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ON A SPECIAL CONFIGURATION OF LINES AND POINTS IN $\mathbb{P}^N$

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ABSTRACT. This note concerns some arrangements of lines in $\mathbb{P}^N(\mathbb{C})$ and the condition under which there exists a hyperplane intersecting transversely every line of the given arrangement at a unique point.

1. INTRODUCTION.

In this note we want to address the following combinatorial problem. Let us fix a set $\mathcal{L}$ of r disjoint lines $\{L_1, L_2, \ldots, L_r\}$ in $\mathbb{P}^N(\mathbb{C})$. Let us pick r distinct points $\{P_1, \ldots, P_r\}$ such that $P_i \in L_i$ for $i = 1, ..., r$. Under which conditions can one find a hyperplane through $P_1, \ldots, P_r$ that intersects each line L_i exactly at P_i ? It easy to see that there are situations in which no such hyperplane exists. For instance, let $\langle .. \rangle$ denote the linear span of a given subset, and assume that $\dim(\langle L_1, L_2, L_3, L_4 \rangle) = 3 < N$. Then, for any generic 4-tuple of points P_1, P_2, P_3, P_4, chosen respectively on L_1, L_2, L_3, L_4, every hyperplane in $\mathbb{P}^N$ that contains all of these points must also contain all of the lines L_1, L_2, L_3, L_4.

The above example suggests that the dimension of the linear spans of subsets of $\mathcal{L}$ play a significant role, and that with no additional assumptions on such dimensions one cannot hope to find a general solution. However, if we assume that for <u>any</u> subset $\mathcal{L}' \subseteq \mathcal{L}$ the dimension of the corresponding linear span depends only upon the cardinality of $\mathcal{L}'$, a suitable general result can be achieved. As we shall see, the hypothesis above is satisfied, for instance, when $\mathcal{L}$ is any subset of $r \leq N$ fibres of a rational scroll embedded in $\mathbb{P}^N(\mathbb{C})$.

In Section 2 the main theorem is presented. Section 3 contains a few corollaries showing that, in the situation under consideration, the set of hyperplanes in $\mathbb{P}^{N*}$ (the dual space) satisfying the main hypothesis above with respect to $\mathcal{L}$ is large enough. Section 4 studies the special case in which all lines L_i are contained in a rational ruled surface. Finally Section 5 is devoted to an application of the main theorem which was indeed the original motivation for us to address this problem.

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2. The main theorem

Let us consider a set $\mathcal{L}$ as mentioned in §1. Let us fix a line $L := L_1 \in \mathcal{L}$. Let us pick a second line L_2 such that $\dim(\langle L_1, L_2\rangle) = 3$. Then, let us pick a third line L_3, if it exists, such that $\dim(\langle L_1, L_2, L_3\rangle) = 5$, and so on. We can proceed in this way, say, only for $h \geq 1$ steps to get $L_1, L_2, ..., L_{h+1}$ with $\dim(\langle L_1, L_2, ..., L_{h+1}\rangle) = 2h + 1 \leq N$. Now we pick another line L_{h+2}, if it exists, such that L_{h+2} intersects $\langle L_1, L_2, ..., L_{h+1}\rangle$ at one point only; then we pick another line L_{h+3}, if it exists, intersecting $\langle L_1, L_2, ..., L_{h+1}, L_{h+2}\rangle$ at one point only, and so on. If possible, we can proceed in this way, say, only for another $q \geq 1$ steps to get $L_1, L_2, ..., L_{h+1}, L_{h+2}, ..., L_{h+q+1}$ with $\dim(\langle L_1, L_2, ..., L_{h+1}, L_{h+2}, ..., L_{h+q+1}\rangle) = 2h + q + 1 = N$. Then, independently of the number of the remaining lines, if any, $\dim(\langle L_1, L_2, ..., L_{h+1}, L_{h+2}, ..., L_{h+q+1}, ..., L_p\rangle) = N$ for any $h + q + 2 \leq p \leq r$.

Notice that the function $d : [1, r] \subseteq \mathbb{N} \to \mathbb{N}$ such that $d(n) = \dim(\langle L_1, L_2, ..., L_n\rangle)$ depends upon the order in which our lines were chosen. Here we want to consider only sets $\mathcal{L}$ of r lines in $\mathbb{P}^N$, $r \leq N$, such that d does not depend upon the order. In this case we can prove the following theorem, where $k := h + q$.

Theorem 2.1. *Let (h, k) be a given pair of integers with $1 \leq h \leq k$, $h+k+1 = N$. Let $\mathcal{L} = \{L_1, \ldots, L_r\}$, with $2 \leq r \leq N$, be any set of r distinct and disjoint lines in $\mathbb{P}^N$, such that, for any subset $\{L_1, \ldots, L_\rho\} \subseteq \mathcal{L}$, $(\rho \leq r)$, one has:*

1) $\dim(\langle L_1, \ldots, L_\rho\rangle) = 2\rho - 1$ *when* $1 \leq \rho \leq h + 1$;
2) $\dim(\langle L_1, \ldots, L_\rho\rangle) = \rho + h$ *when* $h + 2 \leq \rho \leq k + 1$;
3) $\dim(\langle L_1, \ldots, L_\rho\rangle) = N$ *when* $k + 2 \leq \rho \leq N$.

Let $W_r := \{(P_1, \ldots, P_r) \in L_1 \times \cdots \times L_r \simeq (\mathbb{P}^1)^{\times r} \mid \dim(\langle P_1, \ldots, P_r\rangle) \leq r - 2\}$.
Then $\dim(W_r) \leq r - 2$, i.e. W_r is a closed subscheme of codimension at least 2 in $(\mathbb{P}^1)^{\times r} := \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (r times). Moreover, if $2 \leq r \leq h+1$ then W_r is empty, if $h + 2 \leq r \leq k + 1$ then $\dim(W_r) \leq r - h - 2$.

Before proving Theorem 2.1 we would like to show that there are concrete situations in which the assumptions of Theorem 2.1 are indeed satisfied.

Lemma 2.1. *Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(h) \oplus \mathcal{O}_{\mathbb{P}^1}(k)$ with $1 \leq h \leq k$, $N = h + k + 1$, and let $S = \mathbb{P}(\mathcal{E})$, be a smooth, rational, surface embedded as a linear scroll in $\mathbb{P}^N(\mathbb{C})$ by its tautological line bundle. Let $L_1, L_2, ..., L_r$ be <u>any</u> set of r lines in $\mathbb{P}^N$ which are fibres of the scroll S, with $2 \leq r \leq N$. Then <u>all</u> the assumptions of Theorem 2.1 hold for $L_1, L_2, ..., L_r$.*

Proof. Let T be the very ample tautological divisor of S. Let $f_{H_1}, ..., f_{H_r}$ be the r fibres of S, over the points $H_1, ..., H_r$ of the base curve $C \simeq \mathbb{P}^1$, corresponding to $L_1, L_2, ..., L_r$. If we consider the linear space of $\mathbb{P}^N$ spanned by any subset of ρ lines in $\{L_1, L_2, ..., L_r\}$, corresponding to ρ points in $\{H_1, ..., H_r\}$, say $H_1, ..., H_\rho$, we have that its dimension is

$$
\begin{aligned}
N - h^0(S, T - f_{H_1} ... - f_{H_\rho}) &= N - h^0(C, \mathcal{E} \otimes \mathcal{O}_C(-H_1 ... - H_\rho)) \\
&= N - h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(h - \rho) \oplus
\end{aligned}
$$

Now, if $1 \leq \rho \leq h$ the dimension is $N - (h - \rho + 1 + k - \rho + 1) = 2\rho - 1$. If $h < \rho \leq k$ the dimension is $N - (k - \rho + 1) = \rho + h$. If $k < \rho$ the dimension is N.

In other words:

$$\dim(\langle L_1, \dots, L_\rho \rangle) = \begin{cases} 2\rho - 1 & \text{if } 1 \le \rho \le h+1 \\ \rho + h & \text{if } h+2 \le \rho \le k+1 \\ N & \text{if } k+2 \le \rho \le N. \end{cases}$$

Hence assumptions $1), 2), 3)$ of Theorem 2.1 hold for $L_1, L_2, ..., L_r$. □

The following remark will be very useful for the proof of Theorem 2.1.

Remark 2.1. Let $\mathcal{L}$ be a set of lines in $\mathbb{P}^N$ satisfying the assumptions of Theorem 2.1. Let $\mathcal{L}' = \{L_1, \dots, L_{r'}\} \subseteq \mathcal{L}$ be any subset of $\mathcal{L}$, with $2 \le r' \le r$, having a corresponding subscheme $W_{r'}$, defined similarly as in Theorem 2.1. Note that $\mathcal{L}'$ satisfies the same assumptions as $\mathcal{L}$, so that to prove Theorem 2.1 one can proceed by induction on r : assuming that $\dim(W_{r'}) \le r' - 2$ for any $\mathcal{L}' \subseteq \mathcal{L}$ with $r' \le r$, we will show that $\dim(W_r) \le r - 2$.

