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MULTIPLY WARPED PRODUCT SUBMANIFOLDS OF A GENERALIZED SASAKIAN SPACE FORM

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Abstract. In this article, we derive some inequalities for second fundamental form and warping functions for multiply warped product submanifolds of a generalized Sasakian space form.

1. Introduction

It is known that the notion of CR-warped product manifold was first introduced by B. Y. Chen([3],[4]). In these papers he obtained the certain sharp inequalities involving warping functions and the squared norm of second fundamental form. After that B. Sahin [11] extended the Chen's result for warped product semi-slant submanifolds of Kaehler manifold. Later I. Mihai [7] improved the same type of inequality in case of contact CR-warped products in Sasakian space form. Moreover Marian-Ioan Munteanu [8] also studied warped product contact CR-submanifolds of Sasakian space form. Some non-existence results were also obtained by M. Atceken [2] for certain class of submanifolds of a Kenmotsu manifold. In the present article we will calculate the inequalities of same kind for different types of multiply warped product submanifolds of generalized Sasakian space form.

Let $N_1, N_2, ..., N_k$ be Riemannian manifolds and let $N = N_1 \times N_2 \times ... \times N_k$ be the cartesian product of $N_1, N_2, ..., N_k$. For each a, denote by $\pi_a : N \longrightarrow N_a$ the canonical projection of N onto N_a. We denote the horizontal lift of N_a in N via π_a by N_a itself.

If $\sigma_2, ..., \sigma_k : N_1 \longrightarrow R^+$ are positive valued functions, then

$$\langle X, Y \rangle = \langle \pi_* X, \pi_* Y \rangle + \sum_{a=1}^{k} (\sigma_a \circ \pi_1)^2 \langle \pi_{a*} X, \pi_{a*} Y \rangle$$

defines a Riemannian metric g on N, called a multiply warped product metric. The product manifold N endowed with this metric is denoted by $N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$.

For a multiply warped product manifold $N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$, let D_a denote the distributions obtained from the vectors tangent to N_a.

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Let $x : N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k \longrightarrow \bar{M}$ be an isometric immersion of of a multiply warped product $N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ into a Riemannian manifold $\bar{M}$. Denote by h the second fundamental form of x. Then the immersion x is called mixed totally geodesic if $h(D_a, D_b) = 0$ holds for distinct $a, b \in \{1, 2, ..., k\}$.

Denote by *trace* h_a the trace of h restricted to N_a, that is

$$trace\ h_a = \sum_{\alpha=1}^{n_a} h(e_\alpha, e_\alpha)$$

for some orthonormal frame fields $e_1, e_2, ..., e_{n_a}$ of D_a.

A submanifold M of an almost contact manifold $\bar{M}$ is called invariant if the almost contact structure ϕ of $\bar{M}$ carries each tangent space of M into itself whereas it is said to be anti-invariant if the almost contact structure ϕ of $\bar{M}$ carries each tangent space of M into its corresponding normal space.

A multiply warped product $N_T \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ in an almost contact metric manifold $\bar{M}$ is called a multiply CR-warped product if N_T is an invariant submanifold and $N_\perp =_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ is an anti-invariant submanifold of $\bar{M}$.

In [6] Chen and Dillen obtained the following inequality for warping functions:

Theorem 1.1. *Let* $x : N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k \longrightarrow \bar{M}^{\ m}$ *be an isometric immersion of a multiply warped product* $N^n = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ *into an arbitrary Riemannian manifold* $\bar{M}$. *Then we have*

$$\sum_{j=2}^{k} n_j \frac{\Delta \sigma_j}{\sigma_j} \leq \frac{n^2}{4} \|H\|^2 + n_1(n - n_1) \max \bar{K}, \qquad n = \sum_{j=1}^{k} n_j,$$

where $\max \bar{K}(p)$ *denotes the maximum of the sectional curvature function of* $\bar{M}^m$ *restricted to 2-planes sections of the tangent space* T_pN *of* N *at* $p = (p_1, ..., p_k)$.

The equality holds identically if and only if the following two statements hold:

(1) x *is a mixed totally geodesic immersion satisfying trace* $h_1 = ... =$ *trace* h_k

(2) at each point $p \in N$, *the sectional curvature function* $\bar{K}$ *of* $\bar{M}$ *satisfies* $\bar{K}(u, v) = \max \bar{K}(p)$ *for each unit vector* u *in* $T_{p_1}(N_1)$ *and each unit vector* v *in* $T_{(p_2,....,p_k)}(N_2 \times .... \times N_k)$.

