} + \frac{1}{2}\nabla_l K_{kb}\right)\right] + b_2\left(\nabla_l K_{ac}g_{bk} - \nabla_l K_{ak}g_{bc}\right) + (b_2 - 2a_2')\left(M_{abk}g_{cl} - M_{abc}g_{kl}\right) + b_2\left[g_{bk}T_{lac} - g_{bc}T_{lak} + g_{ak}T_{lbc} - g_{ac}T_{lbk}\right] + 2b_2'\left[\left(g_{bk}g_{cl} - g_{bc}g_{kl}\right)Y_a + \left(g_{ak}g_{cl} - g_{ac}g_{kl}\right)Y_b + \left(g_{al}g_{bk} + g_{ak}g_{bl}\right)Y_c - \left(g_{al}g_{bc} + g_{ac}g_{bl}\right)Y_k\right] = 0$$

holds on $M \times \{0\}$.

Lemma 3.9. *Under hypothesis (3.1) relations*

$$\begin{aligned}(3.16)\quad \mathbf{A}_{km} = (3a_1B - a_2b_2)\nabla_k X_m + (-2a_2b_1 + \frac{3}{2}a_1b_2 + 2a_2a_1' - 3a_1a_2')\nabla_k Y_m + \\ a_2B(K_{km} - 2K_{mk}) + (3a_1B - 2a_2b_2 + 2a_2a_2')S_{km} + \\ (-a_2b_2 + 2a_2a_2')S_{mk} = 0,\end{aligned}$$

$$\begin{aligned}\mathbf{F}_{kl} + \mathbf{B}_{kl} = 2a_2b_2(L_Xg)_{kl} + (4a_2b_1 - 3a_1b_2 - 4a_2a_1')(L_Yg)_{kl} + \\ 2\left(3a_2b_2 + 3a_1A' - 4a_2a_2'\right)\overline{S}_{kl} + 2a_2B\overline{K}_{kl} = 0.\end{aligned}$$

hold at a point $(x, 0) \in TM$*, where*

$$\mathbf{F}_{mn} = 2a_2B\overline{K}_{mn} + 2(2a_2b_2 + 3a_1A' - 4a_2a_2')\overline{S}_{mn},$$

$$\mathbf{B}_{kl} = 2a_2b_2(L_Xg)_{kl} + (4a_2b_1 - 3a_1b_2 - 4a_2a_1')(L_Yg)_{kl} + 2a_2b_2\overline{S}_{kl}.$$

4. Classification

4.1. **Case 1.** In this section we study relations between Y component of the Killing vector field on TM and the base manifold M (Theorems 4.1, 4.2). Various conditions for Y to be non-zero and relations between X, Y, P, K are proved. Moreover, Theorem 4.3 establishes isomorphism between algebras of Killing vector fields on M and TM for a large subclass of non-degenerate $g-$ natural metrics.

Lemma 4.1. *Under hypothesis (3.1) suppose* $\dim M > 2$ *and* $2(b_1 - a_1')a_2 - a_1b_2 \neq 0$ *at a point* $(x, 0) \in TM$*. Then*

$$(4.1)\qquad (B + A')Y_k = 0,$$

$$(4.2)\qquad 2a\nabla_l K_{km} = [2a_1A' + a_2(b_2 - 2a_2')]\,(g_{lm}Y_k - g_{lk}Y_m),$$

$$(4.3)\qquad 2a\nabla_l P_{km} = -\,[2a_2A' + A(b_2 - 2a_2')]\,(g_{lm}Y_k - g_{lk}Y_m),$$

$$(4.4)\qquad a_1\nabla_m\nabla_l Y_k = A'(g_{ml}Y_k - g_{mk}Y_l),$$

$$(4.5)\qquad a_1Y^rR_{rklm} = A'(g_{km}Y_l - g_{kl}Y_m)$$

hold at the point.

Proof. First suppose $a_1 \neq 0$. Symmetrizing (3.10) in (k, m), making use of the skew-symmetricity of K, then alternating in (k, l) and applying the first Bianchi identity, we get

$$(4.6)\qquad 3a_1Y^rR_{rmkl} + (B - 2A')(g_{lm}Y_k - g_{km}Y_l) = 0.$$

Applying the last identity to (3.10) we find

$$\begin{aligned}6a\nabla_l K_{km} + 2a_1(B + A')g_{km}Y_l + 3\,[2a_1A' + a_2(b_2 - 2a_2')]\,g_{lk}Y_m + \\ [2a_1(2B - A') - 3a_2(b_2 - 2a_2')]\,g_{lm}Y_k = 0,\end{aligned}$$

whence, symmetrizing in (k, m), we obtain (4.1) and, consequently, (4.2).

Suppose now $a_1 = 0$. Substituting in (3.10) we easily state that (4.2) remains true. On the other hand, substituting $a_1 = 0$ into (3.11) and symmetrizing in (k, m) we get

$$2a_2Bg_{km}Y_l + a_2(B + 2A')(g_{lm}Y_k + g_{lk}Y_m) = 0,$$

whence, by contractions with g^{km} and g^{lm}, we obtain

$$BY_l = 0 \text{ and } A'Y_l = 0 \tag{4.7}$$

respectively since $a_2 \neq 0$ must hold. Thus (4.1) holds good.

Since X is a Killing vector field, (3.11), (2.2), (4.1) and (4.6) in the case $a_1 \neq 0$ and (3.11) and (4.7) as well in the case $a_1 = 0$ yield (4.3).

Differentiating covariantly (equation II_1, [9]) and using just obtained identities, we get (4.4). Finally, alternating (4.4) in (l, m), by the use of the Ricci identity (2.1), we obtain (4.5). This completes the proof. □

From (4.5) and Theorem 6.1 by Grycak we infer

Theorem 4.1. *Under hypothesis (3.1) suppose* $\dim M > 2$ *and* $2(b_1 - a_1')a_2 - a_1 b_2 \neq 0$ *on the set* $M \times \{0\} \subset TM$. *If the vector field* $\frac{A'}{a_1} Y^a \partial_a$ *does not vanish on a dense subset of* M *and* M *is semisymmetric, i.e.* $R \cdot R = 0$, *(resp. the Ricci tensor* S *is semisymmetric, i.e.* $R \cdot S = 0$*), then* M *is a space of constant curvature, (resp.* M *is an Einstein manifold).*

Theorem 4.2. *Under hypothesis (3.1) suppose* $\dim M > 2$ *and* $2(b_1 - a_1')a_2 - a_1 b_2 \neq 0$ *at a point* $(x, 0) \in TM$. *Then the* Y *component of the Killing vector field on* TM *satisfies*

$$S_1 Y \left[a_1 R + \frac{B}{2} g \wedge g \right] = 0 \tag{4.8}$$

on M.

Proof. Suppose $a_1 \neq 0$. By (1.1) and (1.2) we have $\overline{S}_{ab} = 0$. Applying this and (1.2), (4.1), (4.2) and (4.7) to Lemma 3.8, after long computations we obtain

$$\begin{aligned} S_1 & \left[3(R_{blck} Y_a + R_{alck} Y_b) + (R_{blak} + R_{albk}) Y_c - (R_{blac} + R_{albc}) Y_k \right] + \\ & S_2 g_{ab} \left(g_{kl} Y_c - g_{cl} Y_k \right) + S_3 \left[(g_{al} g_{bk} + g_{ak} g_{bl}) Y_c - (g_{al} g_{bc} + g_{ac} g_{bl}) Y_k \right] + \\ & \qquad S_4 \left[(g_{bk} g_{cl} - g_{bc} g_{kl}) Y_a + (g_{ak} g_{cl} - g_{ac} g_{kl}) Y_b \right] = 0, \end{aligned} \tag{4.9}$$

where

$$S_1 = a_1 \left[2a_2 a_1' - a_1 \left(b_2 + 2a_2' \right) \right],$$

$$\begin{aligned} S_2 = & -2 \left[b_2 \left(-Ab_1 + 3a_2 b_2 + 5a_1 A' - Aa_1' - 4a_2 a_2' \right) + 2b_1 (Aa_2' - a_2 A') + \right. \\ & \left. 2(a_1 A' + Aa_1' - 2a_2 a_{2'}) a_2' \right] = \\ & -2 \left[b_2 \left(-Ab_1 + 3 \left(a_2 b_2 + a_1 A' - Aa_1' \right) + 2a' \right) + 2b_1 (Aa_2' - a_2 A') + 2a' a_2' \right], \end{aligned}$$

$$\begin{aligned} S_3 = & -3a_1 b_2 A' - 2Ab_2 a_1' + 2a_2 A' a_1' + 4a_2 a_2' b_2 - 2a_1 A' a_2' + 4ab_2' = \\ & 2A'(a_2 a_1' - a_1 a_2') - b_2 (2a' + a_1 A') + 4ab_2', \end{aligned}$$

$$\begin{aligned} S_4 = & b_2 \left(-2Ab_1 + 6a_2 b_2 + 7a_1 A' - 4Aa_1' - 4a_2 a_2' \right) - 4a_2 b_1 A' + 2a_2 A' a_1' + \\ & a_2' \left(4Ab_1 + 2a_1 A' + 4Aa_1' - 8a_2 a_2' \right) + 4ab_2' \end{aligned}$$

and

$$S_2 - S_3 + S_4 = 0$$

identically.