As suggested by Remark 2.1, the proof of Theorem 2.1 will proceed by induction on r, and will make use of a few preliminary Lemmata. The following Lemma collects two simple observations that will facilitate the induction process.

Lemma 2.2. *In the assumptions of Theorem 2.1, let $r \ge 3$ and let m be any fixed positive integer. Assume that $\dim(W_{r'}) \le r' - m$ for any subset of $r' < r$ lines in $\mathcal{L}$. Then, in order to prove that $\dim(W_r) \le r - m$, one can assume that for any generic configuration $(P_1, \dots, P_r) \in L_1 \times L_2 \times \dots \times L_r \simeq (\mathbb{P}^1)^{\times r}$ in W_r the following facts are true:*
1) $\dim(\langle P_1, \dots, P_r \rangle) = r - 2$
2) $\dim(\left\langle P_1, \dots, \widehat{P_i}, \dots, P_r \right\rangle) = r - 2$ *for any i, where $\widehat{P_i}$ is deleted.*

Proof. To prove that we can assume 1), let us consider $W'_r := \{(P_1, \dots, P_r) \in L_1 \times L_2 \times \dots \times L_r \simeq (\mathbb{P}^1)^{\times r} | \dim(\langle P_1, \dots, P_r \rangle) \le r - 3\} \subseteq W_r$ (if $r = 3$ $W'_r = \emptyset$). If we project any r-uple of W'_r onto any product of $r - 1$ lines chosen in $\mathcal{L}$ we get a $(r - 1)$-tuple of the set W_{r-1} corresponding to those $r - 1$ lines. By assumption $\dim(W_{r-1}) \le r - 1 - m$, hence $\dim(W'_r) \le r - 1 - m + 1 = r - m$. Therefore if $W_r = W'_r$ then $\dim(W_r) \le r - m$, so that we can always assume that $W_r \supsetneq W'_r$, i.e. fact 1).

To prove that we can assume 2), choose any $i \in \{1, \dots, r\}$ and let us consider the closed subscheme $W_{r-1} \subseteq W_r$ corresponding to the subset $\{L_1, \dots, \widehat{L_i}, \dots, L_r\} \subsetneq \mathcal{L}$, where $\widehat{L_i}$ is removed. Obviously $W_{r-1} \times L_i \subseteq W_r$. By assumption $\dim(W_{r-1}) \le r-1-m$, hence $\dim(W_{r-1} \times L_i) \le r-1-m+1 = r-m$. Therefore if $W_r = W_{r-1} \times L_i$ then $\dim(W_r) \le r - m$, so that we can always assume that $W_r \supsetneq W_{r-1} \times L_i$. As this is true for any $i \in \{1, \dots, r\}$ we can assume fact 2). □

Lemma 2.3. *Let $L_1, \dots, L_{h+1}$ be disjoint lines in $\mathbb{P}^{2h+1}$, with $h \ge 1$, such that their linear span has maximal dimension, i.e. $\langle L_1, \dots, L_{h+1} \rangle = \mathbb{P}^{2h+1}$. For any $Q \in \mathbb{P}^{2h+1}$ let $1 \le t_Q \le h+1$ be the minimum number of lines among the L_i's necessary to have Q contained in their linear span, which has dimension $2t_Q - 1$. Let $W_{h+1}(Q) := \{(P_1, \dots, P_{h+1}) \in L_1 \times L_2 \times \dots \times L_{h+1} \simeq (\mathbb{P}^1)^{\times (h+1)} | \dim(\langle Q, P_1, \dots, P_{h+1} \rangle) \le h\}$. Then $0 \le \dim(W_{h+1}(Q)) \le h + 1 - t_Q$ and if $\dim(W_{h+1}(Q)) = 0$ then $W_{h+1}(Q)$ is a single point.*

Proof. The proof will be conducted in detail for $h = 2$. The general case is handled exactly in the same fashion. As the given lines have a linear span of maximal dimension, it is possible to choose a coordinate system in the ambient space $\mathbb{P}^{2h+1=5}$ such that its $2h+2 = 6$ fundamental points belong, pairwise, to the $h+1 = 3$ given lines. In this situation, let us consider the $(h+2 = 4, 2h+2 = 6)$ matrix M whose first $3 = h+1$ rows are given by the coordinates of points on the lines L_1, L_2, L_3, and where the last row consists of the coordinates of Q :

$$M = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_3 & \beta_3 \\ x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}.$$

For any Q, $W_3(Q)$ is given by all possible choices of pairs $(\alpha_i : \beta_i) \neq (0,0)$ for which $\operatorname{rk}(M) \leq 3$. It is easy to see that, for any Q, there exists at least one such choice of pairs $(\alpha_i : \beta_i)$, namely $(\alpha_i : \beta_i) = (x_{2i-2}, x_{2i-1})$ for all pairs $(x_{2i-2}, x_{2i-1}) \neq (0,0)$), hence $\dim(W_3(Q)) \geq 0$.

To get the other side of the stated inequality notice that, as $(\alpha_i : \beta_i) \neq (0:0)$, M can always be transformed into the following matrix

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \lambda_3 \\ y_0 & y_2 & y_4 & y_1 & y_3 & y_5 \end{bmatrix},$$

where $\operatorname{rk}(M_1) = \operatorname{rk}(M)$, $\lambda_i = \alpha_i/\beta_i$ or $\lambda_i = \beta_i/\alpha_i$ respectively when $\beta_i \neq 0$ or $\alpha_i \neq 0$, and $(y_0, \ldots, y_5)$ is a permutation of $(x_0, \ldots, x_5)$. M_1 can then be further transformed, keeping its rank unaltered:

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \lambda_3 \\ 0 & 0 & 0 & y_1 - \lambda_1 y_0 & y_3 - \lambda_2 y_2 & y_5 - \lambda_3 y_4 \end{bmatrix}$$

It is $\operatorname{rk}(M_2) \leq 3$ if and only if:

$$\begin{cases} \lambda_1 y_0 & = y_1 \\ \lambda_2 y_2 & = y_3 \\ \lambda_3 y_4 & = y_5. \end{cases}$$

As $\dim(W_3(Q)) \geq 0$, the above system must have at least one solution. If no equation is identically satisfied, then there exists only one solution $(\lambda_1, \lambda_2, \lambda_3)$, corresponding to a triplet of points, one for each line; in this case $\dim(W_3(Q)) = 0$ and the Lemma is proved. If there is only one identically satisfied equation, say $y_0 = y_1 = 0$, then $\dim(W_3(Q)) = 1$ (you can choose an arbitrary point on the first line, but then the other two are determined) and in this case $Q \in \langle L_2, L_3 \rangle$, hence $t_Q = 2$ and the Lemma is proved. If exactly two equations are identically satisfied, say $y_0 = y_1 = y_2 = y_3 = 0$, then $\dim(W_3(Q)) = 2$ (you can choose arbitrary points on the first two lines, while the last point is uniquely determined), and in this case $Q \in \langle L_3 \rangle$ hence $t_Q = 1$ and the Lemma is proved. As $(y_0, \ldots, y_5)$ is a permutation of projective coordinates of Q, not all equations can be identically satisfied, so that the Lemma is proved for $h = 2$. □

Lemma 2.4. *Under the assumptions of Theorem 2.1, further assume that* $r \geq h+2$. *Let* $\mathcal{L}' = \{L'_1, L'_2, \ldots, L'_{h+1}\}$ *be any subset of* $h+1$ *lines chosen from the given set* $\mathcal{L} = \{L_1, \ldots, L_r\}$. *Let* $L \in \mathcal{L} \setminus \mathcal{L}'$. *Then* L *intersects the* $(2h+1)$*-dimensional linear space* $\langle L'_1, L'_2, \ldots, L'_{h+1} \rangle$ *only at one point* Q *and such* Q *does not belong to any linear space spanned by any proper subset of* $\mathcal{L}'$. *Moreover, there exists a unique choice of* $P_i \in L'_i$, *such that* $\dim(\langle Q, P_1, \ldots, P_{h+1} \rangle) \leq h$.