They also derived inequality between the squared norm of the second fundamental form h and the gradient of warping functions.

Recently, in [9][10] A. Olteanu achieved the similar results in case of multiply CR-warped product submanifolds of a Kenmotsu space form and doubly warped product submanifolds of generalized Sasakian space form. In the present article we obtain the similar inequalities for multiply CR-warped product submanifolds of a generalized Sasakian space form.

2. Preliminaries

A $(2m+1)$-dimensional Riemannian manifold $\bar{M}$ is said to be an almost contact metric manifold if it admits an endomorphism ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying the following properties:

$$\phi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi),$$

$$g(X, \phi Y) = -g(\phi X, Y).$$

for any vector fields X, Y on $\bar{M}$.

A plane section π in $T_p\bar{M}$ is called a ϕ-section if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a ϕ-section is called a ϕ-sectional curvature. A Sasakian (resp. Kenmotsu and cosymplectic) manifold with constant ϕ-sectional curvature is a Sasakian (resp. Kenmotsu and cosymplectic) space form. In 2004 [1] Alegre, Blair and Carriazo introduced a space form called generalized Sasakian space form which generalized Sasakian, Kenmotsu and cosymplectic space forms.

An almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$ is said to be generalized Sasakian space form if there exist three functions f_1, f_2 and f_3 on $\bar{M}$ such that the curvature tensor $\bar{R}$ is given by [1]

$$\begin{aligned} \bar{R}(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{2.1}$$

Let $\bar{\nabla}$ denote the connection on generalized Sasakian space form $\bar{M}$. If M is a submanifold of a generalized Sasakian space form $\bar{M}$, we denote the induced metric on M by the same symbol g whereas the induced connection on M by ∇. The Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

for each $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ denotes the induced connection on the normal bundle $T^{\perp}M$, h and A_N are the second fundamental form and the shape operator of the immersion of M into $\bar{M}$. The relation between h and A_N is given as

$$g(h(X,Y),N) = g(A_N X, Y).$$

The covariant derivative of the second fundamental form h is defined as

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^{\perp} h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for any $X, Y, Z \in TM$.If $\bar{R}$ and R are the curvature tensors of the connections $\bar{\nabla}$ and ∇ on $\bar{M}$ and M respectively, the Gauss and Codazzi equation are mentioned as

$$\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,Z), g(Y,W)) + g(h(X,W), h(Y,Z)) \tag{2.2}$$

$$(\bar{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \tag{2.3}$$

respectively.

Let $N = N_T \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ be a multiply CR-warped product submanifold of a generalized Sasakian space form. Let D be the invariant distribution (i.e. $TN_T = D$) such that its orthogonal complementary distribution $D^{\perp}$ is anti-invariant (i.e. $TN_{\perp} = T(N_2 \times ... \times N_k) = D^{\perp}$). We take $\{\xi\}$ to be tangent to TN.Then we have the following decompositions

$$TN = D \oplus D^{\perp} \oplus \{\xi\}, \tag{2.4}$$

$$T^{\perp}N = \phi D^{\perp} \oplus \lambda, \tag{2.5}$$

where λ denotes the orthogonal complementary distribution of $\phi D^{\perp}$ and is an invariant normal subbundle of $T^{\perp}N$.

For any vector field $X \in TN$ we put

$$(\bar{\nabla}_X \phi)Y = P_X Y + Q_X Y, \tag{2.6}$$

where PX (resp. QX) denotes the tangential (resp. normal) component of ϕX. Also let $p \in N$ and $\{e_1, .., e_n, e_{n+1}, ..., e_{2m+1}\}$ an orthonormal basis of the tangent space $T_p\bar{M}$, such that $e_1, ...e_n$ are tangents to N at p and . We denote by H the mean curvature vector, that is

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

N is minimal if H vanishes identically. We also set

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

Let $K(e_i \wedge e_j)$, $1 \leq i, j \leq n$, denote the sectional curvature of the plane section spanned by e_i and e_j. Then the scalar curvature of N is given by

$$\tau = \sum_{i<j} K(e_i \wedge e_j). \tag{2.7}$$

For a differentiable function f on N, the Laplacian Δf of f is defined as

$$\Delta f = \sum_{j=1}^{n}\{e_j(e_j f) - \nabla_{e_j} e_j f\}.$$

A submanifold N tangent to ξ is said to be invariant (resp. anti-invariant) submanifold if $\phi(T_pN) \subset T_pN$, $\forall p \in N$ (resp. $\phi(T_pN) \subset T_p^{\perp}N$, $\forall p \in N$).

