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PARACOMPLEX LIGHTLIKE SUBMANIFOLDS OF ALMOST PARAHERMITIAN MANIFOLDS

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ABSTRACT. We prove that if an almost parahermitian structure of the ambient space induces the identity endomorphism of the radical distribution on a paracomplex lightlike submanifold, then there exists the global lightlike transversal bundle, which is uniquely determined with respect to a screen distribution and a screen transversal bundle of the lightlike submanifold. As an application, we give a sufficient condition that paracomplex lightlike submanifolds in a parakähler manifold are minimal.

1. INTRODUCTION

Let M be a submanifold in a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Let g be the induced symmetric $(0,2)$-tensor field on M from $\widetilde{g}$. Then if the intersection $\mathrm{Rad}\,(TM)$ of the tangent bundle TM and normal bundle $TM^{\perp}$ of M is a smooth r-dimensional distribution, then (M, g) is called an r-lightlike submanifold in $(\widetilde{M}, \widetilde{g})$. The distribution $\mathrm{Rad}\,(TM)$ is called the radical distribution of (M, g). We note that 0-lightlike submanifolds are semi-Riemannian. The geometry of r-lightlike submanifolds with $r > 0$ is much different from that of semi-Riemannian submanifolds.

In the case of $r > 0$, we can take a semi-Riemannian complementary distribution $S(TM)$ (resp. vector bundle $S(TM^{\perp})$) of $\mathrm{Rad}\,(TM)$ in TM (resp. $TM^{\perp}$), which is not uniquely determined in general. The fixed distribution $S(TM)$ (resp. vector bundle $S(TM^{\perp})$) is called the screen distribution (resp. screen transversal vector bundle). Following Duggal and Bejancu [3], we can take a lightlike transversal bundle $\mathrm{ltr}\,(TM)$ on an open subset U of M, which depends on $S(TM)$, $S(TM^{\perp})$ and a local basis $\xi = (\xi_1, \ldots, \xi_r)$ of $\mathrm{Rad}\,(TM)$ on U. In general, $\mathrm{ltr}\,(TM)$ is locally constructed on M. Then we obtain the decomposition $T\widetilde{M}|_U = TM|_U \oplus \mathrm{tr}\,(TM)$, where $\mathrm{tr}\,(TM) := S(TM^{\perp}) \oplus \mathrm{ltr}\,(TM)$ is called a transversal bundle. The theory of lightlike submanifolds is to study *properties* of (M, g) which are independent of $S(TM)$ and $S(TM^{\perp})$, using the decomposition above. As one of such properties,

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Sakaki introduced the notion of minimal lightlike submanifolds in [5], modifying the definition of minimal lightlike submanifolds in Bejan and Duggal [2].

In this paper, we study a paracomplex lightlike submanifold M of an almost parahermitian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. Let J be the induced endomorphism of TM from $\widetilde{J}$. In Section 2, we give notions and results we need from paracomplex geometry, and recall some basic facts on lightlike submanifolds following [3]. We prove in Section 3 that if $J|_{\mathrm{Rad}\,(TM)} = \pm I_{\mathrm{Rad}\,(TM)}$, then there exists the uniquely and globally determined lightlike transversal vector bundle $\mathrm{ltr}\,(TM)$ with respect to fixed $S(TM)$ and $S(TM^{\perp})$. As an application, in Section 4, we can see that co-isotropic paracomplex lightlike submanifolds with $J|_{\mathrm{Rad}\,(TM)} = \pm I_{\mathrm{Rad}\,(TM)}$ in a parakähler manifold are minimal in the sense of Sakaki [5].

2. Preliminaries

In this paper, we assume that all manifolds are connected, paracompact and differentiable of class C^{∞}. Let E be a vector bundle over a manifold N and $\mathrm{End}\,(E)$ be the vector bundle of which the fiber on $p \in N$ is $\mathrm{End}\,(E_p)$. The identity endomorphism of E is denoted by I_E. We denote the algebra of smooth functions on N by $\mathcal{F}(N)$, and the $\mathcal{F}(N)$-module of smooth sections of E by $\Gamma(E)$. For a section $P \in \Gamma(\mathrm{End}\,(E))$ and a linear connection ∇^E of E, we define the differential $\nabla^E P$ of P with respect to ∇^E by

$$(\nabla^E_X P)(s) := \nabla^E_X(P(s)) - P(\nabla^E_X s), \quad \forall X \in \Gamma(TN),\ \forall s \in \Gamma(E),$$

where TN is the tangent bundle of N.

An endomorphism $\widetilde{J} \in \mathrm{End}\,(T\widetilde{M})$ is an *almost product structure* of a manifold $\widetilde{M}$, if $\widetilde{J}^2 = I_{T\widetilde{M}}$ and $\widetilde{J} \neq \pm I_{T\widetilde{M}}$. If there exists a nondegenerate metric on $\widetilde{M}$ such that

$$\widetilde{g}(\widetilde{J}X, \widetilde{J}Y) = -\widetilde{g}(X, Y), \quad \forall X, Y \in \Gamma(T\widetilde{M}),$$

then we say that $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ is an *almost parahermitian manifold.* If $\widetilde{J}$ is integrable, we say that $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ is a *parahermitian manifold.* We note that $\widetilde{J}$ is integrable, if and only if the Nijenhuis tensor N of $\widetilde{J}$ given by

$$N(X, Y) := [\widetilde{J}X, \widetilde{J}Y] - \widetilde{J}[\widetilde{J}X, Y] - \widetilde{J}[X, \widetilde{J}Y] + [X, Y],$$

vanishes identically on $\widetilde{M}$. For a torsion-free affine connection $\hat{\nabla}$ on $\widetilde{M}$, putting $(\hat{\nabla}\widetilde{J})(X, Y) := (\hat{\nabla}_X \widetilde{J})(Y)$ for any $X, Y \in T\widetilde{M}$, we obtain

$$N(X, Y) = (\hat{\nabla}\widetilde{J})(\widetilde{J}X, Y) - (\hat{\nabla}\widetilde{J})(\widetilde{J}Y, X) + \widetilde{J}(\hat{\nabla}\widetilde{J})(Y, X) - \widetilde{J}(\hat{\nabla}\widetilde{J})(X, Y).$$

We say that an almost parahermitian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ is a *parakählerian manifold* if $\widetilde{J}$ is parallel with respect to the Levi-Civita connection $\widetilde{\nabla}$ of the semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$: $(\widetilde{\nabla}_X \widetilde{J})(Y) = 0$ for any $X, Y \in T\widetilde{M}$. Thus, if $(\widetilde{M}, \widetilde{J}, \widetilde{g})$ is parakählerian, then $\widetilde{J}$ is integrable.