Proof. Assumption 1) of Theorem 2.1 gives that $\mathcal{L}'$ spans a $(2h+1)$-dimensional linear subspace. Any other line $L \in \mathcal{L} \setminus \mathcal{L}'$, cuts this subspace only at one point Q, by assumption 2). Moreover, Q can not belong to any linear space spanned by a proper subset of $\mathcal{L}'$, otherwise the union of this proper subset and L would contradict assumption 1) of Theorem 2.1. Therefore Lemma 2.3, gives a unique choice of points $P_i \in L'_i$ such that $\dim(\langle Q, P_1, \ldots, P_{h+1} \rangle) \leq h$. □

The above Lemmata will now be combined to provide a proof for Theorem 2.1.

Proof. (of Theorem 2.1).

It is convenient to divide the proof into 4 cases, according to the relative sizes of r, h and k.

Case 1: $2 \leq r \leq h+1$. In this case W_r is actually empty. To see this, choose a coordinate system in $\mathbb{P}^N$ such that $2r$ points among its $N+1$ fundamental points belong, pairwise, to the r given lines. This is possible by assumption 1). As in the proof of Lemma 2.3, consider the following $(r, N+1)$ matrix whose rows are given by the coordinates of points on each of the given r lines:

$$\begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \alpha_2 & \beta_2 & \ldots & 0 & 0 & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & \alpha_r & \beta_r & 0 & \ldots & 0 \end{bmatrix}.$$

As $(\alpha_i : \beta_i) \neq (0:0)$ for all i, it is clear that there always exists a non singular, rank r, submatrix and thus $W_r = \emptyset$.

Case 2: $2 \leq r \leq k+1$. In this case, induction on r will show that

$$\dim(W_r) \leq r - h - 2. \tag{2.1}$$

This slightly stronger inequality implies the statement and it will be useful in proving the remaining cases. If $k = h$, or $2 \leq r \leq h+1$, there is nothing to prove after Case 1, so we can assume $h < k$ and $r \geq h+2$ (note that this implies $r \geq 3$). Our inductive hypothesis is that the desired inequality (2.1) holds for each subset of r' lines contained in $\mathcal{L}$, with $2 \leq r' < r$ (recall remark 2.1). Moreover, for a generic $(P_1, \ldots, P_r) \in W_r$, it is enough to consider the cases that $\dim(\langle P_1, \ldots, P_r \rangle) = r-2$ by part 1) of Lemma 2.2.

Fix any order for $\mathcal{L}$ and, recalling Lemma 2.4, let $Q_1, Q_2, \ldots, Q_{r-h-1}$ be the points of intersection of each of the last $r-h-1$ lines with the linear subspace spanned by the first $h+1$ lines. Let us choose a coordinate system in $\mathbb{P}^N$ such that its first $2(h+1)$ fundamental points belong, pairwise, to the first $h+1$ lines, and such that each one of the remaining fundamental points belongs to one of the remaining lines. Notice that these remaining fundamental points are certainly distinct from $Q_1, Q_2, \ldots, Q_{r-h-1}$. The rows of the following $(r, N+1)$ matrix M are given by the coordinates of the points of the r given lines :

$$M = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & \alpha_2 & \beta_2 & \dots & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \alpha_{h+1} & \beta_{h+1} & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ & & & & \gamma_1 \underline{a}_1 & & & \delta_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ & & & & \gamma_2 \underline{a}_2 & & & 0 & \delta_2 & \dots & 0 & \dots & \dots & \dots \\ & & & & \dots & & & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & \gamma_{r-h-1} \underline{a}_{r-h-1} & & & 0 & 0 & \dots & \delta_{r-h-1} & 0 & \dots & 0 \end{bmatrix}$$

where every row vector $\underline{a}_i$ is determined by the coordinates of Q_i.

By looking at M one sees that an r-tuple $(P_1, \dots, P_r)$ belongs to W_r (i.e. $\operatorname{rk}(M) < r$) if and only if $\delta_i = 0$ for at least one i, i.e. if and only if at least a point among $P_{h+2}, \dots, P_r$ coincides with one of the points $Q_1, Q_2, \dots, Q_{r-h-1}$. For each of these $r - h - 1$ possible equalities one gets a different component of W_r. Thus it suffices to prove (2.1) for the component with maximal dimension. Without loss of generality, let us assume that the maximal dimension is achieved for $Q_{r-h-1} = P_r := \overline{P}$, the point, on the last line, with coordinates $(\gamma_{r-h-1} : \delta_{r-h-1}) = (1 : 0)$.

Let $Z_{r-1} = \{(P_1, \dots, P_{r-1}) \in L_1 \times L_2 \times \dots \times L_{r-1} | \dim(\langle P_1, \dots, P_{r-1}, \overline{P} \rangle) = r - 2\}$. From the above discussion we have that $\dim(W_r) \leq \dim Z_{r-1}$. Note that we can assume $\dim(\langle P_1, \dots, P_{r-1}, \overline{P} \rangle) = r - 2$ because Lemma 2.2, part 1), guarantees it for a generic point of W_r, hence it is also true for a generic point of the component of maximal dimension of W_r. If $Z_{r-1} \subseteq W_{r-1}$ we would have $\dim(W_r) \leq \dim(Z_{r-1}) \leq \dim(W_{r-1}) \leq r-1-h-2 \leq r-h-2$, by induction, and we would be done. If not, the generic $(r-1)$-tuple $(P_1, \dots, P_{r-1}) \in Z_{r-1}$ is such that $\dim(\langle P_1, \dots, P_{r-1} \rangle) = r-2$. This fact implies that in any matrix M corresponding to a generic point of W_r with $P_r = \overline{P}$, it must be $\delta_{r-h-1} = 0$, and $\delta_i \neq 0$ for any $i \neq r - h - 1$. Hence, as $\operatorname{rk}(M) < r$, the submatrix M_1 consisting of the first $h + 1$ rows and the last one must have $\operatorname{rk}(M_1) < h + 2$. By Lemma 2.3, by recalling that $\overline{P}$ can not belong to the linear subspace spanned by any proper subset of $L_1, \dots, L_{h+1}$, this can happen only for a unique choice of points $(P_1, \dots, P_{h+1})$. Therefore in any matrix M corresponding to a generic point of W_r with $P_r = \overline{P}$ all parameters appearing in the first $h+1$ rows and in the last one are fixed. Only $r - (h+1) - 1$ parameters remain free in M and we are done.

Case 3: $2 \leq r \leq N$ and $1 \leq r - (h + 1) \leq h + 1$. In this case, inequality $\dim(W_r) \leq r - 2$ will be established by induction on r, keeping always in mind Remark 2.1. Having established Cases 1 and 2 we can assume that $k + 2 \leq r \leq N$ and, by Lemma 2.2 part 1), we can also assume that $\dim(\langle P_1, \dots, P_r \rangle) = r - 2$ for the generic r-tuple of W_r. Fix an order for $\mathcal{L}$ and let us divide any r-tuple in W_r into two non empty subsets: $(P_1, \dots, P_r) = (P_1, \dots, P_{h+1})(P_{h+2}, \dots, P_r)$. As $(P_1, \dots, P_r) \in W_r$ we have: $\dim(\langle P_1, \dots, P_{h+1} \rangle \cup \langle P_{h+2}, \dots, P_r \rangle) \leq r - 2$ and, by Case 1, $\dim(\langle P_1, \dots, P_{h+1} \rangle) = h$ and $\dim(\langle P_{h+2}, \dots, P_r \rangle) = r - (h + 1) - 1$. Hence $\dim(\langle P_1, \dots, P_{h+1} \rangle \cap \langle P_{h+2}, \dots, P_r \rangle) \geq h + r - (h + 1) - 1 - r + 2 = 0$ and therefore there always exists at least a point $Q \in \langle P_1, \dots, P_{h+1} \rangle \cap \langle P_{h+2}, \dots, P_r \rangle$. Moreover, as for the generic r-tuple of W_r it is true that $\dim(\langle P_1, \dots, P_r \rangle) = \dim(\langle P_1, \dots, P_{h+1} \rangle \cup \langle P_{h+2}, \dots, P_r \rangle) = r - 2$, we can also say that for the generic r-tuple of W_r there exists a unique point $Q \in \langle P_1, \dots, P_{h+1} \rangle \cap \langle P_{h+2}, \dots, P_r \rangle$.