Lemma 2.1. [5] *Let $a_1, ..., a_n$ be n real numbers and let k be an integer in $[2, n-1]$. Then for any partition $(n_1, ..., n_k)$ of n, we have*

$$\sum_{1\leq i_1<j_1\leq n_1} a_{i_1}a_{j_1} + \sum_{n_1+1\leq i_2<j_2\leq n_1+n_2} a_{i_2}a_{j_2} + ... + \sum_{n_1+....+n_{k-1}+1\leq i_k<j_k\leq n} a_{i_k}a_{j_k}$$
$$\geq \frac{1}{2k}\left[(a_1 + ... + a_n)^2 - k\left(a_1^2 + ... + a_n^2\right)\right],$$

with the equality holding if and only if $a_1 + ... + a_{n_1} = ... = a_{n_1+....+n_{k-1}+1} + ... + a_n$.

3. Multiply CR-warped product submanifolds of generalized Sasakian space form

In 2008, Chen and Dillen [6] studied an interesting inequality between the warping functions $\sigma_2, ..., \sigma_k$ and the squared norm of the second fundamental form $\|h\|^2$ of a multiply CR-warped product submanifold $N = N_T \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ in a general Kaehler manifold. Later in 2010, A Olteanu [10] established the same inequality in case of a multiply CR-warped product submanifolds in a Kenmotsu space form. In the present article we achieve the inequality for multiply CR-warped product submanifolds in a generalized Sasakian space form.

From the decomposition (2.5) we may write

$$h(X,Y) = h_{\phi D^\perp}(X,Y) + h_\lambda(X,Y).$$

Also it is known that for multiply CR-warped submanifold of a Riemannian manifold we have

$$\nabla_X Z = \sum_{a=2}^{k}(X(\log\sigma_a))Z^a, \tag{3.1}$$

for any vector fields X in D and Z in $D^\perp$, where Z^a denotes the N_a-component of Z.

For proving the main inequality we need the following lemmas.

Lemma 3.1. *Let $N = N_T \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ be a multiply CR-warped product submanifold of a generalized Sasakian space form $\bar{M}$. Then we have*

(i) $h_{\phi D^\perp}(\phi X, Z) = \sum_{a=2}^{k}(X(\log\sigma_a))\phi Z^a + \phi P_Z \phi X$,

(ii) $g(Q_Z X, \phi W) = g(P_Z \phi X, W)$,

(iii) $g(h(\phi X, Z), \phi h(X,Z)) = \|h_\lambda(X,Z)\|^2 - g(Q_X Z, \phi h_\lambda(X,Z))$

for any vector fields X in D and Z, W in $D^\perp$, where Z^a denotes the N_a-component of Z.

Proof. From Gauss formula we can write

$$\nabla_Z \phi X + h(\phi X, Z) = P_Z X + Q_Z X + \phi\nabla_Z X + \phi h(Z,X)$$

or

$$h(\phi X, Z) = P_Z X + Q_Z X + \phi(\sum_{a=2}^{k}(X(\log\sigma_a))Z^a)$$

$$+\phi h(Z,X) - \sum_{a=2}^{k}(\phi X(\log\sigma_a))Z^a. \tag{3.2}$$

Comparing tangential parts in the above equation and then taking inner product with $W \in D^\perp$, we get

$$h_{\phi D^\perp}(Z,X) = -\sum_{a=2}^{k}(\phi X(\log\sigma_a))\phi Z^a + \phi P_Z X,$$

which may be written as

$$h_{\phi D^\perp}(\phi X, Z) = \sum_{a=2}^{k}(X(\log\sigma_a))\phi Z^a + \phi P_Z \phi X, \quad \forall\ X \in D, Z \in D^\perp.$$

This is part (i) of the lemma.