Let (E, g^E) be a semi-Riemannian vector bundle over a manifold M, that is, g^E is a nondegenerate bundle metric of E. An endomorphism $J^E \in \mathrm{End}\,(E)$ is a *parahermitian structure*, if $(J^E)^2 = I_E$ and $g^E(J^E s_1, J^E s_2) = -g^E(s_1, s_2)$ for any $s_1, s_2 \in \Gamma(E)$. The triplet (E, g^E, J^E) is called a *parahermitian vector bundle.* We note that the rank of a parahermitian vector bundle is even and the metric is neutral. Thus, it follows that any almost parahermitian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ has even dimension, say $2m$, and the index ind $\widetilde{M}$ is equal to m. Thus $(\widetilde{M}, \widetilde{g})$ is a *neutral* semi-Riemannian manifold.

Putting $E^{\pm} := \mathrm{Ker}\,(J^E \mp I_E)$ for a parahermitian vector bundle (E, g^E, J^E) with rank $2k$, we obtain the non-orthogonal direct decomposition $E = E^+ \oplus E^-$, where $E^{\pm}$ is the eigenspace of J^E corresponding to the eigenvalue ± 1. Then rank $E^{\pm} = k$ and the subbundles are *totally lightlike* (*isotropic*), that is, g^E vanishes on each of them. Then we denote the $E^{\pm}$-component of $s \in E$ by $s^{\pm}$.

A tangent vector v of $(\widetilde{M}, \widetilde{g})$ is said to be *spacelike*, *timelike*, or *null* according as we have $v = 0$ or $\widetilde{g}(v, v) > 0$, $\widetilde{g}(v, v) < 0$, or $\widetilde{g}(v, v) = 0$ and $v \neq 0$. It is easy that $\widetilde{J}(v)$ is perpendicular to v for any $v \in T\widetilde{M}$, and $\widetilde{J}(v)$ is timelike (resp. spacelike) for any spacelike (resp. timelike) tangent vector $v \in T\widetilde{M}$.

We recall some basic results on lightlike submanifolds of a semi-Riemannian manifold. With respect to this class of submanifolds, we refer the monograph by Duggal and Bejancu [3]. (See Duggal and Sahin [4] also.)

Let $(\widetilde{M}, \widetilde{g})$ be an n-dimensional semi-Riemannian manifold with index t. An m-dimensional submanifold M of $(\widetilde{M}, \widetilde{g})$ is said to be r-*lightlike* if the subset of TM:

$$\mathrm{Rad}\,(TM) := \bigcup_{p \in M} \mathrm{Rad}\,(T_pM), \quad \text{where} \quad \mathrm{Rad}\,(T_pM) := T_pM \cap T_pM^{\perp},$$

is a smooth distribution on M of rank r called the *lightlike distribution*. Then we see that the rank r of $\mathrm{Rad}\,(TM)$ satisfies

$$r \leq \min\{t,\ n-t,\ m,\ n-m\}. \tag{2.1}$$

It follows that M is r-lightlike if and only if the induced tensor field g on M by $\widetilde{g}$ has a constant rank $m - r$. In the case of $r = 0$, i.e., the distribution $\mathrm{Rad}\,(TM)$ is zero, (M, g) is a semi-Riemannian submanifold.

By the definition above, we see that the normal bundle $TM^{\perp}$ of M is not complementary to TM in $T\widetilde{M}$ along M if $r > 0$. Then we take two vector bundles $S(TM)$ and $S(TM^{\perp})$, whose existences are consequences of the paracompactness of M, such that

$$TM = S(TM) \oplus_{\mathrm{orth}} \mathrm{Rad}\,(TM) \quad \text{and} \quad TM^{\perp} = \mathrm{Rad}\,(TM) \oplus_{\mathrm{orth}} S(TM^{\perp}),$$

which $\oplus_{\mathrm{orth}}$ stands for orthogonal direct sum of vector bundles. We call $S(TM)$ and $S(TM^{\perp})$ a *screen distribution* and a *screen transversal vector bundle* of M, respectively. We note that both $S(TM)$ and $S(TM^{\perp})$ are nondegenerate vector subbundles of $T\widetilde{M}$ along M.

For a fixed screen distribution $S(TM)$, we can take the complementary orthogonal vector subbundle $S(TM)^{\perp}$ in $T\widetilde{M}$ along M:

$$T\widetilde{M}|_M = S(TM) \oplus_{\mathrm{orth}} S(TM)^{\perp},$$

which is automatically nondegenerate. Since another fixed object $S(TM^{\perp})$ is a vector subbundle of $S(TM)^{\perp}$, we can take the complementary orthogonal vector subbundle $S(TM^{\perp})^{\perp}$ in $S(TM)^{\perp}$ such that

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{\mathrm{orth}} S(TM^{\perp})^{\perp}.$$

We note that $\mathrm{Rad}\,TM \subset S(TM^{\perp})^{\perp}$.