Now, let us consider $\langle L_1, \dots, L_{h+1} \rangle$ and $\langle L_{h+2}, L_{h+3}, \dots, L_r \rangle$. As $1 \leq r - (h + 1) \leq h + 1$ by assumption 1) we can say that $\dim(\langle L_1, \dots, L_{h+1} \rangle) = 2(h + 1) - 1$ and $\dim(\langle L_{h+2}, L_{h+3}, \dots, L_r \rangle) = 2(r - h - 1) - 1$. As $k + 2 \leq r \leq N$ we can say that $\dim(\langle L_1, \dots, L_{h+1} \rangle \cup \langle L_{h+2}, L_{h+3}, \dots, L_r \rangle) = \dim(\langle L_1, \dots, L_r \rangle) = N$. Hence, if we define $A := \langle L_1, \dots, L_{h+1} \rangle \cap \langle L_{h+2}, L_{h+3}, \dots, L_r \rangle$, we have that $\dim(A) = 2h + 1 + 2r - 2h - 3 - N = 2r - 2 - N \leq r - 2$. Moreover, as we saw that for the generic r-tuple of W_r there exists a (unique) point $Q \in \langle P_1, \dots, P_{h+1}\rangle \cap \langle P_{h+2}, \dots, P_r\rangle \subseteq A$, we can also say that A is not empty (unless W_r is empty, in which case there is nothing to prove). The linear space A contains all intersections points of lines $L_{h+1}, L_{h+2}, \dots, L_r$ with $\langle L_1, \dots, L_{h+1}\rangle$, and these intersection points surely exist by assumption 2). Therefore A can not be contained in a linear subspace of $\langle L_1, \dots, L_{h+1}\rangle$ spanned by a proper subset of these lines because no one of those points belong to such a space, thanks to Lemma 2.4. Lemma 2.3 then implies that, for a generic $Q \in A$, there exists a unique $(h+1)$-tuple of points $P_1, \dots, P_{h+1}$, such that $\dim(\langle Q, P_1, \dots, P_{h+1}\rangle) \leq h$.

Let us introduce in $L_1 \times L_2 \times \cdots \times L_r \times A \simeq (\mathbb{P}^1)^{\times r} \times \mathbb{P}^{2r-2-N}$ the following (non empty) incidence variety:

$$\begin{aligned} J := &\{(P_1, \dots, P_r, Q) \in (\mathbb{P}^1)^{\times r} \times A | Q \in \langle P_1, \dots, P_{h+1}\rangle \cap \langle P_{h+2}, \dots, P_r\rangle\} \\ = &\{(P_1, \dots, P_r, Q) \in (\mathbb{P}^1)^{\times r} \times A | \dim(\langle Q, P_1, \dots, P_{h+1}\rangle) \leq h \\ &\text{and} \dim(\langle Q, P_{h+2}, \dots, P_r\rangle) \leq r-(h+1)-1\}. \end{aligned}$$

Let $p : J \to (\mathbb{P}^1)^{\times r}$ and $f : J \to A$ be the natural projections. It is $p(J) \subseteq W_r$ because if $(P_1, \dots, P_r, Q) \in J$ then the points $(P_1, \dots, P_r)$ can not be linearly independent in $\mathbb{P}^N$. On the other hand we have seen that for any r-tuple of W_r there exist at least a point $Q \in \langle P_1, \dots, P_{h+1}\rangle \cap \langle P_{h+2}, \dots, P_r\rangle \subseteq A$ and that for the generic r-tuple of W_r there exist a unique point Q. Hence $\mathrm{Im}(p) = W_r$ and $\dim(J) = \dim(W_r)$. Then $\dim(W_r) = \dim(J) = \dim(\mathrm{Im}(f)) + \dim($*generic fibre of* $f)$.

Let us consider any point $Q \in A$. As $Q \in \langle L_1, \dots, L_{h+1}\rangle$, Lemma 2.3 implies that there exists at least an $(h+1)$-tuple of points $(P_1, \dots, P_{h+1})$ such that $\dim(\langle Q, P_1, \dots, P_{h+1}\rangle) \leq h$. As $Q \in \langle L_{h+2}, L_{h+3}, \dots, L_r\rangle$, Lemma 2.3 implies that there exists at least an $(r-h-1)$-tuple of points $(P_{h+2}, \dots, P_r)$ such that $\dim(\langle Q, P_{h+2}, \dots, P_r\rangle) \leq r-h-2$. Therefore $\mathrm{Im}(f) = A$.

In order to estimate the dimension of a generic fiber of f, let Q be now a generic point of A. Lemma 2.3 implies that A can not be contained in a linear subspace of $\langle L_1, \dots, L_{h+1}\rangle$ spanned by a proper subset of these lines and that there exists a unique $(h+1)$-tuple of points $(P_1, \dots, P_{h+1})$ such that $\dim(\langle Q, P_1, \dots, P_{h+1}\rangle) \leq h$. Hence to get a bound for $\dim(f^{-1}(Q))$ it suffices to consider the $(r-h-1)$-tuples of points $P_{h+2}, \dots, P_r$ such that $\dim(\langle Q, P_{h+2}, \dots, P_r\rangle) \leq r-h-2$. With the notation introduced in the proof of Lemma 2.3, it is true that
$\dim(f^{-1}(Q)) = \dim(W_{r-h-1}(Q)) =$
$\dim(\{(P_{h+2}, \dots, P_r) \in L_{h+1} \times L_{h+2} \times \cdots \times L_r | \dim(\langle Q, P_{h+2}, \dots, P_r\rangle) \leq r-h-2\})$.

If A is not contained in a linear subspace of $\langle L_{h+1}, \dots, L_r\rangle$ spanned by a proper subset of these lines, Lemma 2.3 gives that for the generic point $Q \in A$ there exists only one $(r-h-1)$-tuple of points $(P_{h+2}, \dots, P_r)$ such that $\dim(\langle Q, P_{h+2}, \dots, P_r\rangle) \leq r-h-2$. In this case $\dim(f^{-1}(Q)) = 0$ and therefore $\dim(W_r) = \dim(J) = \dim(\mathrm{Im}(f)) + \dim($*generic fibre of* $f) = \dim(A) = 2r-2-N \leq r-2$ and we are done.

If A is contained in at least one linear subspace spanned by a proper subset of $\{L_{h+1}, \dots, L_r\}$, let $2t-1$ be the dimension of the space, spanned by t lines, with the minimal dimension among them. Note that $1 \leq t < r-(h+1) \leq h+1$. For all $Q \in A$ Lemma 2.3 gives $\dim[W_{r-h-1}(Q)] \leq r-h-1-t$. Then we have

$\dim(W_r) = \dim(J) = \dim(\mathrm{Im}(f)) + \dim(generic\ fibre\ of\ f) \le 2t-1+r-h-1-t = t+r-h-2 < h+1+r-h-2 = r-1$, i.e. $\dim(W_r) \le r-2$.

Case 4: $2 \le r \le N$ and $h+2 \le r-(h+1) < k+1$. Because of Cases 1, 2 and 3 we can assume $k+2 \le r \le N$ and, by Lemma 2.2 part 1), we can also assume that $\dim(\langle P_1, \dots, P_r\rangle) = r-2$ for the generic r-tuple of W_r. From Case 2 we have $\dim(W_{r-h-1}) \le r-h-1-h-2 = r-2h-3$.

As before, fix an order for $\mathcal{L}$ and let us divide every r-tuple in W_r into two non empty subsets $(P_1, \dots, P_r) = (P_1, \dots, P_{h+1})(P_{h+2}, \dots, P_r)$. As $(P_1, \dots, P_r) \in W_r$ we have that $\dim(\langle P_1, \dots, P_{h+1}\rangle \cup \langle P_{h+2}, \dots, P_r\rangle) \le r-2$ and, from Case 1, $\dim(\langle P_1, \dots, P_{h+1}\rangle) = h$. Moreover, Lemma 2.2 part 2) gives $\dim(\langle P_{h+2}, \dots, P_r\rangle) = r-h-2$ for the generic r-tuple of W_r.

Thus $\dim(\langle P_1, \dots, P_{h+1}\rangle \cap \langle P_{h+2}, \dots, P_r\rangle) \ge h+r-h-2-r+2 = 0$ and therefore there always exists at least one point $Q \in \langle P_1, \dots, P_{h+1}\rangle \cap \langle P_{h+2}, \dots, P_r\rangle$. Moreover, as $\dim(\langle P_1, \dots, P_r\rangle) = \dim(\langle P_1, \dots, P_{h+1}\rangle \cup \langle P_{h+2}, \dots, P_r\rangle) = r-2$, for a generic r-tuple of W_r, it follows that there exists a unique point $Q \in \langle P_1, \dots, P_{h+1}\rangle \cap \langle P_{h+2}, \dots, P_r\rangle$.