Now comparing normal parts of equation (3.2) we get

$$h(\phi X, Z) - \phi h_\lambda(Z,X) = Q_Z X + \sum_{a=2}^{k}(X(\log\sigma_a)\phi Z^a) \tag{3.3}$$

By taking inner product with ϕW of the above equation we obtain

$$g(h_{\phi D^\perp}(\phi X,Z),\phi W)=g(Q_ZX,\phi W)+\sum_{a=2}^{k}(X(\log\sigma_a)g(\phi Z^a,\phi W)).$$

Simplifying the above equation by using part (i) of the lemma we arrive at

$$g(P_Z\phi X,W)=g(Q_ZX,\phi W),$$

which proves part (ii).

Taking inner product of equation (3.3) by $\phi h(X,Z)$ gives part (iii) of the lemma. □

Theorem 3.1. *Let $N=N_T\times_{\sigma_2}N_2\times...\times_{\sigma_k}N_k$ be multiply CR-warped product submanifold of a generalized Sasakian space form $\bar{M}$ with $P_{D^\perp}D\in D$, then the squared norm of the second fundamental form satisfies*

$$\|h\|^2\geq\sum_{a=2}^{k}n_a^2\,\|(\nabla\log\sigma_a)\|^2+\|P_{D^\perp}D\|^2.$$

Proof. Let $\{X_\circ=\xi,X_1,X_2,..,X_p,X_{p+1}=\phi X_1,...,X_{2p}=\phi X_p\}$ be a local orthonormal frame of vector fields on N_T and $\{Z_1,Z_2,...,Z_q\}$ be such that Z_{Δ_a} is a basis for some N_a , $a=2,...,k$ where $\Delta_2=\{1,2,..,n_2\}$, ..., $\Delta_k=\{n_2+n_3+...+n_{k-1}+1,...,n_1+n_2+...+n_k\}$ and $n_2+n_3+....+n_k=q$. Then we have

$$\begin{aligned}\|h\|^2 &= \sum_{i,j=1}^{2p}g(h(X_i,X_j),h(X_i,X_j))+\sum_{i=1}^{2p}g(h(X_i,\xi),h(X_i,\xi))\\ &+\sum_{i=1}^{2p}\sum_{a=2}^{k}g(h(X_i,Z_{\Delta_a}),h(X_i,Z_{\Delta_a}))+\sum_{a,b=2}^{k}g(h(Z_{\Delta_a},Z_{\Delta_b}),h(Z_{\Delta_a},Z_{\Delta_b}))\\ &+\sum_{a=2}^{k}g(h(Z_{\Delta_a},\xi),h(Z_{\Delta_a},\xi)).\end{aligned}$$

The above equation implies

$$\|h\|^2\geq\sum_{i=1}^{2p}\sum_{a=2}^{k}g(h(X_i,Z_{\Delta_a}),h(X_i,Z_{\Delta_a})).$$

Now using part (i) of lemma-3.1 we get

$$\|h\|^2\geq\sum_{i=1}^{2p}\sum_{a=2}^{k}g(n_a(\phi X_i(\log\sigma_a))\phi Z_{\Delta_a}+\phi P_{Z_{\Delta_a}}X_i,n_a(\phi X_i(\log\sigma_a))\phi Z_{\Delta_a}+\phi P_{Z_{\Delta_a}}X_i).$$

In view of the assumption $P_{D^\perp}D\in D$, the above inequality takes the form

$$\|h\|^2\geq\sum_{a=2}^{k}n_a^2\,\|\nabla\log\sigma_a\|^2\,\|Z_{\Delta_a}\|^2+\|P_{D^\perp}D\|^2.$$

By Cauchy-Schwartz inequality the above equation becomes

$$\sum_{a=2}^{k}n_a^2\,\|\nabla\log\sigma_a\|^2\,\|Z_{\Delta_a}\|^2+\|P_{D^\perp}D\|^2\geq\sum_{a=2}^{k}n_a^2\,\|(\nabla\log\sigma_a)Z_{\Delta_a}\|^2+\|P_{D^\perp}D\|^2$$

Therefore

$$\|h\|^2 \geq \sum_{a=2}^{k} n_a^2 \left\|(\nabla \log \sigma_a)\right\|^2 + \left\|P_{D^\perp} D\right\|^2 .$$