For a local basis $\xi = (\xi_1, \dots, \xi_r)$ of $\mathrm{Rad}\,(TM)$ on an open subset U of M, we can take local sections $N_1, \dots, N_r$ of $S(TM^\perp)^\perp$ on U such that

$$\widetilde{g}(\xi_i, N_j) = \delta_{ij} \quad \text{and} \quad \widetilde{g}(N_i, N_j) = 0 \quad \text{for any } i, j = 1, 2, \dots, r.$$

Then we obtain a complementary vector bundle $\mathrm{ltr}\,(TM) := \mathrm{Span}\,\{N_1, \dots, N_r\}$ to $\mathrm{Rad}\,(TM)$ in $S(TM^\perp)^\perp$ on U (cf. [3]). We call $\mathrm{ltr}\,(TM)$ the *lightlike transversal bundle*. This enables us to consider the vector bundle:

$$\mathrm{tr}\,(TM) := S(TM^\perp) \oplus_{\mathrm{orth}} \mathrm{ltr}\,(TM),$$

which is a complementary vector bundle to TM in $T\widetilde{M}$ along $U \subset M$. We call *tr* (TM) the *transversal vector bundle*. Then we have the following decompositions:

$$\begin{aligned} T\widetilde{M}|_U &= TM \oplus \mathrm{tr}\,(TM) \\ &= (S(TM) \oplus_{\mathrm{orth}} \mathrm{Rad}\,(TM)) \oplus (S(TM^\perp) \oplus_{\mathrm{orth}} \mathrm{ltr}\,(TM)) \\ &= S(TM) \oplus_{\mathrm{orth}} S(TM)^\perp \oplus_{\mathrm{orth}} (\mathrm{Rad}\,(TM) \oplus \mathrm{ltr}\,(TM)), \end{aligned}$$

where $\oplus$ stands for non-orthogonal direct sum of vector bundles. We note that *tr* (TM) is never orthogonal to TM, if $r > 0$.

Let M be a submanifold of an almost parahermitian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. We say that M is a *paracomplex* submanifold if the tangent space T_pM at any point p of M is $\widetilde{J}$-invariant in $T_p\widetilde{M}$, that is, $\widetilde{J}(T_pM) = T_pM$ for any $p \in M$. Then the normal vector bundle $TM^\perp$ is also $\widetilde{J}$-invariant.

For a paracomplex r-lightlike submanifold M of $(\widetilde{M}, \widetilde{g}, \widetilde{J})$, M has the induced symmetric $(0,2)$-tensor field g from $\widetilde{g}$ and the induced endomorphism J from $\widetilde{J}$ on M. We note that J is not necessarily $J \neq \pm I_{TM}$.

3. Paracomplex lightlike submanifolds of almost parahermitian manifolds

Let $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ be a $2n$-dimensional almost parahermitian manifold with index n. Let M be an m-dimensional paracomplex r-lightlike submanifold of $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ and J (resp. g) the induced endomorphism from $\widetilde{J}$ (resp. symmetric $(0,2)$-tensor field from $\widetilde{g}$) on M. We note that the dimension m of M is not necessary even, in contrast to the theory of nondegenerate paracomplex submanifolds in almost parahermitian manifolds.

Theorem 3.1. *Let (M, J, g) be a paracomplex r-lightlike submanifold of an almost parahermitian manifold $(\widetilde{M}, \widetilde{J}, \widetilde{g})$. Then we have the following assertions*:

(i) *The lightlike distribution* $\mathrm{Rad}\,(TM)$ *is J-invariant.*
(ii) *There exists a J-invariant screen distribution $S(TM)$ on M.*
(iii) *There exists a $\widetilde{J}$-invariant screen transversal bundle $S(TM^\perp)$ on M.*

Moreover, the induced metrics of $S(TM)$ and $S(TM^\perp)$ are parahermitian. Thus these are neutral.

Proof. (i) Since TM and $TM^\perp$ are $\widetilde{J}$-invariant, the intersection $\mathrm{Rad}\,(TM) := TM \cap TM^\perp$ is also $\widetilde{J}$-invariant in TM.

In order to prove (ii) (resp. (iii)), we take a positive definite metric l (resp. $l^\perp$) of TM (resp. $TM^\perp$) whose existence is a consequence of the paracompactness of

M. Put

$$k(X,Y) := l(X,Y) + l(JX,JY),$$
$$\text{(resp.}\quad k^{\perp}(V,W) := l^{\perp}(V,W) + l^{\perp}(\widetilde{J}V,\widetilde{J}W)),$$

where $X, Y \in TM$ (resp. $V, W \in TM^{\perp}$). Since TM (resp. $TM^{\perp}$) is J-invariant (resp. $\widetilde{J}$-), k (resp. $k^{\perp}$) is also a positive definite metric. We can take as a screen distribution $S(TM)$ (resp. screen transversal bundle $S(TM^{\perp})$) of M the complementary orthogonal distribution to $\mathrm{Rad}\,(TM)$ in TM (resp. the complementary orthogonal subbundle to $\mathrm{Rad}\,(TM)$ in $TM^{\perp}$) with respect to k (resp. $k^{\perp}$). It is easy to see that

$$k(JX,\xi) = k(X,J\xi) = 0, \qquad k^{\perp}(\widetilde{J}V,\xi) = k^{\perp}(V,\widetilde{J}\xi) = 0,$$

where any $X \in S(TM)$, $\xi \in \mathrm{Rad}\,(TM)$, $V \in S(TM^{\perp})$. Therefore $S(TM)$ (resp. $S(TM^{\perp})$) is J-invariant (resp. $\widetilde{J}$-invatiant). This completes the proof of our assertion (ii) (resp. (iii)). Since $S(TM)$ (resp. $S(TM^{\perp})$) is complementary to $\mathrm{Rad}\,(TM)$ in TM (resp. $TM^{\perp}$), the induced tensor from $\widetilde{g}$ is nondegenerate. In particular, $S(TM)$ and $S(TM^{\perp})$ are parahermitian bundles with respect to the induced objects from $\widetilde{g}$ and $\widetilde{J}$. □

Remark 3.1. Theorem 3.1 is a generalization of Theorem 4.2 in [1].