Symmetrizing (4.9) in (a,b,l) we get

$$(S_2+2S_3)\left[(g_{al}g_{bk}+g_{ak}g_{lb}+g_{ab}g_{kl})\,Y_c-(g_{al}g_{bc}+g_{ac}g_{lb}+g_{ab}g_{cl})\,Y_k\right]=0,$$

whence, by contraction with $g^{al}g^{bk}$, we find $(n-1)(n+2)(S_2+2S_3)Y_c=0$. Therefore, symmetrizing (4.9) in (a,b,c) and using the last result, we obtain

$$Y_aT_{bckl}+Y_bT_{cakl}+Y_cT_{abkl}=0,$$

where

$$T_{bckl}=T_{cbkl}=T_{klbc}=\\ 2S_1(R_{bkcl}+R_{blck})-(S_3+S_4)\left[g_{bc}g_{kl}-\frac{1}{2}(g_{bl}g_{ck}+g_{bk}g_{cl})\right].$$

Hence, by the use of the Walker's Lemma 6.1, we get

$$Y_aT_{bckl}=0. \tag{4.10}$$

Alternating (4.10) in (l,c) and applying the Bianchi identity we obtain

$$Y_a\left[4S_1R_{bkcl}+(S_3+S_4)\,(g_{bl}g_{kc}-g_{bc}g_{kl})\right]=0.$$

Transvecting the last equation with Y^b, by the use of (4.7), we easily get

$$\left[4BS_1+a_1(S_3+S_4)\right]Y_a=0,$$

whence (4.8) results.

On the other hand, from the proof of Lemma 4.1 it follows that $a_1(0)=0$ implies $B(0)Y_a=0$. Thus, by continuity, (4.8) holds good on M. □

Corollary 4.1. *Under assumptions of the above theorem we have on M :*

$$\begin{aligned}(S_2+2S_3)\,Y &= 0 \text{ if } a_1\neq 0,\\ \left[4BS_1+a_1(S_3+S_4)\right]Y &= 0.\end{aligned}$$

Notice that multiplying the first equation by a_1 and adding to the second one we obtain

$$a_1\,(b_2a'-2ab_2')\,Y=0.$$

Lemma 4.2. *Under hypothesis (3.1) suppose* $\dim M>2$ *and* $2(b_1-a_1')a_2-a_1b_2\neq 0$ *at a point* $(x,0)\in TM$.

If $a_1a_2\neq 0$, *then*

$$\mathbf{A}_{km}=\left[2a_2\,(b_1-a_1')-\frac{3}{2}a_1\,(b_2-2a_2')\right]Y_{k,m}+\\ (3a_1B-a_2b_2)\,P_{km}+3a_2BK_{km}=0. \tag{4.11}$$

If $a_2=0$ *and* $a_1b_2\neq 0$ *then*

$$\frac{1}{3}\mathbf{A}_{km}=-\frac{1}{2}a_1\,(b_2-2a_2')\,Y_{k,m}+a_1BP_{km}=0. \tag{4.12}$$

If $a_1=0$ *and* $(b_1-a_1')a_2\neq 0$ *then*

$$(n+1)BK_{kn}-b_2P_{kn}+2(b_1-a_1')Y_{k,n} = 0, \tag{4.13}$$

$$3BK_{ln}-(n-1)b_2P_{ln}+2(n-1)(b_1-a_1')Y_{l,n} = 0. \tag{4.14}$$

Proof. If $a_1 a_2 \neq 0$, we apply (1.1) and (1.2) to (3.16) to obtain (4.11).

If $a_2 = 0$ but $a_1 \neq 0$, then also there must be $b_2 \neq 0$. Substituting $a_2 = 0$ into (3.16) and applying (1.1) and (1.2) we get (4.12).

Finally, the last two identities one obtains substituting $a_1 = 0$ into (3.12), contracting with g^{km} and g^{lm} and making use of (1.1) and (1.2). □

Taking into account (4.13) and (4.14) together with the equation (equation II_1, [9]) which, in virtue of (1.1), writes

$$AK_{km} + a_2 P_{km} - a_1 Y_{k,m} = 0$$

we find that $B \neq 0$ implies $P = K = \nabla Y = 0$ on M. We conclude with the following

Theorem 4.3. *Let TM, $dimTM > 4$, be endowed with a non-degenerate $g-$ natural metric G, such that $a_1 = 0$, $(b_1 - a_1')a_2 \neq 0$ and $B \neq 0$ on $M \times \{0\} \subset TM$. Let V be an open subset of TM such that $M \times \{0\} \subset V$. If V admits a Killing vector field, then it is a complete lift of a Killing vector field on M. Consequently, Lie algebras of Killing vector fields on M and $V \subset TM$ are isomorphic.*

Besides, for $B = 0$, we have

Theorem 4.4. *Let TM, $dimTM > 4$, be endowed with a non-degenerate $g-$ natural metric G, such that $a_1 = 0$, $(b_1 - a_1')a_2 \neq 0$ and $B = 0$ on $M \times \{0\} \subset TM$. Then*

$$\begin{aligned} a_2 P + AK &= 0, \\ b_2 P - 2(b_1 - a_1')\nabla Y &= 0 \end{aligned}$$

hold on $M \times \{0\} \subset TM$.

Hence, for $B = 0$, $A \neq 0$ and $b_2 \neq 0$, a theorem similar to the former one can be deduced.

The next theorem gives further restrictions on the vector Y to be non-zero.

Theorem 4.5. *Under hypothesis (3.1) suppose* $\dim M > 2$ *and* $2(b_1 - a_1')a_2 - a_1 b_2 \neq 0$ *at a point $(x, 0) \in TM$. If $a_1 \neq 0$, then the Y component of the Killing vector field on TM satisfies*

$$Q_2 Y = \{a_1 b_2 [A(b_2 - 2a_2') - 2a_2 B] - 4aB(b_1 - a_1'))\} Y = 0,$$

$$B' Y = 0,$$

$$B[a_1 a_2 (b_2 + 2a_2') - 2Aa_1 a_1' + aa_1'] Y = 0.$$

Proof. We apply Lemma 3.7. By the use of (1.1), (1.2), (4.1) - (4.4) and (3.6) the components of the tensors K and L defined by (3.14) and (3.15) can be written as

$$K_{kal} = \frac{[aB(b_1 - a_1') + 2a_1 a_2 B b_2 - Aa_1 b_2 (b_2 - 2a_2')]}{aa_1} (2g_{kl} Y_a - g_{ka} Y_l - g_{la} Y_k),$$

$$\begin{aligned} L_{abl} = 3BT_{lab} + 3B'(g_{bl} Y_a + g_{al} Y_b) = \\ - \frac{3B[A(b_1 - a_1') - a_2 b_2]}{a} g_{ab} Y_l + \\ \frac{3[B(a_2 b_2 - 2Aa_1' + 2a_2 a_2') + 2aB']}{2a} (g_{al} Y_b + g_{bl} Y_a). \end{aligned}$$