Hence we finish the proof. □

We now prove the main inequality of this section

Theorem 3.2. *Let $N = N_T \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ be a multiply CR-warped product submanifold of a generalized Sasakian space form $\bar{M}$ with $P_{D^\perp} D \in D$, then*

$$\|h\|^2 \geq \sum_{a=2}^{k} \left[n_a \left\{ (\Delta \log \sigma_a) + n_a \left\|\nabla \log \sigma_a\right\|^2 \right\} g(Z^a, Z^a) \right] + 2 f_2 pq.$$

Proof. Let $X \in D$ and $Z \in D^\perp$, then from equation (2.1) we have

$$\bar{R}(X, \phi X, Z, \phi Z) = -2 f_2 g(X, X) g(Z, Z). \tag{3.4}$$

On the other hand from equation (2.3)

$$\bar{R}(X, \phi X, Z, \phi Z) = g(\nabla^\perp_X h(\phi X, Z), \phi Z) - g(h(\nabla_X \phi X, Z), \phi Z)$$

(3.5)
$$-g(h(\phi X, \nabla_X Z), \phi Z) - g(\nabla^\perp_{\phi X} h(X, Z), \phi Z) + g(h(\nabla_{\phi X} X, Z), \phi Z) + g(h(X, \nabla_{\phi X} Z), \phi Z).$$

We will calculate each term of the above expression to get the required inequality. Now

$$g(\nabla^\perp_X h(\phi X, Z), \phi Z) = X g(h(\phi X, Z), \phi Z) - g(h(\phi X, Z), \bar{\nabla}_X \phi Z). \tag{3.6}$$

Using equation (3.3) we get the value of first term of the above equation as

$$X g(h(\phi X, Z), \phi Z) = X \{ \sum_{a=2}^{k} (X (\log \sigma_a) g(Z^a, Z^a)) \}$$

or equivalently

$$X g(h(\phi X, Z), \phi Z) = \sum_{a=2}^{k} \left[\left\{ X (X \log \sigma_a) + 2 (X \log \sigma_a)^2 \right\} g(Z^a, Z^a) \right] . \tag{3.7}$$

Also the second term in equation (3.6) is obtained as

$$g(h(\phi X, Z), \bar{\nabla}_X \phi Z) = g(h(\phi X, Z), (\bar{\nabla}_X \phi) Z) + g(h(\phi X, Z), \phi(\bar{\nabla}_X Z)).$$

Using equation (3.1) and part-(iii) of lemma-3.1, it follows that

$$\begin{aligned} g(h(\phi X, Z), \bar{\nabla}_X \phi Z) &= g(h(\phi X, Z) - \phi h(X, Z), Q_X Z) \\ &\quad + \sum_{a=2}^{k} ((X \log \sigma_a) g(h(\phi X, Z), \phi Z^a)) + \left\|h_\lambda(X, Z)\right\|^2 . \end{aligned}$$

Now making use of equation (3.3) and part-(ii) of lemma-3.1 in the above equation we get finally

$$g(h(\phi X, Z), \bar{\nabla}_X \phi Z) = \sum_{a=2}^{k} ((X \log \sigma_a)^2 g(Z^a, Z^a)) - \left\|Q_X Z\right\|^2 + \left\|h_\lambda(X, Z)\right\|^2 . \tag{3.8}$$

Putting equations (3.7) and (3.8) in equation (3.6) we have

$$g(\nabla^{\perp}_{X}h(\phi X,Z),\phi Z)=\sum_{a=2}^{k}\left[\left\{X(X\log\sigma_a)+(X\log\sigma_a)^2\right\}g(Z^a,Z^a)\right]$$

$$+\left\|Q_X Z\right\|^2-\left\|h_\lambda(X,Z)\right\|^2. \tag{3.9}$$

Similarly we obtain

$$g(\nabla^{\perp}_{\phi X}h(\phi X,Z),\phi Z)=\sum_{a=2}^{k}\left[\left\{\phi X(\phi X\log\sigma_a)+(\phi X\log\sigma_a)^2\right\}g(Z^a,Z^a)\right]$$

$$+\left\|Q_{\phi X} Z\right\|^2-\left\|h_\lambda(X,Z)\right\|^2. \tag{3.10}$$

Again using part-(i) of lemma-3.1 we evaluate

$$g(h(\phi X,\nabla_X Z),\phi Z)=\sum_{a=2}^{k}\left\{(X\log\sigma_a)^2 g(Z^a,Z^a)\right\}. \tag{3.11}$$

