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RESEARCH ARTICLE / ARAŞTIRMA MAKALESİ

A Note on Ring Source over Semi-Infinite Rigid Pipe

Yarı Sonsuz Rijit Boru Üzerindeki Halka Kaynak Üzerine Bir Not

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Abstract

In this study, semi-infinite rigid pipe is considered. An analytical solution is presented for the diffraction problem of acoustic waves spreading from a ring source by semi-infinite pipe. Applying the boundary and continuity conditions in conjunction with the Fourier transform technique, the boundary value problem is solved analytically. The influence of the problem parameters on the diffraction phenomenon is displayed graphically.

Keywords: Ring Source, Diffraction, Fourier Transform, Pipe.

Öz

Bu çalışmada, yarı sonsuz rijit bir boru ele alınmıştır. Bir halka kaynaktan yayılan akustik dalgaların yarı sonsuz boru ile kırınımı problemi için analitik bir çözüm sunulmuştur. Sınır ve süreklilik koşullarını Fourier dönüşüm tekniği ile birlikte uygulayarak, sınır değer problemi analitik olarak çözülmüştür. Problem parametrelerinin kırınım fenomeni üzerindeki etkisi grafiksel olarak gösterilmiştir.

Anahtar Kelimeler: Halka Kaynak, Kırınım, Fourier Dönüşümü, Boru

I. INTRODUCTION

Diffraction of acoustic waves is an important problem which has been extensively studied in the literature so far. In particular, the problem of diffraction of sound waves by semi-infinite pipes has been used as a model for many engineering applications, such as noise reduction in architectural and experimental aerodynamics, in road transportation, in modern aircraft jet and turbofan engines, etc. For this reason, it is essential to investigate more accurate mathematical models for such engineering problems.

Levine & Schwinger was the first who considered the problem of sound radiation from a semi-infinite circular unflanged duct of infinitely thin hard walls [1]. An analytical solution was obtained based on the Wiener-Hopf technique [2]. Later, different geometries were investigated rigorously by the help of Wiener-Hopf technique [3-6]. Acoustic absorbing material was used in some of these studies for reducing noise.

The goal of this work is to consider the diffraction of acoustic waves emanating from a ring source by semi-infinite rigid pipe. Pipe walls are assumed to be infinitely thin. The ring source is located out of the pipe ($\rho = b > a, z = -c, c > 0$). The total field have angular symmetry which makes the problem simpler than the asymmetric case because of the ring source [7,8]. By applying the Fourier transform, we obtain a Wiener-Hopf equation which depends on the boundary conditions and continuity relations. Then, numerical solution is obtained approximately for various values of the problem parameters such as pipe radius, pipe extension, ring source location etc. The influence of these factors on the diffraction phenomenon is presented graphically. At the end of the analysis same geometry is considered with infinite pipe for validation of the results. Time dependency is assumed to be $e^{-i\omega t}$ and suppressed throughout this work, where ω is the angular frequency.

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II. ANALYSIS

2.1.Problem Formulation

We consider the diffraction of acoustic waves by semi-infinite rigid pipe (Fig. 1). The total field will be independent of azimuth θ everywhere in cylindrical coordinate system (ρ, θ, z) due to the symmetry of the problem geometry and of the ring source. The velocity potential u will be used to obtain acoustic pressure p and velocity v via $p = -\rho_0 (\partial/\partial t) u$ and $v = \mathrm{grad} u$, where ρ_0 is the density of the medium.

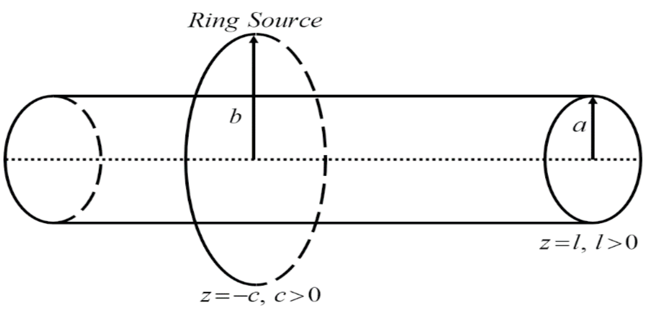

Figure 1. Geometry of the problem

It is suitable to state the total field as follows for analysis intents

$$u^T(\rho, z) = \begin{cases} u_1(\rho, z), & \rho > b \\ u_2(\rho, z), & a < \rho < b, \\ u_3(\rho, z), & \rho < a, \ z > l \\ u_4(\rho, z), & \rho < a, \ z < l \end{cases} \tag{2.1}$$