As in the previous case let us consider $\langle L_1, \dots, L_{h+1}\rangle$ and $\langle L_{h+2}, \dots, L_r\rangle$. As $h+2 \le r-(h+1) < k+1$ by assumptions 1) and 2) we have $\dim(\langle L_1, \dots, L_{h+1}\rangle) = 2(h+1)-1$ and $\dim(\langle L_{h+2}, L_{h+3}, \dots, L_r\rangle) = r-h-1+h = r-1$. As $k+2 \le r \le N$ we have $\dim(\langle L_1, \dots, L_{h+1}\rangle \cup \langle L_{h+2}, L_{h+3}, \dots, L_r\rangle) = \dim(\langle L_1, \dots, L_r\rangle) = N$. As in the previous case, let $A = \langle L_1, \dots, L_{h+1}\rangle \cap \langle L_{h+2}, L_{h+3}, \dots, L_r\rangle$. It is $\dim(A) = 2h+1+r-1-N = 2h+r-N$. Moreover, as for the generic r-tuple of W_r there exists a (unique) point $Q \in \langle P_1, \dots, P_{h+1}\rangle \cap \langle P_{h+2}, \dots, P_r\rangle \subseteq A$, A is non empty, unless W_r is empty, in which case there is nothing to prove. Note that A contains all the intersection points of each of the lines $L_{h+2}, \dots, L_r$ with $\langle L_1, \dots, L_{h+1}\rangle$ and such points certainly exist by assumption 2). Hence A is not contained in any linear subspace of $\langle L_1, \dots, L_{h+1}\rangle$, spanned by a proper subset of these lines because none of the intersections points mentioned above can be contained in such a subspace by Lemma 2.4. Lemma 2.3 then gives, for a generic point $Q \in A$, a unique $(h+1)$-tuple of points $P_1, \dots, P_{h+1}$, such that $\dim(\langle Q, P_1, \dots, P_{h+1}\rangle) \le h$.

As in the previous case, let us introduce in $L_1 \times L_2 \times \cdots \times L_r \times A \simeq (\mathbb{P}^1)^{\times r} \times \mathbb{P}^{2h+r-N}$ the following (non empty) incidence variety:

$$\begin{aligned} J :=& \{(P_1, \dots, P_r, Q) \in (\mathbb{P}^1)^{\times r} \times A | Q \in \langle P_1, \dots, P_{h+1}\rangle \cap \langle P_{h+2}, \dots, P_r\rangle\} \\ =& \{(P_1, \dots, P_r, Q) \in (\mathbb{P}^1)^{\times r} \times A | \dim(\langle Q, P_1, \dots, P_{h+1}\rangle) \le h \\ & \quad \text{and} \dim(\langle Q, P_{h+2}, \dots, P_r\rangle) \le r-(h+1)-1\}. \end{aligned}$$

Let $p : J \to (\mathbb{P}^1)^{\times r}$ and $f : J \to A$ be the natural projections. Note that $p(J) \subseteq W_r$ because if $(P_1, \dots, P_r, Q) \in J$ then $(P_1, \dots, P_r)$ are not linearly independent in $\mathbb{P}^N$. On the other hand we have seen that for every r-tuple of W_r there exists at least a point $Q \in \langle P_1, \dots, P_{h+1}\rangle \cap \langle P_{h+2}, \dots, P_r\rangle \subseteq A$ and that for the generic r-tuple of W_r there exists a unique such Q. Hence $\mathrm{Im}(p) = W_r$ and $\dim(J) = \dim(W_r)$. Then $\dim(W_r) = \dim(J) = \dim(\mathrm{Im}(f)) + \dim(generic\ fibre\ of\ f) \le \dim(A) + \dim[f^{-1}(\overline{Q})] = 2h+r-N + \dim[f^{-1}(\overline{Q})]$ where $\overline{Q}$ is now any fixed point of $\mathrm{Im}(f)$. Pick $\overline{Q} := \langle L_1, \dots, L_{h+1}\rangle \cap L_{h+2}$. Obviously $\overline{Q} \in A$. Moreover, as $\overline{Q}$ is the intersection point of L_{h+2} with $\langle L_1, \dots, L_{h+1}\rangle$, we know that it does not belong to any linear subspace of $\langle L_1, \dots, L_{h+1}\rangle$ spanned by a proper subset of these lines. Hence there exists a unique $(h+1)$-tuple of points $P_1, \dots, P_{h+1}$, such that $\dim(\langle \overline{Q}, P_1, \dots, P_{h+1}\rangle) \le h$. Choosing $P_{h+2} = \overline{Q}$ one sees that there exists also a $(r-h-1)$-tuple of points $(P_{h+2},\dots,P_r)\in L_{h+2}\times L_{h+3}\times\cdots\times L_r$ such that $\dim(\langle\overline{Q},P_{h+2},\dots,P_r\rangle)\le r-(h+1)-1$. Hence $\overline{Q}\in \mathrm{Im}(f)$ and, to estimate $\dim(f^{-1}(\overline{Q}))$, consider the $(r-h-1)$-tuples of points $P_{h+2},\dots,P_r$ such that $\dim(\langle\overline{Q},P_{h+2},\dots,P_r\rangle)\le r-h-2$, i.e. the set $Z(\overline{Q}):=\{(P_{h+2},\dots,P_r)\in L_{h+2}\times L_{h+3}\times\cdots\times L_r \,|\, \dim(\langle\overline{Q},P_{h+2},\dots,P_r\rangle)\le r-h-2\}$. Note that, as $r-(h+1)\ge h+2\ge 3$, we have $r\ge h+4$. Hence in $Z(\overline{Q})$ there are at least pairs of points.

Notice that, for the generic $(r-h-1)$-tuple $(P_{h+2},\dots,P_r)\in Z(\overline{Q})$, we have $\dim(\langle\overline{Q},P_{h+2},\dots,P_r\rangle)=r-h-2=\dim(\langle P_{h+2},\dots,P_r\rangle)$. Indeed the generic $(r-h-1)$-tuple $(P_{h+2},\dots,P_r)\in Z(\overline{Q})$ is a proper subset of a generic r-tuple of W_r and by Lemma 2.2, part 2), we have $\dim(\langle P_{h+2},\dots,P_r\rangle)=r-h-2$. On the other hand $\dim(\langle P_{h+2},\dots,P_r\rangle)\le\dim(\langle\overline{Q},P_{h+2},\dots,P_r\rangle)\le r-h-2$ by the definition of $Z(\overline{Q})$. Then one can define a map $\psi:\mathcal{Z}\to W_{r-h-1}$, where $\mathcal{Z}$ is a non empty Zariski-open subset of $Z(\overline{Q})$, by setting $\psi(P_{h+2},\dots,P_r)=(P,P_{h+3},\dots,P_r)$, where $(P_{h+2},\dots,P_r)$ is a generic element of $Z(\overline{Q})$ and P is the unique intersection, in $\langle\overline{Q},P_{h+2},\dots,P_r\rangle=\langle P_{h+2},\dots,P_r\rangle$ of the line L_{h+2} with the linear subspace $\langle P_{h+3},\dots,P_r\rangle$. Notice that $\langle P_{h+3},\dots,P_r\rangle$ has codimension 1 in $\langle\overline{Q},P_{h+2},\dots,P_r\rangle=\langle P_{h+2},\dots,P_r\rangle$ and it does not contain L_{h+2}. Obviously $(P,P_{h+3},\dots,P_r)\in W_{r-h-1}$. The generic fibre of ψ is contained in L_{h+2} and therefore it has dimension 1 at most. It follows that $\dim[Z(\overline{Q})]\le\dim(W_{r-h-1})+1$. So we get: $\dim[Z(\overline{Q})]\le\dim(W_{r-h-1})+1\le r-2h-3+1=r-2h-2$ by induction. Hence $\dim(W_r)\le 2h+r-N+\dim[f^{-1}(\overline{Q})]\le 2h+r-N+\dim[Z(\overline{Q})]\le 2h+r-N+r-2h-2=2r-N-2\le r-2$ and we are done. □

3. Corollaries of the main theorem

In this section we give a list of 5 corollaries of Theorem 2.1. The first two corollaries contain our answer to the question in Section 1. The third one proves a property of the open Zariski set A_r which is defined in the previous corollaries. The last two show that, under the assumption $r+1\le N$, we can say more about the hyperplanes cutting $P_1,\dots,P_r$ on the lines of $\mathcal{L}$.