Substituting into (3.13) and applying (3.5), (3.7) and (4.7) we get

$$\begin{aligned}(4.15)\quad & Q_1\left[\left(R_{bkcl}+R_{blck}\right)Y_a+\left(R_{ckal}+R_{clak}\right)Y_b+\left(R_{akbl}+R_{albk}\right)Y_c\right]+\\ & Q_2\left[\left(g_{al}g_{bc}+g_{bl}g_{ca}+g_{cl}g_{ab}\right)Y_k+\left(g_{ak}g_{bc}+g_{bk}g_{ca}+g_{ck}g_{ab}\right)Y_l\right]+\\ & Q_3 g_{kl}\left(g_{bc}Y_a+g_{ca}Y_b+g_{ab}Y_c\right)+\\ & Q_4\left[\left(g_{bl}g_{kc}+g_{bk}g_{lc}\right)Y_a+\left(g_{cl}g_{ka}+g_{ck}g_{la}\right)Y_b+\left(g_{al}g_{kb}+g_{ak}g_{lb}\right)Y_c\right]=0,\end{aligned}$$

where

$$Q_1=-\frac{3a_2\left(a_1b_2-2a_2a_1'+2a_1a_2'\right)}{a},$$

$$Q_2=\frac{\left[a_1b_2(A(b_1-a_1')-2Ba_2)-4aB(b_1-a_1')\right)]}{aa_1},$$

$$Q_3=2\frac{4aB(b_1-a_1')-a_1\left[A(b_2-2a_2')+B(a_2b_2-6Aa_1'+6a_2a_2')\right]}{aa_1},$$

$$Q_4=\frac{3\left[B\left(a_2b_2-2Aa_1'+2a_2a_2'\right)+2aB'\right]}{a}.$$

Contracting (4.15) with g^{ab}, by the use of (4.7), we get

$$\begin{aligned}(4.16)\quad & g_{kl}\left(-\frac{4BQ_1}{a_1}+(n+2)Q_3+2Q_4\right)Y_c-2Q_1R_{kl}Y_c+\\ & \left(\frac{2BQ_1}{a_1}+(n+2)Q_2+2Q_4\right)\left(g_{cl}Y_k+g_{kc}Y_l\right)=0.\end{aligned}$$

Symmetrizing in (c,k,l) we obtain

$$T_{kl}Y_c+T_{lc}Y_k+T_{ck}Y_l=0,$$

where

$$(4.17)\qquad T_{kl}=T_{lk}=g_{kl}\left[(n+2)\left(2Q_2+Q_3\right)+6Q_4\right]-2Q_1R_{kl}.$$

Then the Walker lemma yields $T_{kl}=0$ or $Y_c=0$. Subtracting (4.17) from (4.16) and contracting with g^{kl} we get

$$(4.18)\qquad \left[a_1\left((n+2)Q_2+2Q_4\right)+2BQ_1\right]Y_c=0.$$

In the same way, by contraction of (4.15) with g^{cl}, we find

$$(4.19)\qquad \left\{g_{bk}\left[(n+5)Q_2+3Q_3+2(n+2)Q_4\right]+2Q_1R_{bk}\right\}Y_c=0$$

and

$$(4.20)\qquad \left[a_1\left((n+3)\,Q_2+Q_3\right)-2BQ_1\right]Y_k=0.$$

At last, by contraction of (4.15) with g^{kl}, we obtain

$$(4.21)\qquad \left[g_{bc}\left(2Q_2+nQ_3+2Q_4\right)-2Q_1R_{bc}\right]Y_a=0.$$

Eliminating the Ricci tensor between (4.17), (4.21) and (4.19) we find

$$\left[3(n+3)Q_2+(n+5)(Q_3+2Q_4)\right]Y_c=0,$$

$$\left[(n+1)Q_2+2Q_3+2Q_4\right]Y_c=0.$$

The system consisting of (4.18), (4.20) and the above two equations is undetermined and equivalent to $Y=0$ or $Q_2=0$ and $2BQ_1+a_1Q_3=0$ and $Q_3+2Q_4=0$. Hence $2Q_2+Q_3+2Q_4$ yields the second identity, while $a_1(Q_3+2Q_4)-(2BQ_1+a_1Q_3)$ gives the third one. □

Remark 4.1. From (4.15) one can deduce the identity

$$Q_1 Y \left[a_1 R + \frac{B}{2} g \wedge g \right] = 0.$$

4.2. **Case 2.** The next theorem partially improves the result of Tanno ([17]) concerned with Killing vector field on (TM, g^C), where the complete lift g^C of g is a $g-$ natural metric with $a_2 = 1$, all others being zero. (In Tanno's paper the Killing vector on (TM, g^C) is of the form $\iota C^{[X]} + X^C + Y^v + (u^r P_r^t)\partial_t^h$, where Y and P satisfy some additional conditions). Furthermore, we prove in the section some sufficient conditions for X and Y to be either infinitesimal affine transformation or infinitesimal isometry.

Theorem 4.6. *Let X be an infinitesimal affine vector field on some open $U \subset M$. If*

$$a_2 = const \neq 0, \;\; b_3 = const, \;\; all\ others\ equal\ 0$$

on $\pi^{-1}(U) \subset TM$, then $\iota C^{[X]} + X^C$ is a Killing vector field on $\pi^{-1}(U)$.

Proof. It follows from the results of subsection 5.3.3. □

Lemma 4.3. *Under hypothesis (3.1) suppose* $\dim M > 2$ *and* $2(b_1 - a_1')a_2 - a_1 b_2 = 0$ *at a point $(x, 0) \in TM$. Moreover, let either $a_1 a_2 b_2 \neq 0$ or $a_2 \neq 0$, $b_2 = 0$, $b_1 - a_1' = 0$. Then*

$$(a_1 B - 2a_2 b_2 - 3a_1 A' + 4a_2 a_2') \left[(L_X g) - \frac{1}{n} Tr(L_X g) g \right] = 0,$$

$$a_2 (b_1 - a_1') \left[(L_Y g) - \frac{1}{n} Tr(L_Y g) g \right] = 0,$$

$$a_1 \left[a_2' \left(L_Y g \right) + A' \left(L_X g \right) \right] = 0,$$

$$\left[a_1 (B - 3A') + A(b_1 - a_1') - 2a_2 (b_2 - 2a_2') \right] (L_X g) = 0.$$

Proof. First consider the case $a_1 a_2 b_2 \neq 0$. By the use of (1.3) - (1.5) and the equality $a_1 b_2 = 2a_2(b_1 - a_1')$ Lemma 3.9 yields

$$\mathbf{F} = 2(a_1 B - 2a_2 b_2 - 3a_1 A' + 4a_2 a_2')(L_X g),$$

$$\mathbf{B} = -2a_2 (b_1 - a_1')(L_Y g),$$

whence, by ([9], Lemma 19 or [10], Lemma 54), the first two equalities result. Moreover, by Lemma 3.9 we have

$$\mathbf{F} + \mathbf{B} = -2a_2(b_1 - a_1')(L_Y g) + 2(a_1 B - 2a_2 b_2 - 3a_1 A' + 4a_2 a_2')(L_X g) = 0, \tag{4.22}$$

and

$$\mathbf{A}_{km} = 3a_2 B K_{km} + (3a_1 B - a_2 b_2) P_{km} + (a_1 B - 2a_2 a_2')(L_X g)_{km} + \left[a_2 \left(b_1 - a_1' \right) - 3a_1 a_2' \right] \nabla_k Y_m = 0.$$

Symmetrizing in (k, m) and transforming the obtained equation in the same manner as before we find

$$\left[a_2 \left(b_1 - a_1' \right) - 3a_1 a_2' \right] (L_Y g) - (a_1 B - 2a_2 b_2 + 4a_2 a_2')(L_X g) = 0. \tag{4.23}$$

Now from (4.22) and (4.23) we easily deduce the third equality. Finally, the last one is obtained by applying (1.4) to (4.22).

The proof of the second case can be obtained in the same way. The statements differ only in that $b_2 = 0$. □

Corollary 4.2. *If* $a_1(a_2A' - a_2'A) \neq 0$, *then* $L_X g = 0$.