The unknown fields $u_j(\rho, z), j = 1-4$, satisfy the wave equation for $z \in (-\infty, \infty)$

$$\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{\partial^2}{\partial z^2} + k^2\right] u_j(\rho, z) = 0 \quad , \quad \mathrm{j} = 1-4 \tag{2.2}$$

with wave number $k = \omega / c_0$ and the speed of sound c_0. For the determination of unknown fields, we need boundary conditions and continuity relations, one can write these equations from the geometry of the problem.

$$\frac{\partial}{\partial \rho} u_1(b, z) - \frac{\partial}{\partial \rho} u_2(b, z) = \delta(z + c) \quad , \quad -\infty < z < \infty \tag{2.3}$$

$$u_1(b, z) = u_2(b, z) \quad , \quad -\infty < z < \infty \tag{2.4}$$

$$\frac{\partial}{\partial \rho} u_2(a, z) \quad , \quad z < l \tag{2.5}$$

$$u_2(a, z) = u_3(a, z) \quad , \quad z > l \tag{2.6}$$

$$\frac{\partial}{\partial \rho} u_2(a, z) = \frac{\partial}{\partial \rho} u_3(a, z) \quad , \quad z > l \tag{2.7}$$

$$u_3(\rho, l) = u_4(\rho, l) \quad , \quad \rho < a \tag{2.8}$$

$$\frac{\partial}{\partial z} u_3(\rho, l) = \frac{\partial}{\partial z} u_4(\rho, l) \quad , \quad \rho < a \tag{2.9}$$

where δ is dirac delta function.

2.2. Derivation of the Wiener-Hopf Equation

Consider the Fourier transform of the wave equation satisfied by the diffracted field $u_1(\rho, z)$ in the region $\rho > b$, for $z \in (-\infty, \infty)$, namely,

$$\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + K^2(\alpha)\right] F(\rho, \alpha) = 0 \tag{2.10}$$

where $K(\alpha)$ is a square root function

$$K(\alpha) = \sqrt{k^2 - \alpha^2} \tag{2.11}$$

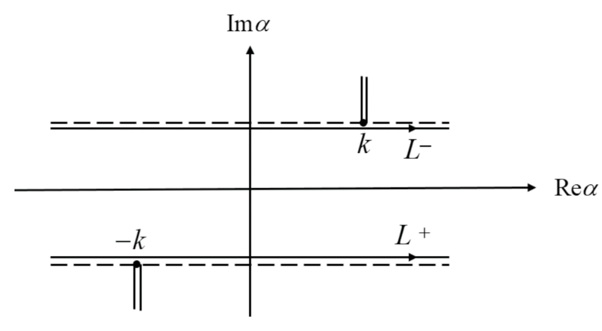

Figure 2. Complex α-plane

which as defined in the complex α-plane cut as shown in Fig. 2 and $F(\rho, \alpha)$ is the Fourier transform of the field $u_1(\rho, z)$ defined to be

$$F(\rho, \alpha) = \int_{-\infty}^{\infty} u_1(\rho, z) e^{i\alpha z} dz \tag{2.12}$$

The solution of equation (2.10) reads

$$F(\rho, \alpha) = A(\alpha) H_0^{(1)}(K\rho) \tag{2.13}$$

where $A(\alpha)$ is spectral coefficient to be determined and $H_0^{(1)}$ is the Hankel function of first type. In the region $a < \rho < b$, $u_2(\rho, z)$ satisfies the wave equation in the range $z \in (-\infty, \infty)$. In a similar way the solution can be given as

$$e^{i\alpha l}(G^-(\rho, \alpha) + G^+(\rho, \alpha)) = B(\alpha) J_0(K\rho) + C(\alpha) Y_0(K\rho) \tag{2.14}$$

where

$$G^-(\rho,\alpha) = \int_{-\infty}^{l} u_2(\rho,z) e^{i\alpha(z-l)} dz \tag{2.15}$$

$$G^+(\rho,\alpha) = \int_{l}^{\infty} u_2(\rho,z) e^{i\alpha(z-l)} dz \tag{2.16}$$