By Theorem 3.1, since $S(TM)$ and $S(TM^{\perp})$ are parahermitian, the rank of both $S(TM)$ and $S(TM^{\perp})$ are even. Thus, we obtain the following corollary:

Corollary 3.1. *Let (M, g, J) be a paracomplex r-lightlike submanifold of an almost parahermitian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. If the dimension of M is odd (resp. even), then r is odd (resp. even). Hence, there exist no odd-dimensional paracomplex semi-Riemannian submanifolds.*

In this paper, we call submanifolds with real codimension one *hypersurfaces.* From the inequality (2.1), we have

Corollary 3.2. *Any paracomplex r-lightlike hypersurface (M, g, J) of an almost parahermitian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ is 1-lightlike.*

Remark 3.2. corollary 3.1 and 3.2 are generalizations of Theorem 4.1 in [1]. We note that lightlike submanifolds with real codimension two are called “hypersurfaces” in Section 4 of [1].

Lemma 3.1. *Let (M, g, J) be a paracomplex r-lightlike submanifold of an almost parahermitian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. There exists a local basis $\xi = (\xi_1, \ldots, \xi_r)$ of* $\mathrm{Rad}\,(TM)$ *such that $J(\xi_i) = +\xi_i$ or $-\xi_i$, that is, ξ_i is a local eigensection of J.*

Proof. For an everywhere nonzero local section $\zeta \in \Gamma(\mathrm{Rad})$ on an open subset $U \subset M$, if $\zeta \wedge J(\zeta) = 0$ on U, then we take $\xi_1 := \zeta$. Otherwise, we can put $\xi_1 := \zeta + J(\zeta)$. For l $(1 \leq l < r)$, we assume that $\xi_1, \xi_2, \ldots, \xi_l$ are eigensections which are linearly independent on an open set $U' \subset U$, that is, $\xi_1 \wedge \cdots \wedge \xi_l \neq 0$ on U'. There exists a local section $\zeta \in \Gamma(\mathrm{Rad}(TM))$ such that $\zeta \notin \mathrm{Span}\,\{\xi_1, \ldots, \xi_l\}$ on U'. If $\zeta \wedge J(\zeta) = 0$ on U', then we take $\xi_{l+1} := \zeta$. Otherwise, we put $\xi_{\pm} := \zeta \pm J(\zeta)$.

Then, it follows that $\xi_+ \notin \mathrm{Span}\{\xi_1, \dots, \xi_l\}$ or $\xi_- \notin \mathrm{Span}\{\xi_1, \dots, \xi_l\}$ on U'. Indeed, in case of $\xi_+ \in \mathrm{Span}\{\xi_1, \dots, \xi_l\}$, we can see

$$\begin{aligned} 0 &= \xi_+ \wedge \xi_1 \wedge \cdots \wedge \xi_l \\ &= \zeta \wedge \xi_1 \wedge \cdots \wedge \xi_l + J(\zeta) \wedge \xi_1 \wedge \cdots \wedge \xi_l, \end{aligned}$$

therefore, we get $J(\zeta) \wedge \xi_1 \wedge \cdots \wedge \xi_l = -\zeta \wedge \xi_1 \wedge \cdots \wedge \xi_l \neq 0$. Thus, we get

$$\xi_- \wedge \xi_1 \wedge \cdots \wedge \xi_l = 2(\zeta \wedge \xi_1 \wedge \cdots \wedge \xi_l) \neq 0 \quad \text{on } U'.$$

So $\xi_- \notin \mathrm{Span}\{\xi_1, \dots, \xi_l\}$ on U'. By the inductively way, we can obtain a required local basis $\xi = (\xi_1, \dots, \xi_r)$ of $\mathrm{Rad}(TM)$. □

Theorem 3.2. *Let (M, g, J) be a paracomplex r-lightlike submanifold of an almost parahermitian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. For $(M, g, J, S(TM), S(TM^\perp))$ and a local basis $\xi = (\xi_1, \dots, \xi_r)$ of $\mathrm{Rad}\,(TM)|_U$ as in* Lemma 3.1*, where U is an open set of M, there exist local smooth sections $\eta_1, \dots, \eta_r$ of $S(TM^\perp)^\perp|_U$ such that*

$$\widetilde{J}(\eta_i) = -\varepsilon_i \eta_i, \quad \widetilde{g}(\xi_i, \eta_j) = \delta_{ij}, \quad \widetilde{g}(\eta_i, \eta_j) = 0, \tag{3.1}$$

where $\varepsilon_i \in \{+1, -1\}$ is an eigenvalue of ξ_i for J, that is, the signature defined by $J(\xi_i) = \varepsilon_i \xi_i$, and $i, j \in \{1, \dots, r\}$.

Proof. By [3], for a local basis $\xi = (\xi_1, \dots, \xi_r)$ of $\mathrm{Rad}\,(TM)$ on $U \subset M$, we can take local sections $N_1, \dots, N_r$ of $S(TM^\perp)^\perp$ on U such that

$$\widetilde{g}(\xi_i, N_j) = \delta_{ij}, \quad \widetilde{g}(N_i, N_j) = 0 \quad \text{for any } i, j \in \{1, 2, \dots, r\}.$$

We define

$$\eta_i := \frac{1}{2}(N_i - \varepsilon_i \widetilde{J}(N_i)) \quad \text{for } i \in \{1, \dots, r\}.$$

It is easy to check $\widetilde{J}(\eta_i) = -\varepsilon_i \eta_i$ for any $i \in \{1, \dots, r\}$. Moreover, we have

$$\begin{aligned} 2\widetilde{g}(\xi_i, \eta_j) &= \widetilde{g}(\xi_i, N_j - \varepsilon_j \widetilde{J}(N_j)) = \widetilde{g}(\xi_i, N_j) - \varepsilon_j \widetilde{g}(\xi_i, \widetilde{J}(N_j)) \\ &= \widetilde{g}(\xi_i, N_j) + \varepsilon_j \widetilde{g}(\widetilde{J}(\xi_i), N_j) = 2\delta_{ij}. \end{aligned}$$

Thus $\widetilde{g}(\xi_i, \eta_j) = \delta_{ij}$ for any $i, j \in \{1, \dots, r\}$.