Similarly we have

$$g(h(X,\nabla_X Z),\phi Z)=-\sum_{a=2}^{k}\left\{(\phi X\log\sigma_a)^2 g(Z^a,Z^a)\right\}. \tag{3.12}$$

Using again part-(i) of lemma-3.1 we have

$$g(h(\nabla_{\phi X}X,Z),\phi Z)=-\sum_{a=2}^{k}\left\{(\phi\nabla_{\phi X}X\log\sigma_a)^2 g(Z^a,Z^a)\right\}. \tag{3.13}$$

Similarly we have

$$g(h(\nabla_X\phi X,Z),\phi Z)=-\sum_{a=2}^{k}\left\{(\phi\nabla_X\phi X\log\sigma_a)^2 g(Z^a,Z^a)\right\}.$$

The last equation may further be simplified by using the fact that N_T is totally geodesic in N, equation (3.1) and Gauss equation as

$$g(h(\nabla_X\phi X,Z),\phi Z)=\sum_{a=2}^{k}\{(\nabla_X X\log\sigma_a)g(Z^a,Z^a)\}$$

$$+\sum_{a=2}^{k}\{(\nabla_{\phi X}\phi X\log\sigma_a)g(Z^a,Z^a)\}-\sum_{a=2}^{k}\{(\phi\nabla_{\phi X}X\log\sigma_a)g(Z^a,Z^a)\}. \tag{3.14}$$

Substituting equations (3.9)-(3.14) into equation (3.5) it becomes

$$\bar{R}(X,\phi X,Z,\phi Z)=\sum_{a=2}^{k}\{(X(X\log\sigma_a)+\phi X(\phi X\log\sigma_a)-\nabla_X X\log\sigma_a-\nabla_{\phi X}\phi X\log\sigma_a)g(Z^a,Z^a)\}$$

$$+\left\|Q_X Z\right\|^2+\left\|Q_{\phi X}Z\right\|^2-\left\|h_\lambda(X,Z)\right\|^2-\left\|h_\lambda(\phi X,Z)\right\|^2.$$

Putting the bases of $N_T, N_2, ...N_k$ and summing over them we find

$$\sum_{i=1}^{p}\sum_{\Delta_a}\bar{R}(X_i,\phi X_i,Z_{\Delta_a},\phi Z_{\Delta_a})=\sum_{a=2}^{k}n_a(\Delta\log\sigma_a)g(Z^a,Z^a)$$

$$+\left\|Q_D D^{\perp}\right\|^2-\left\|h_\lambda(D,D^{\perp})\right\|^2. \tag{3.15}$$

On the other hand equation (3.4) gives

$$\sum_{i=1}^{p}\sum_{\Delta_a}\bar{R}(X_i,\phi X_i,Z_{\Delta_a},\phi Z_{\Delta_a})=-2f_2pq. \tag{3.16}$$

Combining equations (3.15) and (3.16) we get

$$\left\|h_\lambda(D,D^{\perp})\right\|^2=\sum_{a=2}^{k}n_a(\Delta\log\sigma_a)g(Z^a,Z^a)+\left\|Q_D D^{\perp}\right\|^2+2f_2pq. \tag{3.17}$$

It is easy to see from part-(i) of lemma-3.1 that

$$\left\|h_{\phi D^{\perp}}(D,D^{\perp})\right\|^2=\sum_{a=2}^{k}n_a^2\left\|\nabla\log\sigma_a\right\|^2g(Z^a,Z^a)+\left\|P_{D^{\perp}}D\right\|^2. \tag{3.18}$$

Therefore equations (3.17) and (3.18) give

$$\begin{aligned}\|h\|^2 &= \sum_{a=2}^{k}n_a(\Delta\log\sigma_a)g(Z^a,Z^a)+\left\|Q_D D^{\perp}\right\|^2+2f_2pq\\ &\quad+\sum_{a=2}^{k}n_a^2\left\|\nabla\log\sigma_a\right\|^2g(Z^a,Z^a)+\left\|P_{D^{\perp}}D\right\|^2,\end{aligned}$$

which proves that

$$\|h\|^2\geq\sum_{a=2}^{k}\left[n_a\left\{(\Delta\log\sigma_a)+n_a\left\|\nabla\log\sigma_a\right\|^2\right\}g(Z^a,Z^a)\right]+2f_2pq.$$

□

4. Anti-invariant multiply warped product submanifolds of a generalized Sasakian space form

In 2008 Chen and Dillen [6] obtained a sharp inequality for warping functions $\sigma_2,....,\sigma_k$ of a multiply warped product submanifold $N=N_1\times_{\sigma_2}N_2\times...\times_{\sigma_k}N_k$ isometrically immersed in an arbitrary Riemannian manifold and the squared mean curvature $\|H\|^2$. Later in 2010, A. Olteanu [10] established the similar inequality in case of anti-invariant submanifold of a Kenmotsu space form. In this paper we achieve the same type of inequality for anti-invariant submanifold of a Generalized Sasakian space form.