Owing to analytical properties of Fourier integrals, $G^+(\rho,\alpha)$ and $G^-(\rho,\alpha)$ are regular functions in the upper half plane $(Im\alpha > Im(-k))$ and in the lower half plane $(Im\alpha < Imk)$, respectively. Consider now the Fourier transform of (2.5)

$$\dot{G}^-(a,\alpha) = 0 \tag{2.17}$$

where the dot specifies the derivative with respect to ρ. From the definition of ring source given in (2.3) and (2.4) we get

$$A(\alpha)H_1^{(1)}(Kb) = B(\alpha)J_1(Kb) + C(\alpha)Y_1(Kb) - e^{-i\alpha c}/K(\alpha) \tag{2.18}$$

$$A(\alpha)H_0^{(1)}(Kb) = B(\alpha)J_0(Kb) + C(\alpha)Y_0(Kb) \tag{2.19}$$

One can obtain the relation $B(\alpha)$ and $C(\alpha)$ from (2.18) and (2.19).

$$C(\alpha) - iB(\alpha) = -\frac{\pi b}{2} H_0^{(1)}(Kb) e^{-i\alpha c} \tag{2.20}$$

By taking the derivative of (2.14) with respect to ρ and using (2.17), we obtain

$$e^{i\alpha l}\dot{G}^+(a,\alpha) = -B(\alpha)KJ_1(Ka) - C(\alpha)KY_1(Ka) \tag{2.21}$$

In the region $\rho < a, z > l$ the field $u_3(\rho,z)$ satisfies the wave equation for $z \in (l,\infty)$. By taking Fourier transform we get

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + K^2(\alpha)\right] H^+(\rho,\alpha) = f(\alpha) - i\alpha g(\alpha) \tag{2.22}$$

where

$$f(\rho) = \frac{\partial}{\partial z} u_3(\rho,l) \quad , \quad g(\rho) = u_3(\rho,l) \tag{2.23}$$

In (2.22), $H^+(\rho,\alpha)$ is a regular function in the upper half of the complex α-plane which is defined as

$$H^+(\rho,\alpha) = \int_{l}^{\infty} u_3(\rho,z) e^{i\alpha(z-l)} dz \tag{2.24}$$

Particular solution to (2.22) can be found easily by using Green's function which satisfies the wave equation

$$H^+(\rho,\alpha) = \frac{1}{KJ_1(Ka)}\left\{-\dot{G}^+(a,\alpha)J_0(K\rho) + \int_0^a \big(f(t) - i\alpha g(t)\big) Q(t,\rho,\alpha)\, t\,dt\right\} \tag{2.25}$$

where

$$Q(t,\rho,\alpha) = K(\alpha)\frac{\pi}{2}\begin{cases} J_0(K\rho)[J_1(Ka)Y_0(Kt) - Y_1(Ka)J_0(Kt)], & \rho < t \\ J_0(Kt)[J_1(Ka)Y_0(K\rho) - Y_1(Ka)J_0(K\rho)], & \rho > t \end{cases} \tag{2.26}$$

The left hand side of (2.25) is analytic in the upper half plane, the right hand side have poles at $\alpha = \alpha_m, m = 1,2,\ldots$

In order that the right hand side of (2.25) be also regular at $\alpha = \alpha_m$, we should have

$$J_1(j_m) = 0 \ , \ \alpha_m = \sqrt{k^2 - (j_m/a)^2} \ , \quad \alpha_0 = k \ , \ \mathrm{Im}\alpha_m > \mathrm{Im}k \tag{2.27}$$

$$\dot{G}^+(a,k) = \frac{a}{2}[f_0 - ikg_0] \tag{2.28}$$

$$\dot{G}^+(a,\alpha_m) = \frac{a}{2}J_0(j_m)[f_m - i\alpha_m g_m] \tag{2.29}$$

where

$$f_0 = \frac{2}{a^2}\int_0^a f(\rho)\rho\, d\rho \tag{2.30}$$

$$f_m = \frac{2}{a^2 J_0^2(j_m)}\int_0^a f(\rho) J_0\left(\frac{j_m}{a}\rho\right)\rho\, d\rho \tag{2.31}$$

$$g_0 = \frac{2}{a^2}\int_0^a g(\rho)\rho\, d\rho \tag{2.32}$$

$$g_m = \frac{2}{a^2 J_0^2(j_m)}\int_0^a g(\rho) J_0\left(\frac{j_m}{a}\rho\right)\rho\, d\rho \tag{2.33}$$