With respect to the local null frame $\xi_1, \dots, \xi_r, N_1, \dots, N_r$ of $S(TM^\perp)^\perp$,

$$\begin{aligned} \widetilde{J}(N_i) &= \sum_{j=1}^{r} \Big(\widetilde{g}(\widetilde{J}(N_i), N_j)\xi_j + \widetilde{g}(\widetilde{J}(N_i), \xi_j) N_j \Big) \\ &= \sum_{j=1}^{r} \widetilde{g}(\widetilde{J}(N_i), N_j)\xi_j - \varepsilon_i N_i. \end{aligned} \tag{3.2}$$

Applying $\widetilde{J}$ to the above equation, we have

$$N_i = \sum_{j=1}^{r} \varepsilon_j \, \widetilde{g}(\widetilde{J}(N_i), N_j)\xi_j - \varepsilon_i \widetilde{J}(N_i). \tag{3.3}$$

Substituting (3.3) into (3.2), we obtain

$$\sum_{j=1}^{r} (1 - \varepsilon_i \varepsilon_j) \, \widetilde{g}(\widetilde{J}(N_i), N_j)\xi_j = 0.$$

Consequently we can see

$$\widetilde{g}(\widetilde{J}(N_i), N_j) = 0 \quad \text{for any } i, j \text{ such that } \varepsilon_j = -\varepsilon_i. \tag{3.4}$$

On the other hand,

$$\begin{aligned}4\widetilde{g}(\eta_i,\eta_j) &= \widetilde{g}(N_i-\varepsilon_i\widetilde{J}(N_i), N_j-\varepsilon_j\widetilde{J}(N_j))\\ &= (\varepsilon_j-\varepsilon_i)\ \widetilde{g}(\widetilde{J}(N_i),N_j)\\ &= \begin{cases} 0 & \text{for any } i,j \text{ such that } \varepsilon_j=\varepsilon_i,\\ 2\varepsilon_j\ \widetilde{g}(\widetilde{J}(N_i),N_j) & \text{for any } i,j \text{ such that } \varepsilon_j=-\varepsilon_i.\end{cases}\end{aligned}$$

By the equation above and (3.4), we obtain $\widetilde{g}(\eta_i,\eta_j)=0$ for any $i,j\in\{1,\dots,r\}$. This completes the proof of our assertion. □

For a paracomplex r-lightlike submanifold $(M,g,J,S(TM),S(TM^\perp))$ and a local basis $\xi=(\xi_1,\dots,\xi_r)$ of $\mathrm{Rad}\,(TM)$ as in Lemma 3.1, by virtue of Theorem 3.2, we can define over U:

$$\mathrm{ltr}\,(TM) := \mathrm{ltr}\,(TM,\xi) := \mathrm{Span}\,\{\eta_1,\dots,\eta_r\},$$
$$\mathrm{tr}\,(TM) := \mathrm{tr}\,(TM,\xi) := S(TM^\perp)\oplus \mathrm{ltr}\,(TM).$$

We obtain the following:

Theorem 3.3. *Let (M,g,J) be a paracomplex r-lightlike submanifold of an almost parahermitian manifold $(\widetilde{M},\widetilde{g},\widetilde{J})$. For $(M,g,J,S(TM),S(TM^\perp))$ and a basis $\xi=(\xi_1,\dots,\xi_r)$ of* $\mathrm{Rad}\,(TM)|_U$ *as in* Lemma 3.1, *where U is an open set of M, there exist local decompositions of vector bundles over U:*

$$\begin{aligned}T\widetilde{M}|_U &= TM\oplus \mathrm{tr}\,(TM)\\ &= (S(TM)\oplus_{\mathrm{orth}}\mathrm{Rad}\,(TM))\oplus(S(TM^\perp)\oplus_{\mathrm{orth}}\mathrm{ltr}\,(TM))\\ &= S(TM)\oplus_{\mathrm{orth}}S(TM^\perp)\oplus_{\mathrm{orth}}(\mathrm{Rad}\,(TM)\oplus\mathrm{ltr}\,(TM)),\end{aligned}$$

where $S(TM)$, $S(TM^\perp)$, $\mathrm{Rad}\,(TM)$ and $\mathrm{ltr}\,(TM)$ are $\widetilde{J}$-invariant, and $S(TM)$, $S(TM^\perp)$ and $(\mathrm{Rad}\,(TM)\oplus\mathrm{ltr}\,(TM))$ are parahermitian vector bundles over U.

According to the $\widetilde{J}$-invariant decomposition over U: $T\widetilde{M}|_U = TM\oplus\mathrm{tr}\,(TM)$ as in Theorem 3.3, we have the Gauss formula and the Weingarten formula:

$$\begin{aligned}\widetilde{\nabla}_XY &= \nabla_XY+h(X,Y), && X,Y\in\Gamma(TM),\\ \widetilde{\nabla}_XV &= -A_VX+\nabla^{\mathrm{tr}}_XV, && V\in\Gamma(\mathrm{tr}\,(TM)),\end{aligned}$$

where ∇_XY (resp. $h(X,Y)$) is the tangential (resp. transversal) component of $\widetilde{\nabla}_XY$, and $-A_VX$ (resp. ∇^{tr}_XV) is the tangential (resp. transversal) component of $\widetilde{\nabla}_XV$. We note that the induced connection ∇ is not necessary a metric connection in case of $r>0$ and refer details for [3] and [4].