Corollary 3.1. *With the same assumptions of Theorem 2.1 there exists a non empty, Zariski-open set $A_r\subseteq L_1\times L_2\times\cdots\times L_r\simeq(\mathbb{P}^1)^{\times r}$ such that, for every $(P_1,\dots,P_r)\in A_r$, it is $\dim(\langle P_1,\dots,P_r\rangle)=r-1$, and the generic hyperplane of $\mathbb{P}^N$ passing through $P_1,\dots,P_r$ does not contain any line of $\mathcal{L}$.*

Proof. Let $J_1,J_2,\dots,J_r$ be the r varieties defined by removing, respectively, the first, the second,…, the r^{th} factor of $(\mathbb{P}^1)^{\times r}$. Let $p_1,p_2,\dots,p_r$ be the natural projections $p_i:(\mathbb{P}^1)^{\times r}\to J_i$. By Theorem 2.1 we know that $\dim(W_r)\le r-2$ in $(\mathbb{P}^1)^{\times r}$, hence $p_i^{-1}(p_i(W_r))$ is a closed subscheme of dimension $\le r-1$ in $(\mathbb{P}^1)^{\times r}$, for any $i=1,\dots,r$. In $(\mathbb{P}^1)^{\times r}$, let A_r be the complement of the union of the r closed subschemes $p_i^{-1}(p_i(W_r))$. Obviously A_r is a non empty Zariski-open set in $(\mathbb{P}^1)^{\times r}$ and $\dim(\langle P_1,\dots,P_r\rangle)=r-1$ for every r-tuple $(P_1,\dots,P_r)\in A_r$ because $(P_1,\dots,P_r)\notin W_r$. Choose $L_t\in\mathcal{L}$ and, by contradiction, let us assume that every hyperplane in $\mathbb{P}^N$ passing through $P_1,\dots,P_r$ contains L_t. This would imply that there exists a point $Q\in L_t$ $(Q\neq P_t)$ such that $\dim(\langle P_1,\dots,P_{t-1},Q,P_{t+1},\dots,P_r\rangle)=r-2$ and therefore $(P_1,\dots,P_{t-1},Q,P_{t+1},\dots,P_r)\in W_r$. In fact: if all the hyperplanes passing through $P_1,\dots,P_r$ contain L_t, this line belongs to $\langle P_1,\dots,P_r\rangle$, which is the intersection of all hyperplanes passing through $P_1,\dots,P_r$; in the

$(r-1)$-dimensional linear space $\langle P_1,\dots,P_r\rangle$ there is the $(r-2)$-dimensional subspace $\langle P_1,\dots,P_{t-1},P_{t+1},\dots,P_r\rangle$ and the line L_t cuts this subspace at a point Q. But $(P_1,\dots,P_{t-1},Q,P_{t+1},\dots,P_r)$ cannot belong to W_r because $(P_1,\dots,P_r)\in p_t^{-1}(p_t(P_1,\dots,P_{t-1},Q,P_{t+1},\dots,P_r))$ and if $(P_1,\dots,P_{t-1},Q,P_{t+1},\dots,P_r)\in W_r$ the r-tuple $(P_1,\dots,P_r)$ would belong to the complement of A_r. □

Corollary 3.2. *With the same assumptions of Theorem 2.1, there exists a non empty, Zariski-open set* $\mathcal{H}\subseteq\mathbb{P}^{N*}$ *whose points correspond to hyperplanes in* $\mathbb{P}^N$ *cutting the set of lines* $L_1,\dots,L_r$ *only at an* r*-tuple of points* $P_1,\dots,P_r$*, with* $(P_1,\dots,P_r)\in A_r$*; moreover, for any non empty Zariski-open sets* $\mathcal{H}'\subseteq\mathcal{H}$ *and* $A'_r\subseteq A_r$ *and for any* <u>*generic*</u> $(P_1,\dots,P_r)\in A'_r$ *there is at least a point in* $\mathcal{H}'$ *corresponding to a hyperplane in* $\mathbb{P}^N$ *cutting the set of lines* $L_1,\dots,L_r$ *only at the* r*-tuple of points* $P_1,\dots,P_r$*.*

Proof. To prove Corollary 3.2, let us consider the incidence variety:

$I=\{(H,P_1,\dots,P_r)\in\mathbb{P}^{N*}\times(\mathbb{P}^1)^{\times r}|P_1,\dots,P_r\in H\}$

and its natural projections $\alpha: I\to\mathbb{P}^{N*}$ and $\beta: I\to(\mathbb{P}^1)^{\times r}$. Note that α is surjective and the dimension of the generic fibre of α is zero because a generic hyperplane of $\mathbb{P}^N$ intersects every line of $\mathcal{L}$ at one point only; thus $\dim(I)=N$. For any fixed r-tuple of points $(P_1,\dots,P_r)\in A_r$, there exists a linear subspace $\Lambda_{(P_1,\dots,P_r)}$ in $\mathbb{P}^{N*}$, given by the hyperplanes of $\mathbb{P}^N$ passing through $P_1,\dots,P_r$; we have $\dim(\Lambda_{(P_1,\dots,P_r)})=N-r$, because $P_1,\dots,P_r$ are linearly independent. It follows that $\dim(\beta^{-1}(A_r))=r+N-r=N$ for the non empty Zariski-open subset $\beta^{-1}(A_r)\subseteq I$, and therefore $I=\overline{\beta^{-1}(A_r)}$. Moreover, as every hyperplane either cuts every line $L_1,\dots,L_r$ at one point only or it contains the line entirely, the generic hyperplane of $\Lambda_{(P_1,\dots,P_r)}$ contains only the fixed r-tuple. If it contains other r-tuples it will then contain at least one of the lines in $\mathcal{L}$ but this is not possible as $(P_1,\dots,P_r)\in A_r$.

The above discussion shows that a generic point of $\beta^{-1}(A_r)$ can be represented as a pair $\{H,(P_1,\dots,P_r)\}$ where H is a hyperplane cutting every $L_1,\dots,L_r$ only at the points $P_1,\dots,P_r$ with $(P_1,\dots,P_r)\in A_r$. Hence there exists a subset $I^\dagger\subseteq\beta^{-1}(A_r)$ given by these pairs and $I^\dagger$ is a non empty Zariski-open set of I. To see this, for any $i=1,\dots,r$, let C_i be the Zariski closed set in $\mathbb{P}^{N*}$ given by all hyperplanes containing L_i. Every $C_i\times(\mathbb{P}^1)^{\times r}$ is a closed set of $\mathbb{P}^{N*}\times(\mathbb{P}^1)^{\times r}$. Let T be the complement of the union of these closed sets in $\mathbb{P}^{N*}\times(\mathbb{P}^1)^{\times r}$, then $I^\dagger$ is the intersection of the non empty Zariski-open set T with I, so that $\overline{I^\dagger}=\overline{\beta^{-1}(A_r)}=I$. Then $\dim(\alpha(I^\dagger))=\dim(\alpha(I))=N$ and therefore the interior of $\alpha(I^\dagger)$ is not empty. Letting $\mathcal{H}$ be the interior of $\alpha(I^\dagger)$, one concludes the proof of the first part of Corollary 3.2. To prove the second part it suffices to change A_r with A'_r : the interior of $\alpha(I'^\dagger)$ will intersect any non empty Zariski-open set $\mathcal{H}'$. □

Corollary 3.3. *With the same assumptions of Theorem 2.1, for every* $L_j\in\mathcal{L}$ *there exists a finite subset of points* $K_j\subsetneq L_j$*, possibly empty, such that for every point* $P_j\in L_j\backslash K_j$*, the intersection* $A_{r,P_j}:=A_r\cap[L_1\times L_2\times\dots,\{P_j\},\dots\times L_r\simeq(\mathbb{P}^1)^{\times(r-1)}]$ *is an open, non empty, Zariski set of* $(\mathbb{P}^1)^{\times(r-1)}$*.*

Proof. To prove Corollary 3.3 it is sufficient to remark that, as A_r is a non empty Zariski-open set in $(\mathbb{P}^1)^{\times r}$, its projection onto any factor $L_j\simeq\mathbb{P}^1$ is a non empty Zariski-open set in L_j. This open set is the complement of a finite set K_j of points (possibly empty). For every point $P_j\in L_j\backslash K_j$, A_r can not be contained in the complement of the closed set $L_1 \times L_2 \times \dots, \{P_j\}, \dots \times L_r$ and A_r intersects this closed set along a non empty Zariski-open subset of it. □

Corollary 3.4. *Let us assume that $r+1 \leq N$, and that there exist $r+1$ lines $L_0, L_1, \dots, L_r$ satisfying the assumptions of Theorem 2.1. Let P be any point on L_0 and let $\mathcal{Z}_P \in \mathbb{P}^{N*}$ be the dual hyperplane of P. Then there exists a non empty Zariski-open set $\mathcal{A}_P \subseteq \mathcal{Z}_P \simeq \mathbb{P}^{N-1}$ such that every hyperplane in $\mathbb{P}^N$ corresponding to a point in $\mathcal{A}_P$ cuts the lines $L_1, \dots, L_r$ only at an r-tuple of points $P_1, \dots, P_r$, with $(P_1, \dots, P_r) \in A_r$.*

Proof. Let us fix $P \in L_0$. By Theorem 2.1 applied to the $r+1$ lines $L_0, L_1, \dots, L_r$, we have $\dim(W_{r+1}) \leq r-1$, hence $\dim(W_{r+1} \cap (\{P\} \times (\mathbb{P}^1)^{\times r} \simeq (\mathbb{P}^1)^{\times r})) \leq r-1$. Therefore there exists a non empty Zariski-open set $B_P \subseteq (\mathbb{P}^1)^{\times r}$ such that $\dim(\langle P, P_1, \dots, P_r \rangle) = r$ for every choice of $(P_1, \dots, P_r) \in B_P$.