4.3. **Case 3.** The main result of the section establishes isomorphism between algebras of Killing vector fields on M and TM for a large subclass of $g-$ metrics (Theorem 4.7). Furthermore, conditions for Y to be non-zero are proved.

Lemma 4.4. *Under hypothesis (3.1) suppose that* $dimM > 2$ *and the following conditions on* a_j, b_j *at a point* $(x,0) \in M$ *are satisfied:* $a_1 = 0$, $b_1 = a_1'$, $a_2 \neq 0$, $b_2 \neq 0$. *Then the relations*

$$(b_2 - 2a_2')\, L_X g = 0, \quad (b_2 - 2a_2')\, Tr\,(\nabla X) = 0, \quad (b_2 - 2a_2')\, TrP = 0, \tag{4.24}$$

$$BK = 0, \quad L_X g + P = 0 \tag{4.25}$$

hold. Moreover P *is symmetric. Finally* $a_3 K = 0$.

Proof. Substituting $a_1 = 0$ and $a_1' = b_1$ into (3.12), then applying (1.8) and (1.6) we find

$$\begin{aligned} b_2\left[2g_{kn}\left((L_Xg)_{lm} + P_{lm}\right) + g_{mn}\left((L_Xg)_{kl} + P_{kl}\right) - g_{km}\left((L_Xg)_{ln} + P_{ln}\right)\right] + \\ g_{ln}\left[-3BK_{km} + (b_2 - 2a_2')\,(L_Xg)_{km}\right] + \\ g_{kl}\left[3BK_{mn} + (b_2 - 2a_2')\,(L_Xg)_{mn}\right] - 2(b_2 - 2a_2')g_{lm}\,(L_Xg)_{kn} = 0. \end{aligned} \tag{4.26}$$

From (1.6) it follows that $P^a_a + 2X^a_{,a} = 0$. Thus contracting (4.26) with g^{lm} and then with g^{kn} we get (4.24) in turn. Consequently, contracting (4.26) with g^{kn}, by the use of (1.6), (1.8) and (4.24), we obtain

$$-3BK_{lm} + (n-1)b_2\left[P_{lm} + (L_Xg)_{lm}\right] = 0.$$

In a similar way, contracting (4.24) with g^{kl}, we find

$$-(n+1)BK_{mn} + b_2\left[P_{mn} + (L_Xg)_{mn}\right] = 0.$$

The last two equations yield (4.25). The final statement is a consequence of (4.25), (equation II_1, [9]) and $a_1 = 0$. □

Lemma 4.5. *Under assumptions of Lemma 4.4 relations*

$$\left[(b_2 - 2a_2')\,(2Ab_1 - 3a_2b_2 - 2a_2a_2') - 2a_2Bb_1\right] Y = 0,$$

$$\left[a_2Bb_1 + Ab_1b_2 - 2a_2\,(b_2a_2' - a_2b_2')\right] Y = 0,$$

$$(b_1b_2 - a_2b_1')Y = 0$$

hold on $M \times \{0\}$.

Proof. We apply Lemma 3.8. Substituting $a_1' = b_1$, $a_1 = 0$, contracting with $g^{ab}g^{cl}$ and applying (1.8) we get

$$-2b_2(n+2){K^r}_{k,r} + 2B{E^r}_{kr} - 2B{E_k}^r{}_r + (n-1)(b_2 - 2a_2'){M^r}_{kr} = 0,$$

whence, by the use of Lemma 3.2 we obtain the first equality. Similarly, contracting with $g^{al}g^{bc}$ we find

$$\begin{aligned} - b_2(n+2){K^r}_{k,r} + B(n+2){E^r}_{kr} - B(n+2){E_k}^r{}_r - b_2n{T^r}_{kr} + b_2{T_k}^r{}_r - \\ 2(n+2)(n-1)Y_k = 0, \end{aligned}$$

whence the second equation results. Finally, the third one follows from Lemma 3.5. □

Lemma 4.6. *Under assumptions of Lemma 4.4 suppose $L_X g = 0$. Then*

$$AY = BY = A'Y = 0$$

at each point $(x,0) \in TM$.

Proof. By (1.7), Y is a Killing vector field on M. Moreover, (3.8) reduces to

$$2A' g_{kl} Y_p + B\,(Y_k g_{lp} + Y_l g_{kp}) = 0,$$

whence we easily deduce $BY = A'Y = 0$. Since an infinitesimal isometry is also an infinitesimal affine transformation, from (3.11), by the use of (2.2) and the above properties, we obtain $AY = 0$. □

Lemma 4.7. *Under assumptions of Lemma 4.4 suppose*

$$P + L_X g = 0. \tag{4.27}$$

Then $\nabla P = 0$ if and only if $BY = A'Y = 0$.

Proof. Substituting into (3.8), symmetrizing in (k,p) and applying (4.27) we get

$$a_2 \nabla_l P_{kp} + B(2g_{kp}Y_l + g_{lp}Y_k + g_{kl}Y_p) + 2A'(g_{kl}Y_p + g_{lp}Y_k) = 0,$$

whence the thesis results. □

A complete lift of a Killing vector field on M to (TM, G) is always a Killing vector field ([9], [10]). Thus we have proved

Theorem 4.7. *Let on TM, $dimTM > 4$, a $g-$ natural metric G*

$$\begin{aligned}
G_{(x,u)}(X^h, Y^h) &= A(r^2)g_x(X,Y) + B(r^2)g_x(X,u)g_x(Y,u),\\
G_{(x,u)}(X^h, Y^v) &= a_2(r^2)g_x(X,Y) + b_2(r^2)g_x(X,u)g_x(Y,u),\\
G_{(x,u)}(X^v, Y^h) &= a_2(r^2)g_x(X,Y) + b_2(r^2)g_x(X,u)g_x(Y,u),\\
G_{(x,u)}(X^v, Y^v) &= 0
\end{aligned}$$

be given, where $a_2 b_2 \neq 0$ everywhere on TM while $b_2 - a_2'$ and either A or B do not vanish on a dense subset of TM. If Z is a Killing vector field on TM, then there exists an open subset U containing M such that Z restricted to U is a complete lift of a Killing vector field X on M, i.e.

$$Z_{|U} = X^C.$$

4.4. **Case 4.** The class under consideration contains the Sasaki metric g^S and the Cheeger-Gromoll one g^{CG}. In ([18]) Tanno proved the following

Theorem 4.8. *Let (M,g) be a Riemannian manifold. Let X be a Killing vector field on M, P be a $(1,1)$ tensor field on M that is skew-symmetric and parallel and Y be a vector field on M that satisfies $\nabla_k \nabla_l Y_p + \nabla_l \nabla_k Y_p = 0$ and (4.31). Then the vector field Z on TM defined by*

$$Z = X^C + \iota P + Y^{\#} = (X^r - \nabla^r Y_s u^s)\partial_r^h + (Y^r + S^r_s u^s)\partial_r^v$$

is a Killing vector field on (TM, g^S). Conversely, any Killing vector field on (TM, g^S) is of this form.

A similar theorem holds for (TM, g^{CG}), ([3]). However, in virtue of Lemma 3.5 and the remark after it, the Y component vanishes.

We shall give a simple sufficient condition for ιP to be a Killing vector field on TM. The rest of the section is devoted to investigations on the properties of the Y component.

Notice that $a \neq 0$ and $a_2 = 0$ require $a_1 A \neq 0$. From (3.8) we get immediately

$$a_1 \left(\nabla_k \nabla_l Y_p + \nabla_l \nabla_k Y_p\right) = 2A' g_{kl} Y_p + B\left(Y_k g_{lp} + Y_l g_{kp}\right). \tag{4.28}$$

Since $b_2 = 0$, symmetrizing (equation II_2, [9]) in (k,p) we get $AE_{lkp} = a_2'(g_{lk}Y_p + g_{lp}Y_k)$. Consequently, in virtue of the properties of the Lie derivative (2.2), (3.11) and (1.9) yield

$$a_1 \nabla_l P_{kp} = a_2'(g_{lp}Y_k - g_{lk}Y_p).$$

Moreover, because of $a_2 = 0$, $b_2 = 0$, $\overline{S}_{pq} = 0$ and $\nabla_l X_q + S_{lq} = P_{lq} = -P_{ql}$, identity (equation I_3, [9]) together with (3.4) yields

$$BP_{kp} = a_2' \nabla_k Y_p,$$

whence, since P is skew-symmetric,

$$a_2' L_Y g = 0 \text{ and } a_2' Tr(\nabla Y) = 0 \tag{4.29}$$

result.