We know from [10] that the following relation holds for warping functions

$$\Delta\sigma_a=\sigma_a\sum_{j=1}^{n_1}K(e_j\wedge X_a) \tag{4.1}$$

for any unit vector tangent X_a tangent to N_a, where $\{e_1,e_2,...,e_{n_1}\}$ is an orthonormal basis of $T_{\pi_1(p)}N_1$.

Case I- When ξ is tangent to N.

Theorem 4.1. *Let ψ be an anti-invariant isometric immersion of an n-dimensional multiply warped $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ into a $(2m+1)$-dimensional generalized Sasakian space form $\bar{M}$ with ξ tangent to N. Then*

$$\sum_{a=2}^{k} n_a \frac{\Delta\sigma_a}{\sigma_a} \leq n_1(n-n_1)f_1 - f_3 + \frac{n^2}{2}\left(1-\frac{1}{k}\right)\|H\|^2 .$$

Proof. Let $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ be an anti-invariant multiply warped product submanifold in a generalized Sasakian space form $\bar{M}$.Then from equation (2.1) and (2.2) the sectional curvature $K_N(X \wedge Y)$ determined by orthonormal vectors X and Y is given by

$$K_N(X \wedge Y) = f_1 - f_3\{\eta(X)^2 + \eta(Y)^2\} + g(h(X,X), h(Y,Y)) - \|h(X,Y)\|^2 . \tag{4.2}$$

Since ξ is tangent to N, using equation (2.7) the above equation becomes

$$\sum_{i \neq j} K_N(e_i \wedge e_j) = 2\tau = n(n-1)f_1 - 2f_3 + n^2\|H\|^2 - \|h\|^2 .$$

We put [6]

$$\eta = 2\tau - n^2\|H\|^2 - n(n-1)f_1 + 2f_3 + \frac{n^2}{k}\|H\|^2 \tag{4.3}$$

and hence we have

$$n^2\|H\|^2 = k(\eta + \|h\|^2).$$

We choose an orthonormal basis $e_1, e_2, ..., e_{2m+1}$ at p such that, for each $j \in \Delta_a$, e_j is tangent to N_a for $a = 1, ..., k$. Moreover we choose the normal vector field e_{n+1} in the direction of the mean curvature vector at p. Therefore the above equation yields

$$\frac{1}{2k}\left[(\sum_{A=1}^{n} a_A)^2 - k\sum_{A=1}^{n} a_A^2\right] = \frac{1}{2}\left[\eta + \sum_{A \neq B}(h_{AB}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{A,B=1}^{n}(h_{AB}^{r})^2\right].$$

Now applying basic lemma-2.1 in the above equation we get

$$\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} + \sum_{\alpha_2<\beta_2} a_{\alpha_2}a_{\beta_2} + ... + \sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k} \geq$$

$$\frac{\eta}{2} + \sum_{A<B}(h_{AB}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{A,B=1}^{n}(h_{AB}^{r})^2.$$

Now using equation (4.1) we may write

$$\sum_{a=2}^{k} n_a \frac{\Delta\sigma_a}{\sigma_a} = \sum_{j\in\Delta_1}\sum_{\beta\in\Delta_2\cup...\cup\Delta_k} K(e_j \wedge e_\beta) + \sum_{1\leq j_1<j_2\leq n_1} K(e_{j_1} \wedge e_{j_2})$$

$$+ \sum_{n_1+1\leq\alpha<\beta\leq n} K(e_\alpha \wedge e_\beta) - \sum_{1\leq j_1<j_2\leq n_1} K(e_{j_1} \wedge e_{j_2}) - \sum_{n_1+1\leq\alpha<\beta\leq n} K(e_\alpha \wedge e_\beta),$$

from which we have

$$\sum_{a=2}^{k} n_a \frac{\Delta\sigma_a}{\sigma_a} \leq \tau - \frac{n(n-1)}{2}f_1 + n_1(n-n_1)f_1 - \frac{\eta}{2}.$$