By Corollary 3.1 we know that there exists a non empty Zariski-open set A_r in $(\mathbb{P}^1)^{\times r}$ such that $\dim(\langle P_1, \dots, P_r \rangle) = r-1$ for every choice of $(P_1, \dots, P_r) \in A_r$ (and the generic hyperplane of $\mathbb{P}^N$ passing through $P_1, \dots, P_r$ does not contain any line of $\mathcal{L}$). Let $C_P = B_P \cap A_r$. Then C_P is a Zariski-open set in $(\mathbb{P}^1)^{\times r}$ such that $\dim(\langle P_1, \dots, P_r \rangle) = r-1$ and $\dim(\langle P, P_1, \dots, P_r \rangle) = r$ for every choice of $(P_1, \dots, P_r) \in C_P$ (and the generic hyperplane of $\mathbb{P}^N$ passing through $P_1, \dots, P_r$ does not contain any line of $\mathcal{L}$).

Now, to prove Corollary 3.4, let us consider the incidence variety:

$I = \{(H, P_1, \dots, P_r) \in \mathcal{Z}_P \times L_1 \times L_2 \times \dots \times L_r \simeq \mathbb{P}^{N-1} \times (\mathbb{P}^1)^{\times r} | P_1, \dots, P_r \in H\}$ and its natural projections $\alpha : I \to \mathcal{Z}_P$ and $\beta : I \to (\mathbb{P}^1)^{\times r}$. As in the proof of Corollary 3.2, the dimension of the generic fibre of α is zero, because a generic hyperplane of $\mathcal{Z}_P$ cuts every line $L_1, \dots, L_r$ at one point only, hence $\dim(I) = N-1$. For every r-tuple $(P_1, \dots, P_r) \in C_P$, $\beta^{-1}(P_1, \dots, P_r)$ is given by the hyperplanes of $\mathcal{Z}_P$ passing through $P_1, \dots, P_r$, i.e. by the hyperplanes of $\mathbb{P}^N$ passing through $P, P_1, \dots, P_r$. As $\dim(\langle P, P_1, \dots, P_r \rangle) = r$ we have that $\dim(\beta^{-1}(P_1, \dots, P_r)) = N-(r+1) \geq 0$. As C_P is a non empty Zariski-open set of $(\mathbb{P}^1)^{\times r}$, $N-(r+1)$ is also the dimension of the generic fibre of β and therefore $\dim(\beta^{-1}(C_P)) = N-(r+1)+r = N-1$ and thus $I = \overline{\beta^{-1}(C_P)}$. Then $\dim(\alpha(\beta^{-1}(C_P))) = N-1$ and therefore its interior $U_0 \subseteq \mathcal{Z}_P$ is not empty. Hence there exists a non empty Zariski-open set $U_0 \subseteq \mathcal{Z}_P$ such that every point of U_0 corresponds to a hyperplane in $\mathbb{P}^N$ containing P and an r-tuple of points $P_1, \dots, P_r$ with $(P_1, \dots, P_r) \in A_r$. On the other hand, for every line $L_i \in \mathcal{L}$, there exists a non empty Zariski-open set $U_i \subseteq \mathcal{Z}_P$ given by the hyperplanes of $\mathcal{Z}_P$ not containing L_i. Let $\mathcal{A}_P = U_0 \cap U_1 \cap \dots \cap U_r$. $\mathcal{A}_P$ is a non empty Zariski-open set in $\mathcal{Z}_P$ such that each one of its points corresponds to a hyperplane in $\mathbb{P}^N$ passing through P and cutting the lines $L_1, \dots, L_r$ at r points $P_1, \dots, P_r$ only, with $(P_1, \dots, P_r) \in A_r$. □

Corollary 3.5. *With the same assumptions of Corollary 3.4, let $A'_r \subseteq A_r$ be any non empty Zariski-open subset. Then for every point $P \in L_0$, there exists a non empty Zariski-open set $\mathcal{A}'_P \subseteq \mathcal{Z}_P \simeq \mathbb{P}^{N-1}$ such that every hyperplane in $\mathbb{P}^N$ corresponding to a point in $\mathcal{A}'_P$ cuts the lines $L_1, \dots, L_r$ only at an r-tuple of points $P_1, \dots, P_r$, with $(P_1, \dots, P_r) \in A'_r$; moreover, for any non empty Zariski-open sets $\mathcal{A}''_P \subseteq \mathcal{Z}_P$ and for any generic $(P_1, \dots, P_r) \in A'_r$ there is at least a point in $\mathcal{A}''_P$ corresponding to a hyperplanes in $\mathbb{P}^N$ cutting the set of lines $L_1, \dots, L_r$ only at the r-tuple of points $P_1, \dots, P_r$.*

Proof. To prove the first part of Corollary 3.5 it suffices to change $A'_r \subseteq A_r$ with A_r in the proof of Corollary 3.4. To prove the second part it suffices to intersect $\mathcal{A}''_P$ with $\mathcal{A}'_P$. □

4. Lines on rational scrolls

Let S be a smooth, rational, scroll surface in $\mathbb{P}^N$ such that $S = \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(h) \oplus \mathcal{O}_{\mathbb{P}^1}(k)$ with $1 \leq h \leq k$, $N = h + k + 1$, and S is embedded in $\mathbb{P}^N$ by its tautological line bundle. Such scrolls are surfaces of minimal degree and projectively normal. By Lemma 2.1 we know that the assumptions of Theorem 2.1 are satisfied when $\mathcal{L} = \{L_1, \ldots, L_r\}$ is *any* set of r lines in $\mathbb{P}^N$ which are fibres of a scroll such S, with $2 \leq r \leq N$. As usual C_0 and f will be the numerical classes of the fundamental section and of any fibre of S, respectively. We have that $-C_0^2 = e = k - h$, where e is the invariant of S (see [2, V.2] for all references about ruled surfaces).

In this section we will always assume that $\mathcal{L} = \{L_1, \ldots, L_r\}$ is a set as above and $r \geq 3$. We will show that Theorem 2.1 can be made more precise for these sets of lines when $r \geq k + 2$ by using the existence of a well known incidence relation I_r, see below, however the theorem cannot be improved in this way.

First of all, let us recall that, by Lemma 2.2 1), to get any bound on the dimension on W_r, when $\mathcal{L} = \{L_1, \ldots, L_r\}$ is a set as above, we can assume that $\dim(\langle P_1, \ldots, P_r \rangle) = r - 2$ for any generic $(P_1, ..., P_r) \in W_r$. Hence let us consider the set $\widehat{W}_r := \{(P_1, \ldots, P_r) \in S^{(r)} \mid P_1, \ldots, P_r$ are distinct, belonging to r distinct lines of S and $\dim(\langle P_1, \ldots, P_r \rangle) = r - 2\}$. Because we can choose r lines amomg the fibres of S in ∞^r ways, we have $\dim(\widehat{W}_r) = \dim(W_r) + r$. Hence, to get a bound on the dimension on W_r, when $\mathcal{L} = \{L_1, \ldots, L_r\}$ is a set as above, it suffices to get a bound for the dimension of $\widehat{W}_r$.

Let G be the Grassmannian $G(r-2, N)$ of the $(r-2)$-dimensional linear spaces of $\mathbb{P}^N$, let $S^{(r)}$ be the r-symmetric product of S. We can consider the incidence variety $I_r \subseteq S^{(r)} \times G$ such that:

$$I_r := \{((P_1, \ldots, P_r), \Pi) \in S^{(r)} \times G \mid P_1, \ldots, P_r \in \Pi\} \qquad (*)$$

with the two natural projections $p : I_r \to S^{(r)}$ and $q : I_r \to G$. Note that $\widehat{W}_r \subseteq \mathrm{Im}(p)$, moreover the fibre of p over $\widehat{W}_r$ is given by only one $(r-2)$-dimensional linear space, so that $\dim(\widehat{W}_r) = \dim[p^{-1}(\widehat{W}_r)]$. Therefore to get bounds on the dimension of $\widehat{W}_r$ it is sufficient to get bounds on the dimension of $p^{-1}(\widehat{W}_r)$ by using q.