Next, Lemma 3.9 yields

$$B\nabla_k X_m - a_2' \nabla_k Y_m + BP_{km} = 0,$$

whence we find

$$B\nabla X = 0.$$

We conclude with

Lemma 4.8. *Suppose (3.1), $dimM > 2$, and $a_2 = 0$, $b_2 = 0$ on $M \times \{0\}$. If $a_2' = 0$ on $M \times \{0\}$, then $BP = 0$ and $\nabla P = 0$ on $M \times \{0\}$.*

By Proposition 5.4 we obtain

Theorem 4.9. *Suppose $a_2(r^2) = 0$, $b_2(r^2) = 0$ and $B(r^2) = 0$ on (TM, G). If M admits non-trivial skew-symmetric and parallel $(0,2)$ tensor field P, then its ι-lift is a Killing vector field on TM.*

Lemma 4.9. *If $a_2 = 0$, $b_2 = 0$ at $(x, 0)$ and $a \neq 0$ everywhere on TM, then*

$$\begin{aligned} 3a_1^2 \left(\nabla^r Y_q R_{rlkp} + \nabla^r Y_p R_{rlkq}\right) = \\ a_1 B\left[\left(2\nabla_q Y_k - \nabla_k Y_q\right) g_{pl} + \left(2\nabla_p Y_k - \nabla_k Y_p\right) g_{ql} - \left(\nabla_p Y_q + \nabla_q Y_p\right) g_{kl}\right] + \\ 2A(b_1 - a_1')\left(2\nabla_l Y_k g_{pq} - \nabla_l Y_p g_{kq} - \nabla_l Y_q g_{kp}\right) \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} 3a_1^2 \nabla^q Y_p R_{qlkr} u^p u^r = \\ 2A(b_1 - a_1')\left[Y_{k,l} r^2 - Y_{p,l} u_k u^p\right] + a_1 B\left[\left(2Y_{k,p} - Y_{p,k}\right) u_l - g_{kl} Y_{p,q} u^q\right] u^p \end{aligned}$$

hold at arbitrary point $(x, 0) \in TM$.

Proof. To prove the lemma it is enough to put $a_2 = 0$, $b_2 = 0$ in (3.12), then multiply by A and apply (1.9). For convenience indices (k, m) are interchanged after that. $\square$

Lemma 4.10. *Suppose (3.1), $dimM > 2$. If neither $\nabla_n Y_m = 0$ nor $\nabla_n Y_m = \frac{T}{n} g_{mn}$, then*

$$\nabla^r Y_q R_{rlkp} + \nabla^r Y_p R_{rlkq} = 0 \tag{4.31}$$

if and only if $B = b = 0$.

Proof. The "only if " part is obvious. Put $T = Y_{r,s} g^{rs}$. Suppose that (4.31) holds. Contracting the right hand side of (4.30) in turn with g^{kl}, $g^{kl}g^{mn}$, g^{lm} and g^{kn}, we get respectively

$$[2Ab + (n-1)a_1 B]\,(Y_{m,n} + Y_{n,m}) - 4AbT g_{mn} = 0,$$

$$(n-1)\,(2Ab - a_1 B)\,T = 0,$$

$$-\,[4Ab + (2n+1)a_1 B]\,Y_{k,n} + [2Ab + (n+2)a_1 B]\,Y_{n,k} + 2AbT g_{kn} = 0,$$

$$-a_1 B Y_{l,m} + 2\,[(n-1)Ab + a_1 B]\,Y_{m,l} - a_1 BT g_{lm} = 0.$$

If $Y_{l,m} - Y_{m,l} \neq 0$, then alternating the last two equations in indices we obtain

$$\begin{aligned} 2Ab + (n+1)a_1 B &= 0, \\ 2(n-1)Ab + 3a_1 B &= 0, \end{aligned}$$

whence $B = b = 0$ for $n \neq 2$ results.

If $Y_{l,m} - Y_{m,l} = 0$, then the suitable linear combination of these equations gives

$$(n-2)\,(2Ab - a_1 B)\,Y_{l,m} + (2Ab - a_1 B)\,T g_{lm} = 0.$$

By the second equation this yields $(2Ab - a_1 B)\,Y_{l,m} = 0$. Applying the last result to the first equality completes the proof. □

Lemma 4.11. *Let (TM, G) be a tangent bundle of a manifold (M, g), $dimM > 2$, with non-degenerate $g-$ natural metric G given by (2.3). Suppose there is given a Killing vector field Z on TM with Taylor expansion (3.2) and (3.3). If the coefficients $a_2(t)$, $b_2(t)$ vanish along M then Y satisfies*

$$A'(2b_1 + a_1')Y = 0, \tag{4.32}$$

$$\{[2B(B + A') - 3AB']\,a_1 + AB(2b_1 + a_1')\}\,Y = 0. \tag{4.33}$$

Proof. Recall that if $a_2(r_0^2) = 0$, then necessary $a_1 A \neq 0$ on some neighbourhood of r_0^2.

From (3.9) we easily get

$$A(\overline{K}_{ab,c} + \overline{K}_{bc,a} + \overline{K}_{ca,b}) + 2(B + A')(g_{bc}Y_a + g_{ca}Y_b + g_{ab}Y_c) = 0. \tag{4.34}$$

From Lemma 3.7, by the use of the assumptions on a_2 and b_2, we find

$$\begin{aligned} &2B\left[\overline{K}_{ab,(k}g_{l)c} + \overline{K}_{bc,(k}g_{l)a} + \overline{K}_{ca,(k}g_{l)b}\right] + \\ &\qquad 3B\left[g_{c(l}T_{k)ab} + g_{a(l}T_{k)bc} + g_{b(l}T_{k)ca}\right] + \\ &6A'g_{kl}M_{abc} - b\left[g_{ab}\left(Y_{c,kl} + Y_{c,lk}\right) + g_{bc}\left(Y_{a,kl} + Y_{a,lk}\right) + g_{ca}\left(Y_{b,kl} + Y_{b,lk}\right)\right] + \\ &\quad 6B'\left[\left(g_{al}g_{kb} + g_{ak}g_{lb}\right)Y_c + \left(g_{bl}g_{kc} + g_{bk}g_{lc}\right)Y_a + \left(g_{al}g_{kc} + g_{ak}g_{lc}\right)Y_b\right] = 0. \end{aligned}$$

Applying (3.8), (3.6) and (3.7) we find

$$\begin{aligned}(4.35)\quad & 2B\left[\overline{K}_{ab,(k}g_{l)c}+\overline{K}_{bc,(k}g_{l)a}+\overline{K}_{ca,(k}g_{l)b}\right]-\\ & \frac{4bB}{a_1}\left[\left(g_{al}g_{bc}+g_{bl}g_{ca}+g_{cl}g_{ab}\right)Y_k+\left(g_{ak}g_{bc}+g_{bk}g_{ca}+g_{ck}g_{ab}\right)Y_l\right]-\\ & \frac{4A'(2b_1+a_1')}{a_1}\left(g_{ab}Y_c+g_{bc}Y_a+g_{ca}Y_b\right)g_{kl}+\frac{6(B'a_1-Ba_1')}{a_1}\times\\ & \left[\left(g_{al}g_{kb}+g_{ak}g_{lb}\right)Y_c+\left(g_{bl}g_{kc}+g_{bk}g_{lc}\right)Y_a+\left(g_{al}g_{kc}+g_{ak}g_{lc}\right)Y_b\right]=0.\end{aligned}$$

Hence, contracting with g^{kl}, we obtain

$$\begin{aligned}(4.36)\quad & B(\overline{K}_{ab,c}+\overline{K}_{bc,a}+\overline{K}_{ca,b})+\\ & \left[3B'-\frac{(B+nA')(2b_1+a_1')}{a_1}\right]\left(g_{bc}Y_a+g_{ca}Y_b+g_{ab}Y_c\right)=0.\end{aligned}$$