Using the continuity relation (2.6) and taking into account (2.14), (2.20) and (2.25), we obtain the following Wiener-Hopf equation:

$$-\frac{a}{2}e^{i\alpha l}G^{-}(a,\alpha)+\frac{e^{i\alpha l}\dot{G}^{+}(a,\alpha)}{K^{2}(\alpha)M(\alpha)}=e^{-i\alpha c}\frac{b}{2}\frac{H_0^{(1)}(Kb)}{KH_1^{(1)}(Ka)}+e^{i\alpha l}\frac{a}{2}\sum_{m=0}^{\infty}\frac{J_0(j_m)}{\alpha_m^2-\alpha^2}[f_m-i\alpha g_m] \tag{2.34}$$

where

$$M(\alpha)=i\pi J_1(Ka)H_1^{(1)}(Ka) \tag{2.35}$$

2.3. Solution of the Wiener-Hopf Equation

Multiplying (2.34) with $e^{-i\alpha l}(k-\alpha)M_-(\alpha)$ and decomposition of the resulting equation, we get

$$\frac{\dot{G}^{+}(a,\alpha)}{(k+\alpha)M_+(\alpha)}=I(\alpha)+\frac{a}{2}\sum_{m=0}^{\infty}\frac{(k+\alpha_m)M_+(\alpha_m)J_0(j_m)[f_m+i\alpha_m g_m]}{2\alpha_m(\alpha+\alpha_m)} \tag{2.36}$$

where

$$I(\alpha)=\frac{b}{2}\frac{1}{2\pi i}\int_{L_+}\frac{H_0^{(1)}(Kb)M_-(\tau)(k-\tau)}{KH_1^{(1)}(Ka)(\tau-\alpha)}e^{-i\tau(c+l)}d\tau \tag{2.37}$$

Integration lines L_+ and L_- are depicted in Fig. 2. According to Jordan's Lemma, the integration line L_+ can be deformed into the branch cut C_1+C_2 through the branch point $\tau=-k$. By using the property of Hankel function and making the following substitution:

$$k+\tau=te^{-i\pi/2} \tag{2.38}$$

the integral can be reduced simple form. When l is large, the main contribution to the integral comes from the end point $t=0$ [9]. Hence $I(\alpha)$ can be approximated by

$$I(\alpha)=\frac{b}{4\pi}M_+(k)e^{ik(c+l)}\xi(a,b,c,l;\alpha) \tag{2.39}$$

where

$$\xi(a,b,c,l;\alpha)=\int_0^{\infty}\frac{2k+it}{k+it+\alpha}\frac{H_0^{(1)}(Kb)H_1^{(2)}(Ka)-H_1^{(1)}(Ka)H_0^{(2)}(Kb)}{KH_1^{(1)}(Ka)H_1^{(2)}(Ka)}e^{-t(c+l)}dt \tag{2.40}$$

$M_+(\alpha)$ and $M_-(\alpha)$ are the split functions regular and free of zeros in the upper and lower halves of the complex α – plane, respectively.

$$M(\alpha)=M_+(\alpha)M_-(\alpha)\quad,\quad M_-(\alpha)=M_+(-\alpha) \tag{2.41}$$

The explicit expression of $M_+(\alpha)$ is given in [10] as follows

$$M_+(\alpha)=\sqrt{\pi iJ_1(ka)H_1^{(1)}(ka)}\exp\left\{i\frac{\alpha a}{\pi}\left[1-\gamma+\ln\left(\frac{2\pi}{ka}\right)+i\frac{\pi}{2}\right]-i\frac{ka}{2}\right\}$$
$$\times\exp\left\{\frac{K(\alpha)a}{\pi}\ln\left(\frac{\alpha+iK(\alpha)}{k}\right)+q(\alpha)\right\}\prod_{m=1}^{\infty}\left(1+\frac{\alpha}{\alpha_m}\right)\exp\left(\frac{i\alpha a}{m\pi}\right) \tag{2.42}$$

where $\gamma=0.57721\ldots$ is the Euler's constant and $q(\alpha)$ is given by

The coefficients f_m and g_m which are required in the evaluation of $\dot{G}^{+}(a,\alpha)$, are obtained from (2.8) – (2.9) and (2.28) – (2.29).