To investigate the fibre of the restriction of q to $p^{-1}(\widehat{W}_r)$, let us put $\overline{W}_r := q[p^{-1}(\widehat{W}_r)]$ and let us consider the fibre over the generic $\Pi \in \overline{W}_r$. Π is a linear space of dimension $r-2$ cutting S at r distinct point belonging to r distinct lines of S. The fibre of q over Π has positive dimension, for instance, when Π contains a curve Γ which is a smooth, irreducible section of S, and in this case the fibre has dimension r, because we could choose any set of r points on Γ. About such a section Γ, we have the following

Lemma 4.1. *Let S be a surface as above. Let Γ be a smooth, irreducible section of S, $\Gamma = C_0 + bf$, such that $\dim(\langle \Gamma \rangle) = N - t$, $t \geq 1$. Then $b = k + 1 - t$ and $1 \leq t \leq h + 1$.*

Proof. Let $H = C_0 + kf$ be the numerical class of the hyperplane section of S. Obviously $b \leq k$, and $k - h \leq b$, as Γ is supposed to be a smooth, irreducible section of S. Let us consider the exact sequence: $0 \to H - \Gamma \to H \to H_{|\Gamma} \to 0$. As $h^1(S, H - \Gamma) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k - b)) = 0$, we have:

$N+1 = h+k+2 = h^0(S,H) = h^0(S,H-\Gamma)+h^0(\Gamma,H_{|\Gamma}) = (k-b+1)+(N-t+1)$.

Hence: $b = k+1-t$ and it must be: $k-h \leq k+1-t \leq k$, i.e. $1 \leq t \leq h+1$. □

Now let us return to the fibre over the generic $\Pi \in \overline{W}_r$. By Lemma 4.1, if Π contains a section Γ as above, given that $\langle\Gamma\rangle \subseteq \Pi$, then $N-t \leq r-2$ with $t \leq h+1$, hence $N+2-r \leq h+1$, hence $h+k+3-r \leq h+1$, hence $r \geq k+2$. It follows that $r \geq k+2$ is exactly the range for which sections as Γ can occur.

Let V be the subvariety of G parametrizing $(r-2)$-dimensional linear spaces of $\mathbb{P}^N$ which are $(r-1)$-secant S. Obviously $\mathrm{Im}(q) \subseteq V$, but $\mathrm{Im}(q) \neq V$ and $\dim(V) = 2(r-1)$, so that $\dim(\mathrm{Im}(q)) < 2r-2$. If $\dim\{q[p^{-1}(\widehat{W}_r)]\} = \dim[p^{-1}(\widehat{W}_r)]$, then $\dim(\widehat{W}_r) = \dim[p^{-1}(\widehat{W}_r)] < 2r-2$, hence $\dim(W_r) < r-2$ thus giving a stronger bound; but we saw above that fibres of the restriction of q to $p^{-1}(\widehat{W}_r)$ can be of positive dimension when $r \geq k+2$, hence we cannot use I_r to improve Theorem 2.1. However we can prove the following

Proposition 4.1. *Let S be a surface as above and let $\widehat{W}_r$ be defined as above. Assume that $N \geq r \geq k+2$, then $\dim(\widehat{W}_r) = 2r-2$ and $\dim(W_r) = r-2$.*

Proof. We know that $\dim(\widehat{W}_r) = \dim(W_r) + r$, so we can consider only $\widehat{W}_r$. By Theorem 2.1 it is sufficient to show that $\dim(\widehat{W}_r) \geq 2r-2$.

Let us put $r = k+2+\eta$ with $0 \leq \eta \leq h-1$. Utilizing again the incidence relation $(*)$ introduced above, we will show that $\dim(\widehat{W}_r) = \dim[p^{-1}(\widehat{W}_r)] \geq e+2\eta+1+r$ for any η with $0 \leq \eta \leq h-1$. By choosing $\eta = h-1$ we will have $\dim[p^{-1}(\widehat{W}_r)] \geq 2r-2$.

Let us fix $k-(e+\eta) = h-\eta \geq 1$ distinct fibres on S and let us consider all hyperplanes in $\mathbb{P}^N$ containing such fibres: $h^0(S, H-(h-\eta)f) = h^0(S, C_0+(e+\eta)f) = e+2\eta+2 \geq 2$. This means that on S there exist a family of dimension at least $e+2\eta+1$ of curves $\Gamma = C_0+(e+\eta)f$, possibly reducible, such that $\dim(\langle\Gamma\rangle) = N - h^0(S, H-\Gamma) = N - h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(h-\eta)) = N-(h-\eta+1) = r-2$. Note that in any $\langle\Gamma\rangle \simeq \mathbb{P}^{r-2}$ there are at most a finite number of curves as Γ, otherwise S would be contained in a projective space of dimension $r-2 < N$.

Now let us recall the incidence variety I_r : by the previous remark we have that the subvariety $V_\Gamma \subseteq \overline{W}_r \subseteq G$ parametrizing subspaces as $\langle\Gamma\rangle$ has dimension at least $e+2\eta+1$, hence $\dim(\overline{W}_r) \geq e+2\eta+1$, moreover the fibre of q over any point of V_Γ has dimension at least r, hence $\dim[p^{-1}(\widehat{W}_r)] \geq e+2\eta+1+r$. □

5. A simple application

To conclude the paper we give a simple application of Corollary 3.2. As mentioned in the introduction, this was the original situation that brought us to consider the problem addressed in this note.

Proposition 5.1. *Let $\{S_1, S_2\}$ be a pair of surfaces in $\mathbb{P}^N$ as in Section 4. Assume that the intersection $S_1 \cap S_2$ in $\mathbb{P}^N$ consists only of r common fibres $L_1, ..., L_r$ and that, at a generic point $P \in L_i$, the tangent planes to S_1 and S_2 at P are distinct.*

Then, for any generic choice of r points $P_1, ..., P_r$, $P_i \in L_i$, there is a hyperplane of $\mathbb{P}^N$ intersecting transversally $S_1 \cap S_2$ only at $P_1, ..., P_r$.

Proof. Apply Corollary 3.2 to $\mathcal{L} := \{L_1, ..., L_r\}$, keeping in mind that the assumptions of Theorem 2.1 are satisfied for any set of r fibres on surfaces as above. □

Remark 5.1. Note that the set up of Proposition 5.1 is achieved, for instance, when every S_j is $\mathbb{P}(\mathcal{E}_{|\Gamma_j})$, where $\mathcal{E}$ is a rank 2 vector bundle over a smooth variety Y, Γ_1 and Γ_2 are rational curves in Y whose intersection is transverse and consists of r distinct points, and $\mathbb{P}(\mathcal{E})$ is embedded in $\mathbb{P}^N$ as a scroll.

Following Remark 5.1, let $\mathcal{E}$ be a rank 2 vector bundle over a smooth surface Y which is rationally connected; let X be $\mathbb{P}(\mathcal{E})$, let T be its tautological divisor and let $\pi : X \to Y$ be the natural projection. In order to prove that the linear system $|T|$ separates two distinct points P and Q of X you can consider a rational smooth curve Γ (if it exists) passing through $\pi(P)$ and $\pi(Q)$, and the surface $S := \mathbb{P}(\mathcal{E}_{|\Gamma})$. If $|T|_{|S}$ is very ample and $|T| \to |T|_{|S}$ is surjective then $|T|$ separates P from Q. The difficult part of this strategy is often to prove the surjectivity (see for instance [1]). The usual exact sequence $0 \to \mathcal{E} \otimes \mathcal{O}_Y(-\Gamma) \to \mathcal{E} \to \mathcal{E}_{|\Gamma} \to 0$ gives the required surjectivity if $h^1(Y, \mathcal{E} \otimes \mathcal{O}_Y(-\Gamma)) = 0$. Unfortunately, this vanishing is not always easy to control. One may choose a set $\{\Gamma = \Gamma_1, ...\Gamma_q\}$ of $q >> 1$ suitable smooth rational curves in order to get $h^1(Y, \mathcal{E} \otimes \mathcal{O}_Y(-\Gamma_1 ... - \Gamma_q)) = 0$ and then use a reducible surface $S' := S_1 \cup ... \cup S_q$, instead of S, with $S_j := \mathbb{P}(\mathcal{E}_{|\Gamma j})$. With this approach one needs to consider elements of $|T|_{S'}$. Even when $|T|_{|S_j}$ is very ample for any j, and $\Gamma_i \cap \Gamma_j$ is a set of distinct points for any i, j, to get sections of $|T|_{S'}$ it is crucial to know what elements of $|T|$ cut $S_i \cap S_j$ only at distinct points. Proposition 5.1 gives the answer.

References

[1] A. Alzati - G.M. Besana: "Criteria for very ampleness of rank two vector bundles over ruled surfaces" Can J. Math. **62** (6) pp 1201-1227 (2010).

[2] R. Hartshorne: "Algebraic Geometry" Springer-Verlag, New York, 1977.

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