If $B\neq 0$, then a linear combination of (4.34) and (4.36) yields $\psi Y=0$ where

$$\psi=2B(B+A')-3AB'+\frac{A(B+nA')(2b_1+a_1')}{a_1}. \tag{4.37}$$

On the other hand, contractions of (4.35) with g^{ak} and then with g^{bl} yield respectively

$$\begin{aligned} & B\left[(n+3)\overline{K}_{bc,l}+\overline{K}^{r}_{c,r}g_{bl}+\overline{K}^{r}_{b,r}g_{cl}\right]=\\ & \frac{2}{a_1}\left[(n+3)bB+A'(2b_1+a_1')\right]g_{bc}Y_l+\\ & \frac{1}{a_1}\left[2bB+3(n+2)(Ba_1'-B'a_1)+2A'(2b_1+a_1')\right]\left(g_{bl}Y_c+g_{cl}Y_b\right)\end{aligned}$$

and

$$2B\overline{K}^{r}_{c,r}=\frac{1}{a_1}\left[4bB+3(n+1)(Ba_1'-B'a_1)+2A'(2b_1+a_1')\right]Y_c.$$

Hence we find

$$\begin{aligned}(4.38)\quad & 2(n+3)a_1B\overline{K}_{bc,l}=4\left[(n+3)bB+A'(2b_1+a_1')\right]g_{bc}Y_l+\\ & \left[3(n+3)(Ba_1'-B'a_1)+2A'(2b_1+a_1')\right]\left(g_{bl}Y_c+g_{cl}Y_b\right)\end{aligned}$$

and

$$\begin{aligned} & (n+3)a_1B(\overline{K}_{bc,l}+\overline{K}_{cl,b}+\overline{K}_{lb,c})-\\ & \left[(n+3)(B(2b_1+a_1')-3a_1B')+4A'(2b_1+a_1')\right]\left(g_{bc}Y_l+g_{cl}Y_b+g_{lb}Y_c\right)=0\end{aligned}$$

If $B\neq 0$, then combining the last relation with (4.36) we obtain (4.32) and, as a consequence of (4.37), equality (4.33). On the other hand, if $B(r_0^2)=0$, then contractions of (4.38) with g^{bc} and g^{bl} yield either $Y^a=0$ or $B'=0$ and $A'(2b_1+a_1')=0$. This completes the proof. □

Lemma 4.12. *For an arbitrary B we have*

$$a_1^2B(\nabla_l\nabla_cY_b+\nabla_l\nabla_bY_c)=-2ABbg_{bc}Y_l-\frac{3}{2}A(Ba_1'-a_1B')(g_{bl}Y_c+g_{cl}Y_b),$$

$$a_1^2 B(\nabla_l \nabla_c Y_b - \nabla_b \nabla_l Y_c) = -B\,(a_1 B + 2Ab)\, g_{bc} Y_l +$$
$$-\frac{1}{2}\left[4A' a_1 B + 3A(Ba_1' - a_1 B')\right] g_{bl} Y_c - \frac{1}{2}\left[2a_1 B^2 + 3A(Ba_1' - a_1 B')\right] g_{cl} Y_b.$$

Proof. We can suppose $B \neq 0$. From (II_1, [9], [10]) we get

$$A \nabla_l \overline{K}_{km} = -a_1\left(\nabla_l \nabla_k Y_m + \nabla_l \nabla_m Y_k\right).$$

Combining this with (4.38), by the use of (4.32), we find the first equality. Hence, by the use of (3.8) and (4.32), we get the second one. On the other hand, if $BY = 0$, then the previous lemma yields $B'Y = 0$. This completes the proof. □

Lemma 4.13. *Under hypothesis (3.1) suppose* $\dim M > 2$ *and* $a_2 = 0$, $b_2 = 0$ *on* $M \times \{0\} \subset TM$. *Then*

$$[Aa_2'(b_1 + a_1') - 2a_1(Ba_2' + Ab_2')]\, Y = 0, \tag{4.39}$$

$$Y\left[a_1 a_2' R - \frac{(Ba_2' + 2Ab_2')}{2} g \wedge g\right] = 0. \tag{4.40}$$

If $a_2' \neq 0$, *then*

$$b_2' \nabla Y = 0, \quad (b_1 - a_1')\nabla Y = 0. \tag{4.41}$$

Proof. For the proof of the first part we apply Lemma 3.8. Substituting $a_2 = 0$, $b_2 = 0$, by the use of (1.9), we get

$$\begin{aligned} a_1\,&[2E^p_{ab}R_{plck} - E^p_{bk}R_{plac} + E^p_{bc}R_{plak} - E^p_{ak}R_{plbc} + E^p_{ac}R_{plbk}] + \\ &B\,[(E_{ckb} - E_{kcb})\, g_{al} + (E_{cak} - E_{kac})\, g_{bl} + \\ &(E_{abk} + E_{bak})\, g_{cl} - (E_{abc} + E_{bac})\, g_{kl}] - 2a_2'\,(M_{abk} g_{cl} - M_{abc} g_{kl}) + \\ &2b_2'\,[(g_{bk} g_{cl} - g_{bc} g_{kl})\, Y_a + (g_{ak} g_{cl} - g_{ac} g_{kl})\, Y_b + \\ &(g_{al} g_{bk} + g_{ak} g_{bl})\, Y_c - (g_{al} g_{bc} + g_{ac} g_{bl})\, Y_k] = 0. \end{aligned}$$

Applying (3.5) - (3.7) and the Bianchi identity we find

$$\begin{aligned} &-a_1^2 a_2'\,[3R_{blck}Y_a + 3R_{alck}Y_b + (R_{akbl} + R_{bkal})Y_c - (R_{acbl} + R_{bcal})Y_k] - \\ &2a_2'\,[A(b_1 + a_1') - a_1 B]\, g_{ab}\,(g_{kl} Y_c - g_{cl} Y_k) - \\ &[2Aa_2'(b_1 + a_1') + a_1(2Ab_2' - Ba_2')]\,[(g_{bc} g_{kl} - g_{bk} g_{cl})\, Y_a + (g_{ac} g_{kl} - g_{ak} g_{cl})\, Y_b] + \\ &a_1(Ba_2' + 2Ab_2')\,[g_{bl}\,(g_{ak} Y_c - g_{ac} Y_k) + g_{al}\,(g_{bk} Y_c - g_{bc} Y_k)] = 0. \end{aligned}$$

Symmetrizing the last relation in (a, b, l) we obtain (4.39). Then, symmetrize in (a, b, k). Since coefficient times Y_c vanishes by (4.39), by the use of the the Walker lemma and (4.39) we get either $Y = 0$ or

$$a_1 a_2'(R_{acbl} + R_{albc}) = (Ba_2' + 2Ab_2')(g_{al} g_{bc} + g_{ac} g_{bl} - 2g_{ab} g_{cl}),$$

whence, alternating in (b, l), we easily obtain

$$a_1 a_2' R_{acbl} = (Ba_2' + 2Ab_2')(g_{al} g_{bc} - g_{ab} g_{cl}).$$

Thus (4.40) is proved.

Suppose now $Y \neq 0$ and $a_2' \neq 0$ on $M \times \{0\}$. Applying (4.40) to (4.30) and eliminating B, by the use of (4.29), we obtain

$$\begin{aligned}(4.42)\quad & 2(b_1 - a_1')\,(Y_{n,l}g_{km} + Y_{m,l}g_{kn} - 2Y_{k,l}g_{mn}) + \\ & \left(2\frac{a_1 b_2'}{a_2'} - b_1 - a_1'\right)(Y_{k,m}g_{ln} + Y_{k,n}g_{lm}) - \\ & \left(4\frac{a_1 b_2'}{a_2'} + b_1 + a_1'\right)(Y_{m,k}g_{ln} + Y_{n,k}g_{lm}) = 0.\end{aligned}$$

Contracting (4.42) with g^{lm}, by the use of (4.29) we get

$$\left[(n+1)\frac{a_1 b_2'}{a_2'} - (b_1 - a_1')\right] Y_{k,n} = 0.$$

On the other hand, by contraction with g^{mn} we obtain

$$\left[\frac{a_1 b_2'}{a_2'} - (n-1)(b_1 - a_1')\right] Y_{k,l} = 0.$$

Hence we easily get either $\nabla Y = 0$ or both $b_2' = 0$ and $b_1 - a_1' = 0$ on $M \times \{0\}$.