We note that $J|_{\mathrm{Rad}\,(TM)}$ is not necessary $J|_{\mathrm{Rad}\,(TM)}\neq\pm I_{\mathrm{Rad}\,(TM)}$. We put $k:=\mathrm{rank}\,(\mathrm{Ker}\,(J|_{\mathrm{Rad}\,(TM)}-I_{\mathrm{Rad}\,(TM)}))$. Hereafter we use the induces i,j, α,β and A,B for the following range respectively:

$$i,j=1,\dots,k;\quad \alpha,\beta=k+1,\dots,r;\quad A,B=1,\dots,r.$$

From now on, we take a local basis of $\mathrm{Rad}\,(TM)$ as in Lemma 3.1 as follows:

$$\xi=(\xi^+;\xi^-)=(\xi^+_1,\dots,\xi^+_k;\xi^-_{k+1},\dots,\xi^-_r),$$

where $J|_{\mathrm{Rad}\,(TM)}(\xi_i^+) = \xi_i^+$, $J|_{\mathrm{Rad}\,(TM)}(\xi_\alpha^-) = -\xi_\alpha^-$. Furthermore, we denote the local basis of $\mathrm{ltr}\,(TM)$ constructed corresponding to ξ in Theorem 3.2 by

$$\eta = (\eta^-; \eta^+) = (\eta_1^-, \dots, \eta_k^-; \eta_{k+1}^+, \dots, \eta_r^+).$$

It follows that $J|_{\mathrm{Rad}\,(\mathrm{TM})}(\eta_i^-) = -\eta_i^-$ and $J|_{\mathrm{Rad}\,(TM)}(\eta_\alpha^+) = \eta_\alpha^+$. We denote the local basis of $S(TM^\perp)^\perp = \mathrm{Rad}\,(TM) \oplus \mathrm{ltr}\,(TM, \xi)$ by $(\xi; \eta)$.

Lemma 3.2. *Let $(M, g, J, S(TM), S(TM^\perp))$ be a paracomplex r-lightlike submanifold of an almost parahermitian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. For local bases $(\xi; \eta)$ and $(\xi'; \eta')$ of $S(TM^\perp)^\perp$ on U and U' respectively, the transition matrix at $p \in U \cap U'$ is*

$$(3.5) \qquad \begin{pmatrix} \xi'^+ & \xi'^- & \eta'^- & \eta'^+ \end{pmatrix}_p = \begin{pmatrix} \xi^+ & \xi^- & \eta^- & \eta^+ \end{pmatrix}_p \begin{bmatrix} A_+ & O & O & B_+ \\ O & A_- & B_- & O \\ O & O & C_- & O \\ O & O & O & C_+ \end{bmatrix},$$

where $A_+, C_- \in GL_k(\mathbb{R})$, $A_-, C_+ \in GL_{r-k}(\mathbb{R})$ and $B_+, {}^tB_- \in M_{k,r-k}(\mathbb{R})$, and these matrices satisfy

$$C_- = {}^tA_+^{-1}, \quad C_+ = {}^tA_-^{-1}, \quad B_- = -A_-{}^tB_+{}^tA_+^{-1}.$$

Proof. Since ξ and ξ' are bases of $\mathrm{Rad}\,(TM)_p$ and eigenvectors of J, we obtain

$$\xi_j'^+ = \sum_{i=1}^k a_{ij}\xi_i^+, \quad \xi_\beta'^- = \sum_{\alpha=k+1}^r a_{\alpha\beta}\xi_\alpha^-,$$

where $j \in \{1, \dots, k\}$ and $\beta \in \{k+1, \dots, r\}$. Then it follows that $A_+ := (a_{ij}) \in GL_k(\mathbb{R})$ and $A_- := (a_{\alpha\beta}) \in GL_{r-k}(\mathbb{R})$. Since $\xi_A^\pm$, $\xi_A'^\pm$, $\eta_A^\pm$ and $\eta_A'^\pm$ $(A \in \{1, \dots, r\})$ are eigenvectors of $\widetilde{J}$ in $S(TM^\perp)_p^\perp$, we obtain

$$\eta_j'^- = \sum_{\alpha=k+1}^r b_{\alpha j}\xi_\alpha^- + \sum_{i=1}^k c_{ij}\eta_i^-, \quad \eta_\beta'^+ = \sum_{i=1}^k b_{i\beta}\xi_i^+ + \sum_{\alpha=k+1}^r c_{\alpha\beta}\eta_\alpha^+,$$

where $j \in \{1, \dots, k\}$ and $\beta \in \{k+1, \dots, r\}$. We put $B_- := (b_{\alpha j})$, $B_+ := (b_{i\beta})$, $C_- := (c_{ij})$ and $C_+ := (c_{\alpha\beta})$. From $\widetilde{g}(\xi_i'^+, \eta_j'^-) = \delta_{ij}$, $\widetilde{g}(\xi_\alpha'^-, \eta_\beta'^+) = \delta_{\alpha\beta}$ and $\widetilde{g}(\xi_i'^+, \eta_\beta'^+) = \widetilde{g}(\xi_\alpha'^-, \eta_j'^-) = 0$,

$$\delta_{ij} = \widetilde{g}(\xi_i'^+, \eta_j'^-) = \sum_{l=1}^k a_{li}c_{lj}, \quad \delta_{\alpha\beta} = \widetilde{g}(\xi_\alpha'^-, \eta_\beta'^+) = \sum_{\gamma=k+1}^r a_{\gamma\alpha}c_{\gamma\beta}.$$

Thus we obtain $C_- = {}^tA_+^{-1}$ and $C_+ = {}^tA_-^{-1}$. Furthermore, using $\widetilde{g}(\eta_i'^+, \eta_\alpha'^-) = 0$, we have

$$0 = \widetilde{g}(\eta_i'^+, \eta_\alpha'^-) = \sum_{j=1}^k b_{j\alpha}c_{ji} + \sum_{\beta=k+1}^r b_{\beta i}c_{\beta\alpha}.$$