$$q(\alpha)=\frac{1}{\pi}\int_0^{\infty}\left[1-\frac{2}{\pi x}\frac{1}{J_1^2(x)+Y_1^2(x)}\right]\ln\left(1+\frac{\alpha a}{\sqrt{(ka)^2-x^2}}\right) \tag{2.43}$$

2.4. Far Field

In the region $\rho>b$ total field can be evaluated from (2.12)

$$u_1(\rho,z)=\frac{1}{2\pi}\int_L A(\alpha)H_0^{(1)}(K\rho)e^{-i\alpha z}d\alpha \tag{2.44}$$

where L is a straight line parallel to the real α-axis, lying in the strip $Im(-k)<Im\alpha<Imk$. Using (2.14), (2.20)

$$u_1(\rho,z)=u_i(\rho,z)+u_r(\rho,z)+u_d(\rho,z) \tag{2.45}$$

where

$$u_i(\rho,z)+u_r(\rho,z)=\frac{b}{4}\int_L \frac{[Y_1(Ka)J_0(Kb)-J_1(Ka)Y_0(Kb)]}{H_1^{(1)}(Ka)}H_0^{(1)}(K\rho)e^{-i\alpha(z+c)}d\alpha \tag{2.46}$$

$$u_d(\rho,z)=-\frac{1}{2\pi}\int_L \frac{\dot{G}^+(a,\alpha)}{KH_1^{(1)}(Ka)}H_0^{(1)}(K\rho)e^{-i\alpha(z-l)}d\alpha \tag{2.47}$$

and replacing $H_0^{(1)}(K\rho)$ by its following asymptotic expression valid for $k\rho \gg 1$.

$$H_0^{(1)}(K\rho)\sim\sqrt{\frac{2}{\pi K\rho}}e^{i(K\rho-\pi/4)} \tag{2.48}$$

and applying the saddle point technique, we get

$$u_1(\rho,z)=u_d(r_1,\theta_1)+u_i(r_2,\theta_2)+u_r(r_2,\theta_2) \tag{2.49}$$

with

$$u_d(r_1,\theta_1)=\frac{i}{\pi}\frac{\dot{G}^+(a,-kcos\,\theta_1)}{sin\,\theta_1 H_1^{(1)}(kasin\,\theta_1)}\frac{e^{ikr_1}}{kr_1} \tag{2.50}$$

$$u_i(r_2,\theta_2)+u_r(r_2,\theta_2)=\frac{kb}{2i}\frac{[Y_1(kasin\,\theta_2)J_0(kbsin\,\theta_2)-J_1(kasin\,\theta_2)Y_0(kbsin\,\theta_2)]}{H_1^{(1)}(kasin\,\theta_2)}\frac{e^{ikr_2}}{kr_2} \tag{2.51}$$

where r_1,θ_1 and r_2,θ_2 are the spherical coordinates and

$$\rho=r_1 sin\theta_1 \quad , \quad z-l=r_1 cos\theta_1 \tag{2.52}$$

$$\rho=r_2 sin\theta_2 \quad , \quad z+c=r_2 cos\theta_2 \tag{2.53}$$

III. RESULTS AND DISCUSSION

In this section, some graphs showing the effect of the parameters of the problem on the diffracted field are presented. Numerical results are produced for the diffracted field as $20log|u_d(r_1,\theta_1)|$ with the observation angle θ changing from 0 to π.