Remark 4.2. If $a_2' \neq 0$ and $Y \neq 0$, then equations (4.41) give a further restriction on the metric G. Namely, if $\nabla Y = 0$, then from (1.9) we infer $K = 0$ while from (4.28) we get $A' = 0$ and $B = 0$ on $M \times \{0\}$. Consequently, (4.33) yields $B' = 0$.

□

Remark 4.3. On the other hand, substituting $b_2' = 0$ and $b_1 = a_1'$ into Lemma 3.5 we get $b_1' = 0$ on $M \times \{0\}$.

5. Lifts properties

5.1. **Vertical lift X^v.**

Proposition 5.1. *The vertical lift $X^v = X^r\partial_r^v$ of a Killing vector field $X = X^r\partial_r$ to (TM, G) with non-degenerate $g-$ natural metrics G is a Killing vector field on TM if and only if $a_j' = 0$ and $b_j = 0$ on TM.*

Proof. Suppose X^v is a Killing vector field. Since X is also the Killing one, ([9], equation 6) yields

$$b_2(X_{r,k}u_l + X_{r,l}u_k)u^r + B(u_kX_l + u_lX_k) + 2u^rX_r(A'g_{kl} + B'u_ku_l) = 0,$$

whence, by contraction with g^{kl} and u^ku^l we obtain

$$2u^rX_r(B + nA' + r^2B') = 0$$

and

$$2r^2u^rX_r(B + A' + r^2B') = 0$$

since X is a Killing vector field on M. Thus $A' = 0$ and the only smooth solution to $B + r^2B' = 0$ on TM is $B = 0$. In similar manner, from ([9], equation 7 and 8) we deduce that $a_1' = a_2' = 0$ and $b_1 = b_2 = 0$ on TM. The "only if" part is obvious. Thus the proposition is proved. □

5.2. $V^a\partial_a^v = u^p\nabla^r Y_p \partial_r^v$. Let Y be a non-parallel Killing vector field on M and consider its lift $u^p\nabla^r Y_p\partial_r^v$ to (TM,G). Then we have $\partial_k^v V^a = \nabla^a Y_k$, $\partial_k^h V^a = u^p\Theta_k(\nabla^a Y_p)$ and from ([9] or [10], equations 6, 7 and 8) we obtain

$$\left(L_{V^a\partial_a^v}G\right)\left(\partial_k^h,\partial_l^h\right) = a_2(\nabla_l\nabla_k Y_p + \nabla_l\nabla_k Y_p)u^p + B(\nabla_k Y_p u^p u_l + \nabla_l Y_p u^p u_k),$$
$$\left(L_{V^a\partial_a^v}G\right)\left(\partial_k^v,\partial_l^h\right) = a_2\nabla_l Y_k + a_1\nabla_l\nabla_k Y_p u^p + b_2\nabla_l Y_p u^p u_k,$$
$$\left(L_{V^a\partial_a^v}G\right)\left(\partial_k^v,\partial_l^v\right) = 0.$$

Hence we deduce

Proposition 5.2. *Let Y be a non-parallel Killing vector field on M satisfying $\nabla\nabla Y = 0$. Then $u^p\nabla^r Y_p\partial_r^v$ is a Killing vector field on TM if and only if $a_2 = b_2 = B = 0$ on TM.*

Proposition 5.3. *Let Y be a non-parallel Killing vector field on M. If $a_2 = b_2 = B = 0$ on TM and $u^p\nabla^r Y_p\partial_r^v$ is a Killing vector field on TM then $\nabla\nabla Y = 0$ on M.*

5.3. ιP.

Proposition 5.4. *Let P be an arbitrary (0,2)-tensor field on (M,g). Then its ι - lift $\iota P = u^r P_r^a\partial_a^v$ to (TM,G) with non-degenerate $g-$ natural metric G satisfies*

$$\begin{aligned}(L_{\iota P}G)\left(\partial_k^h,\partial_l^h\right) &= a_2u^r\left(\nabla_k P_{lr} + \nabla_l P_{kr}\right) + b_2u^pu^r\left(\nabla_k P_{pr}u_l + \nabla_l P_{pr}u_k\right) + \\ &\quad 2(A'g_{kl} + B'u_ku_l)P_{pr}u^pu^r + Bu^r\left(P_{kr}u_l + P_{lr}u_k\right),\\ (L_{\iota P}G)\left(\partial_k^v,\partial_l^h\right) &= a_2P_{lk} + b_2u^rP_{rk}u_l + a_1u^r\nabla_l P_{kr} + b_1\nabla_l P_r^a u_a u^r u_k + \\ &\quad 2(a_2'g_{kl} + b_2'u_ku_l)P_{pr}u^pu^r + b_2u^r\left(P_{kr}u_l + P_{lr}u_k\right),\\ (L_{\iota P}G)\left(\partial_k^v,\partial_l^v\right) &= a_1\left(P_{kl} + P_{lk}\right) + b_1\left[u^p\left(P_{kp} + P_{pk}\right)u_l + u^p\left(P_{lp} + P_{pl}\right)u_k\right] + \\ &\quad 2(a_1'g_{kl} + b_1'u_ku_l)P_{pr}u^pu^r.\end{aligned}$$

Proof. The Proposition follows from ([9] or [10], equations 6, 7 and 8), where we have $H^a = 0$, $V^a = u^rP_r^a$, $\partial_k^v V^a = P_k^a$, $\partial_k^h V^a = u^p\left(\partial_k P_p^a - \Gamma_{pk}^t P_t^a\right)$ and $\partial_k^h V^a + V^r\Gamma_{kr}^a = u^p\nabla_k P_p^a$. □

Hence we easily get

Proposition 5.5. *Let P be a skew-symmetric (0,2)-tensor field on (M,g). Then its ι - lift $\iota P = u^rP_r^a\partial_a^v$ to (TM,G) with non-degenerate $g-$ natural metric G satisfies*

$$\begin{aligned}(L_{\iota P}G)\left(\partial_k^h,\partial_l^h\right) &= a_2\left(u^r\nabla_k P_{lr} + u^r\nabla_l P_{kr}\right) + B\left(u^rP_{kr}u_l + u^rP_{lr}u_k\right),\\ (L_{\iota P}G)\left(\partial_k^v,\partial_l^h\right) &= a_2P_{lk} + b_2u^rP_{lr}u_k + a_1u^r\nabla_l P_{kr},\\ (L_{\iota P}G)\left(\partial_k^v,\partial_l^v\right) &= 0.\end{aligned}$$