Hence, we consequently get $B_- = -A_-{}^tB_+{}^tA_+^{-1}$. □

In case of $k(r-k) \neq 0$, for $(\xi; \eta)$ and $(\xi'; \eta')$ of which is non-vanishing $M_{k,r-k}(\mathbb{R})$-valued function B_+ on $U \cap U'$, we see

$$\mathrm{ltr}\,(TM, \xi) \neq \mathrm{ltr}\,(TM, \xi') \quad \text{on} \quad U \cap U'.$$

In the other hand, when $k(r-k) = 0$, we can obtain the uniquely determined lightlike transversal bundle on M as follows:

Theorem 3.4. *Let (M, J, g) be a paracomplex r-lightlike submanifold of an almost parahermitian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. If $(M, g, J, S(TM), S(TM^{\perp}))$ satisfies*

$$J|_{\mathrm{Rad}\,(TM)} = I_{\mathrm{Rad}\,(TM)} \quad \text{or} \quad J|_{\mathrm{Rad}\,(TM)} = -I_{\mathrm{Rad}\,(TM)}, \tag{3.6}$$

then there uniquely exists the lightlike transversal vector bundle $\mathrm{ltr}\,(TM)$ *over M such that $\widetilde{J}$-invariant. Moreover, if $J|_{\mathrm{Rad}\,(TM)} = \pm I_{\mathrm{Rad}\,(TM)}$, then $\widetilde{J}|_{\mathrm{ltr}\,(TM)} = \mp I_{\mathrm{ltr}\,(TM)}$.*

Proof. By the assumption: $J|_{\mathrm{Rad}\,(TM)} = I_{\mathrm{Rad}\,(TM)}$ or $J|_{\mathrm{Rad}\,(TM)} = -I_{\mathrm{Rad}\,(TM)}$, we have $k(r-k) = 0$. Then, from Lemma 3.2, it follows $B_+ = O$ or/and $B_- = O$ for any $(\xi; \eta)$ and $(\xi'; \eta')$ on U and U' respectively. Therefore, we obtain

$$\mathrm{ltr}\,(TM, \xi) = \mathrm{ltr}\,(TM, \xi') \quad \text{on} \quad U \cap U'.$$

Thus the lightlike transversal bundle is globally and uniquely determined on M. When $J|_{\mathrm{Rad}\,(TM)} = I_{\mathrm{Rad}\,(TM)}$, since all signatures ε_i $(i = 1, \ldots, k)$ in equations (3.1) in Theorem 3.2 are equal to $+1$, we obtain $\widetilde{J}|_{\mathrm{ltr}\,(TM)} = -I_{\mathrm{ltr}\,(TM)}$. By a similar way, we can see $\widetilde{J}|_{\mathrm{ltr}\,(TM)} = I_{\mathrm{ltr}\,(TM)}$, if $J|_{\mathrm{Rad}\,(TM)} = -I_{\mathrm{Rad}\,(TM)}$. We have proved the theorem. □

4. Paracomplex lightlike submanifolds in parakähler manifolds

In this section, we consider minimal lightlike submanifolds in semi-Riemannian manifolds. Sakaki [5] gives a definition of minimal lightlike submanifolds which is independent of the choice of the screen distribution and the screen transversal vector bundle as follows:

Definition 4.1. We say that a lightlike submanifold (M, g) in a semi-Riemannian manifolds $(\widetilde{M}, \widetilde{g})$ is *minimal* if:

(a) $h(X, \xi) = 0$ for any $X \in \Gamma(TM)$, $\xi \in \Gamma(\mathrm{Rad}\,(TM))$, and
(b) $\mathrm{trace}\,(h) = 0$, where the trace is written with respect to g restricted to $S(TM)$.

Remark 4.1. We also refer Bejan and Duggal [2] for another (original) definition of minimal lightlike submanifolds.

From now on, we take a parakähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ as the ambient space. Moreover, for a paracomplex r-lightlike submanifold M of $(\widetilde{M}, \widetilde{g}, \widetilde{J})$, we choose vector bundles $S(TM)$, $S(TM^{\perp})$ and $\mathrm{ltr}\,(TM)$ are $\widetilde{J}$-invariant ones given in Theorem 3.3.

Proposition 4.1. *Let $(M, g, J, S(TM), S(TM^{\perp}))$ be a paracomplex r-lightlike submanifold of a parakähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. Then J is parallel with respect to the induced connection ∇ and the second fundamental form h satisfies $h(X, JY) = \widetilde{J}h(X, Y)$ for any $X, Y \in \Gamma(TM)$.*

Proof. Taking a local basis $\xi = (\xi_1, \ldots, \xi_r)$ as in Lemma 3.1, we fix the $\widetilde{J}$-invariant lightlike transversal bundle $\mathrm{ltr}\,(TM)$. Then, for the induced connection ∇, we have

$$\widetilde{\nabla}_X(\widetilde{J}Y) = (\widetilde{\nabla}_X \widetilde{J})(Y) + \widetilde{J}(\widetilde{\nabla}_X Y) = J(\nabla_X Y) + \widetilde{J}(h(X, Y)).$$

On the other hands, we get

$$\widetilde{\nabla}_X(\widetilde{J}Y) = \nabla_X(JY) + h(X, JY) = (\nabla_X J)(Y) + J(\nabla_X Y) + h(X, JY).$$

Because TM and $\mathrm{tr}\,(TM)$ are $\widetilde{J}$-invariant, we obtain $(\nabla_X J)(Y) = 0$ and $h(X, JY) = \widetilde{J}h(X, Y)$, which complete the proof. □

The decomposition $\mathrm{tr}\,(TM) = S(TM^{\perp}) \oplus \mathrm{ltr}\,(TM)$ introduces

$$h(X, Y) = h^s(X, Y) + h^l(X, Y) \quad \text{for } X, Y \in TM,$$

where h^s (resp. h^l) is called the *screen* (resp. *lightlike*) *second fundamental form* of M. Since $S(TM^{\perp})$ and $\mathrm{ltr}\,(TM)$ are $\widetilde{J}$-invariant, we obtain the following lemma:

Lemma 4.1. *Under the above notations,*

$$h^s(X, JY) = \widetilde{J}h^s(X, Y), \quad h^l(X, JY) = \widetilde{J}h^l(X, Y) \quad \text{for } X, Y \in TM.$$

Lemma 4.2. *Let $(M, g, J, S(TM), S(TM^{\perp}))$ be a paracomplex r-lightlike submanifold in a parakähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. When $J|_{\mathrm{Rad}\,(TM)} = I_{\mathrm{Rad}\,(TM)}$ (resp. $J|_{\mathrm{Rad}\,(TM)} = -I_{\mathrm{Rad}\,(TM)}$), we have for $X, Y \in \Gamma(TM)$,*

$$h^l(X, JY) = -h^l(X, Y) \quad (\text{resp. } h^l(X, JY) = h^l(X, Y)).$$

In particular, we obtain

$$h^l(X^+, Y^-) = 0 \quad \text{for } X^+ \in \Gamma(TM^+) \text{ and } Y^- \in \Gamma(TM^-).$$

Proof. When $J|_{\mathrm{Rad}\,(TM)} = I_{\mathrm{Rad}\,(TM)}$, a local basis $\eta = (\eta_1, \ldots, \eta_r)$ of $\mathrm{ltr}\,(TM)$ as in Theorem 3.2 satisfy $\widetilde{J}\eta_i = -\eta_i$ $(i = 1, \ldots, r)$. Writing the lightlike second fundamental form h^l as follows

$$h^l(X, Y) = \sum_{i=1}^{r} h^l_i(X, Y)\eta_i,$$

we have

$$\begin{aligned} h^l(X, JY) = \widetilde{J}h^l(X, Y) &= \widetilde{J}\left(\sum_{i=1}^{r} h^l_i(X, Y)\eta_i\right) \\ &= \sum_{i=1}^{r} h^l_i(X, Y)\widetilde{J}\eta_i = -\sum_{i=1}^{r} h^l_i(X, Y)\eta_i = -h^l(X, Y). \end{aligned}$$

We can similarly prove, in the case of $J|_{\mathrm{Rad}\,(TM)} = -I_{\mathrm{Rad}\,(TM)}$. □

We call a r-lightlike submanifold (M, g) *co-isotropic* if $r = \mathrm{codim}\, M$. Then we recognize $S(TM^{\perp})$ as the zero vector bundle, hence $h^s = 0$.

Theorem 4.1. *Let $(M, g, J, S(TM))$ be a co-isotropic paracomplex submanifold of a parakähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. If $J|_{\mathrm{Rad}\,(TM)} = \pm I_{\mathrm{Rad}\,(TM)}$, then (M, g) is minimal in the sense of* Definition 4.1.

Proof. Without a loss of generalities, we can assume $J|_{\mathrm{Rad}\,(TM)} = I_{\mathrm{Rad}\,(TM)}$. By the assumption, $S(TM)$ is a parahermitian vector bundle. Thus, we can take a local orthonormal basis $X_1, X_2, \ldots, X_{2s-1}, X_{2s}$ of $S(TM)$ such that $g(X_i, X_j) = (-1)^i\delta_{ij}$ for $i, j = 1, \ldots, 2s$, and $X_{2i} = J(X_{2i-1})$ for $i = 1, \ldots, s$, where $\mathrm{rank}\,(S(TM)) = 2s$

and index $(S(TM)) = s$. Then we obtain

$$\begin{aligned}\text{trace}\,(h) &= \sum_{i=1}^{s} (-h(X_{2i-1}, X_{2i-1}) + h(X_{2i}, X_{2i})) \\ &= \sum_{i=1}^{s} (-h(X_{2i-1}, X_{2i-1}) + h(J(X_{2i-1}), J(X_{2i-1}))) \\ &= \sum_{i=1}^{s} (-h(X_{2i-1}, X_{2i-1}) + h(X_{2i-1}, X_{2i-1})) = 0.\end{aligned}$$

Hence the condition (b) in Definition 4.1 holds.

Since (M, g) is co-isotropic, we have $h^s = 0$. In general, we can see that the lightlike second fundamental form h^l is vanishing on $\text{Rad}\,(TM)$, from [3, p.157, Proposition. 2.2] or [4, p.199. Proposition. 5.1.3]. According the decomposition: $TM = S(TM) \oplus \text{Rad}\,(TM)$, we decompose $X \in TM$ as $X = X_S + X_R$. Moreover, for any $X \in S(TM)$, we decompose X as $X = X^+ + X^-$, where $J(X^\pm) = \pm X$. Then we obtain

$$\begin{aligned}h(X, \xi) &= h^l(X, \xi) = h^l(X_S + X_R, \xi) = h^l(X_S, \xi) + h^l(X_R, \xi) = h^l(X_S, \xi) \\ &= h^l(X_S^+ + X_S^-, \xi) = h^l(X_S^+, \xi) + h^l(X_S^-, \xi).\end{aligned}$$

By virtue of Lemma 4.2 and $\xi \in \text{Rad}\,(TM) = \text{Rad}\,(TM)^+$, $h^l(X_S^-, \xi) = 0$. From Lemma 4.2 and $J|_{\text{Rad}\,(TM)} = I_{\text{Rad}\,(TM)}$ again, we have $h^l(X_S^+, \xi) = h^l(X_S^+, J\xi) = -h^l(X_S^+, \xi)$, thus $h^l(X_S^+, \xi) = 0$. Hence the condition (a) in Definition 4.1 holds. □

Remark 4.2. Sakaki gives examples of minimal lightlike submanifolds in [5, Theorem 5.1]. The examples satisfy the conditions as in Theorem 4.1.

From Corollary 3.1 and Theorem 4.1, we obtain

Corollary 4.1. *Any paracomplex lightlike hypersurfaces in a parakähler manifold are* 1*-lightlike and minimal.*

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