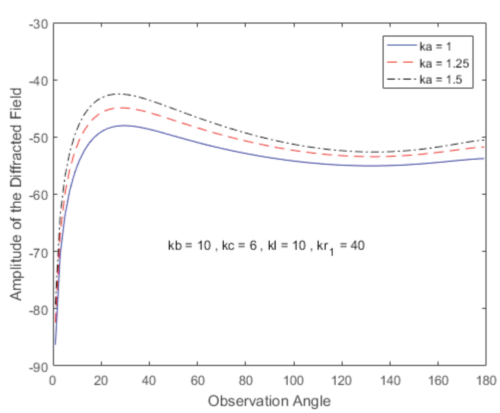

Figure 3. Field of diffraction with different values of ka

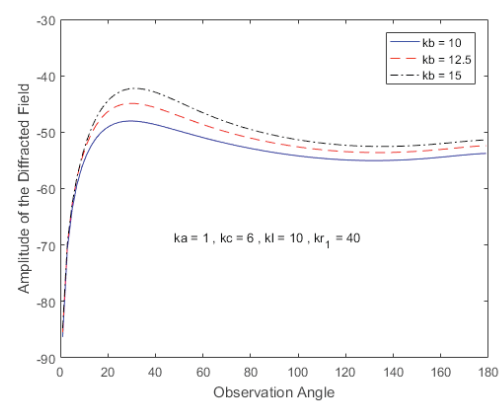

Figure 4. Field of diffraction with different values of kb

Fig. 3 and Fig. 4 show the variation of the diffracted field amplitude against the observation angle for different values of the pipe radius and ring source radius, respectively. Diffracted field amplitude increases with increasing values of pipe and ring source radius, as expected.

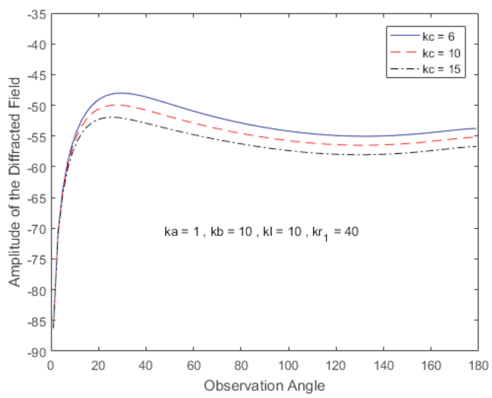

Figure 5. Field of diffraction with different values of kc

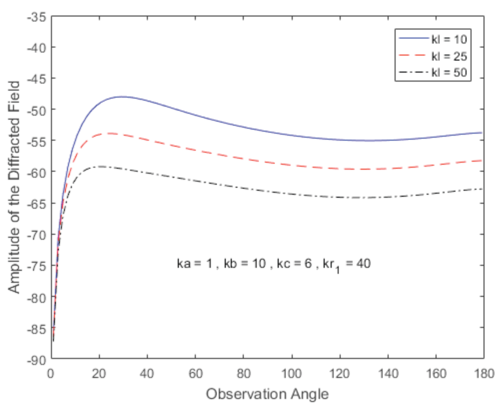

Figure 6. Field of diffraction with different values of kl

Fig. 5 and Fig. 6 display the same effect to the diffracted field for different values of kc and kl. Diffracted field amplitude decreases with increasing values of kc and kl.

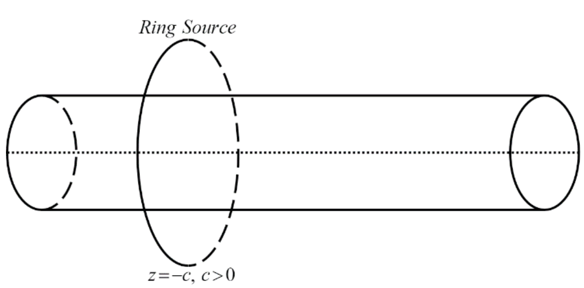

Figure 7. Ring source with an infinite pipe

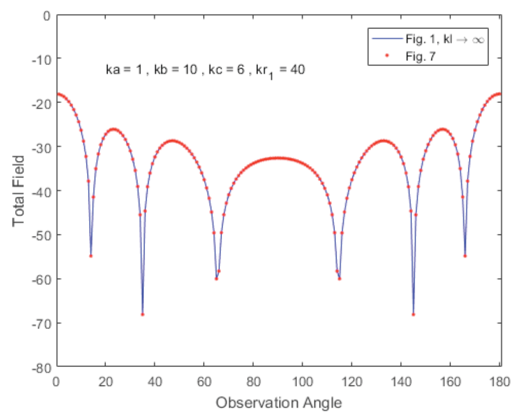

Figure 8. Comparison of the total field with Fig. 7

Fig. 8 depicts an excellent agreement between the Fig. 1 $(kl \to \infty)$ and Fig. 7. In addition, Fig. 8 shows that the mathematical problems encountered are rigorously examined.

IV. CONCLUSION

In the present work, diffraction of sound waves emanating from a ring source by semi-infinite rigid pipe has been investigated by using the Fourier transform technique in conjunction with the Wiener-Hopf technique. The problem is modelled two dimensional due to symmetry of the geometry. An analytical solution is derived by solving the Wiener-Hopf equation. To a better understanding the effect of the parameters of the problem on the diffracted field, graphics are presented.

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