By putting the value of η from equation (4.2) in the above equation we obtain the required inequality. □

We note easily the following corollaries from the theorem as obtained in [9]-

Corollary 4.1. *Let $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ be a multiply warped product into a generalized Sasakian space form $\bar{M}$ such that $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ is an anti-invariant submanifold tangent to ξ. If the warping functions σ_a, $a = 2, ..., k$ are harmonic, then $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ admits no minimal immersion into a generalized Sasakian space form $\bar{M}$ with $f_1 < 0$ and $f_3 > 0$.*

Corollary 4.2. *If the warping functions σ_a, $a = 2, ..., k$ of an anti-invariant multiply warped product submanifold $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ tangent to ξ in a generalized Sasakian space form $\bar{M}$ are eigen functions of the Laplacian with corresponding eigenvalues $\lambda_a > 0$, $a = 2, ..., k$ respectively, then $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ admits no minimal immersion in a generalized Sasakian space form $\bar{M}$ with $f_1 \leq 0$ and $f_3 \geq 0$.*

Corollary 4.3. *Let $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ be an anti-invariant multiply warped product submanifold tangent to ξ in a generalized Sasakian space form $\bar{M}$. If σ_a, $a = 2, .., k_1$ are harmonic and σ_b, $b = 1, ..., k_2$ are eigen functions of the Laplacian with corresponding eigenvalues λ_b, $b = 1, .., k_2$ respectively, where $k_1 + k_2 = k$, then $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ admits no minimal immersion into a generalized Sasakian space form $\bar{M}$ with $f_1 \leq 0$ and $f_3 \geq 0$.*

Case-II When ξ is normal to N.

Theorem 4.2. *Let ψ be an anti-invariant isometric immersion of an n-dimensional multiply warped $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ into a $(2m+1)$-dimensional generalized Sasakian space form $\bar{M}$ with ξ normal to N. Then*

$$\sum_{a=2}^{k} n_a \frac{\Delta \sigma_a}{\sigma_a} \leq n_1(n - n_1) f_1 + \frac{n^2}{2} \left(1 - \frac{1}{k}\right) \|H\|^2 .$$

Proof. Since ξ is normal to N we have from equation (4.2)

$$\sum_{i,j} K_N(e_i \wedge e_j) = 2\tau = n(n-1) f_1 + n^2 \|H\|^2 - \|h\|^2 .$$

Then proceed as in case-I, we obtain the final inequality as

$$\sum_{a=2}^{k} n_a \frac{\Delta \sigma_a}{\sigma_a} \leq n_1(n - n_1) f_1 + \frac{n^2}{2} \left(1 - \frac{1}{k}\right) \|H\|^2 .$$

Hence the theorem is proved. □

We extract the following straight corollaries from this theorem-

Corollary 4.4. *Let $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ be a multiply warped product into a generalized Sasakian space form $\bar{M}$ such that $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ is an anti-invariant submanifold normal to ξ. If the warping functions σ_a, $a = 2, ..., k$ are harmonic, then $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ admits no minimal immersion into a generalized Sasakian space form $\bar{M}$ with $f_1 < 0$.*

Corollary 4.5. *If the warping functions σ_a, $a = 2, ..., k$ of an anti-invariant multiply warped product submanifold $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ normal to ξ in a generalized Sasakian space form $\bar{M}$ are eigen functions of the Laplacian with corresponding eigenvalues $\lambda_a > 0$, $a = 2, ..., k$ respectively, then $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ admits no minimal immersion in a generalized Sasakian space form $\bar{M}$ with $f_1 \leq 0$.*

Corollary 4.6. *Let $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ be an anti-invariant multiply warped product submanifold normal to ξ in a generalized Sasakian space form $\bar{M}$. If σ_a, $a = 2, .., k_1$ are harmonic and σ_b, $b = 1, ..., k_2$ are eigen functions of the Laplacian with corresponding eigenvalues λ_b, $b = 1, .., k_2$ respectively, where $k_1 + k_2 = k$, then $N = N_1 \times_{\sigma_2} N_2 \times ... \times_{\sigma_k} N_k$ admits no minimal immersion into a generalized Sasakian space form $\bar{M}$ with $f_1 \leq 0$.*

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