5.3.1. $\iota C^{[X]}$. Put $C^{[X]} = \left(\left(C^{[X]}\right)_k^h\right) = \left(-g^{hr}\left(L_Xg\right)_{rk}\right) = \left(-\left(\nabla^h X_k + \nabla_k X^h\right)\right)$ on (M,g). Then its ι-lift $\iota C^{[X]} = \left(0,\ u^k\left(C^{[X]}\right)_k^h\right) = \left(0,\ -u^k\left(\nabla^h X_k + \nabla_k X^h\right)\right)$ is a vertical vector field on TM. In adapted coordinates $\left(\partial_k^v,\partial_l^h\right)$ we have

$$\iota C^{[X]} = -u^k\left(\nabla^h X_k + \nabla_k X^h\right)\partial_h^v.$$

Applying ([9] or [10], equations 6, 7 and 8) , we easily get

$$\begin{aligned}&\left(L_{\iota C^{[X]}}G\right)\left(\partial_k^h,\partial_l^h\right)=\\&\quad-a_2u^p\left[\nabla_k\nabla_lX_p+\nabla_l\nabla_kX_p+\nabla_k\nabla_pX_l+\nabla_l\nabla_pX_k\right]-\\&\quad2b_2u^pu^q\left[\nabla_k\nabla_pX_qu_l+\nabla_l\nabla_pX_qu_k\right]-4\left(A'g_{kl}+B'u_ku_l\right)u^pu^q\nabla_pX_q-\\&\quad Bu^p\left[\left(\nabla_kX_p+\nabla_pX_k\right)u_l+\left(\nabla_lX_p+\nabla_pX_l\right)u_k\right],\end{aligned}$$

$$\begin{aligned}&\left(L_{\iota C^{[X]}}G\right)\left(\partial_k^v,\partial_l^h\right)=\\&\quad-a_2\left(\nabla_kX_l+\nabla_lX_k\right)-b_2u^p\left[2\left(\nabla_kX_p+\nabla_pX_k\right)u_l+\left(\nabla_lX_p+\nabla_pX_l\right)u_k\right]-\\&\quad a_1u^p\left(\nabla_l\nabla_kX_p+\nabla_l\nabla_pX_k\right)-2b_1u^pu^q\nabla_l\nabla_pX_qu_k-\\&\quad4\left(a_2'g_{kl}+b_2'u_ku_l\right)u^pu^q\nabla_pX_q,\end{aligned}$$

$$\begin{aligned}&\left(L_{\iota C^{[X]}}G\right)\left(\partial_k^v,\partial_l^v\right)=\\&\quad-2a_1\left(\nabla_kX_l+\nabla_lX_k\right)-2b_1u^p\left[\left(\nabla_kX_p+\nabla_pX_k\right)u_l+\left(\nabla_lX_p+\nabla_pX_l\right)u_k\right]-\\&\quad4\left(a_1'g_{kl}+b_1'u_ku_l\right)u^pu^q\nabla_pX_q.\end{aligned}$$

5.3.2. *Complete lift X^C of X to (TM, G).* We have $X^C=(X^r\partial_r)^C=X^r\partial_r+\partial X^r\delta_r=X^r\partial_r^h+u^p\nabla_pX^r\partial_r^v$. Making use of ([9] or [10], equations 6, 7 and 8) we obtain

$$\begin{aligned}&\left(L_{X^C}G\right)\left(\partial_k^h,\partial_l^h\right)=\\&\quad a_2u^p\left[\nabla_k\nabla_pX_l+X^rR_{rkpl}+\nabla_l\nabla_pX_k+X^rR_{rlpk}\right]+\\&\quad b_2u^pu^q\left[\left(\nabla_k\nabla_pX_q+X^rR_{rkpq}\right)u_l+\left(\nabla_l\nabla_pX_q+X^rR_{rlpq}\right)u_k\right]+\\&\quad A\left(\nabla_kX_l+\nabla_lX_k\right)+Bu^p\left[\left(\nabla_kX_p+\nabla_pX_k\right)u_l+\left(\nabla_lX_p+\nabla_pX_l\right)u_k\right]+\\&\quad2\left(A'g_{kl}+B'u_ku_l\right)u^pu^q\nabla_pX_q,\end{aligned}$$

$$\begin{aligned}&\left(L_{X^C}G\right)\left(\partial_k^v,\partial_l^h\right)=\\&\quad a_1u^p\left[\nabla_l\nabla_pX_k+X^rR_{rlpk}\right]+a_2\left(\nabla_kX_l+\nabla_lX_k\right)+\\&\quad b_2u^p\left[\left(\nabla_kX_p+\nabla_pX_k\right)u_l+\left(\nabla_lX_p+\nabla_pX_l\right)u_k\right]+\\&\quad b_1u^pu^q\left(\nabla_l\nabla_pX_q+X^rR_{rlpq}\right)u_k+2\left(a_2'g_{kl}+b_2'u_ku_l\right)u^pu^q\nabla_pX_q,\end{aligned}$$

$$\begin{aligned}&\left(L_{X^C}G\right)\left(\partial_k^v,\partial_l^v\right)=\\&\quad a_1\left(\nabla_kX_l+\nabla_lX_k\right)+b_1u^p\left[\left(\nabla_kX_p+\nabla_pX_k\right)u_l+\left(\nabla_lX_p+\nabla_pX_l\right)u_k\right]+\\&\quad2\left(a_1'g_{kl}+b_1'u_ku_l\right)u^pu^q\nabla_pX_q.\end{aligned}$$

5.3.3. *$\iota C^{[X]}+X^C$ for an infinitesimal affine transformation.* Suppose that X is an infinitesimal affine transformation on M. Then by (2.2) and the definition

$$\nabla_k\nabla_lX_p+\nabla_k\nabla_pX_l=\nabla_k\nabla_lX_p+X^rR_{rklp}+\nabla_k\nabla_pX_l+X^rR_{rkpl}=0$$

and

$$u^pu^q\nabla_k\nabla_pX_q=-u^pu^qX^rR_{rkpq}=0.$$

Therefore, applying results of previous subsections, we find

$$\left(L_{\iota C^{[X]}+X^C}G\right)\left(\partial_k^h,\partial_l^h\right)=A\left(\nabla_kX_l+\nabla_lX_k\right)-2\left(A'g_{kl}+B'u_ku_l\right)u^pu^q\nabla_pX_q,$$

$$\left(L_{\iota C[X]+X^C} G\right)\left(\partial_k^v, \partial_l^h\right) = \\ -2\left(a_2' g_{kl} + b_2' u_k u_l\right) u^p u^q \nabla_p X_q - b_2 u^p \left(\nabla_k X_p + \nabla_p X_k\right) u_l,$$

$$\left(L_{\iota C[X]+X^C} G\right)\left(\partial_k^v, \partial_l^v\right) = -a_1\left(\nabla_k X_l + \nabla_l X_k\right) - \\ b_1 u^p\left[\left(\nabla_k X_p + \nabla_p X_k\right) u_l + \left(\nabla_l X_p + \nabla_p X_l\right) u_k\right] - 2\left(a_1' g_{kl} + b_1' u_k u_l\right) u^p u^q \nabla_p X_q.$$

6. Appendix

A $(0,4)$ tensor B on a manifold M is said to be a generalized curvature tensor if

$$B(V,X,Y,Z) + B(V,Y,Z,X) + B(V,Z,X,Y) = 0$$

and

$$B(V,X,Y,Z) = -B(X,V,Y,Z), \qquad B(V,X,Y,Z) = B(Y,Z,V,X)$$

for all vector fields V, X, Y, Z on M ([16]).

For a $(0,k)$ tensor T, $k \geq 1$, we define

$$(R \cdot T)(X_1,...,X_k;X,Y) = \nabla_Y \nabla_X T(X_1,...,X_k) - \nabla_X \nabla_Y T(X_1,...,X_k).$$

For more details see for example ([6]) or ([11]).

The Kulkarni-Nomizu product of symmetric $(0,2)$ tensors A and B is given by

$$(A \wedge B)(U,X,Y,V) = \\ A(X,Y)B(U,V) - A(X,V)B(U,Y) + A(U,V)B(X,Y) - A(U,Y)B(X,V).$$

Theorem 6.1. [12] *Let (M,g) be a semi-Riemannian manifold with metric g, $dimM > 2$. Let g_X be a 1-form associated to g, i.e. $g_X(Y) = g(Y,X)$ for any vector field Y.*

If B is generalized curvature tensor having the property $R \cdot B = 0$ and P is a one-form on M satisfying

$$(R \cdot V)(X;Y,Z) = (P \wedge g_X)(Y,Z), \tag{6.1}$$

for some 1-form V, then

$$P\left(B - \frac{TrB}{2n(n-1)} g \wedge g\right) = 0.$$

If A is a symmetric $(0,2)$-tensor on M having the properties $R \cdot A = 0$ and (6.1) then

$$P\left(A - \frac{TrA}{n} g\right) = 0.$$

Lemma 6.1. [19] *Let A_l, B_{hk} where $l,h,k = 1,...,n$ be numbers satisfying*

$$B_{hk} = B_{kh}, \quad A_l B_{hk} + A_h B_{kl} + A_k B_{lh} = 0.$$

Then either $A_l = 0$ for all l or $B_{hk} = 0$ for all h,k.

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