

SOME RELATIONS BETWEEN NORMAL AND RECTIFYING CURVES IN MINKOWSKI SPACE-TIME

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Dedicated to memory of Proffessor Franki Dillen

ABSTRACT. In this paper, we firstly give the necessary and sufficient conditions for null, pseudo null and partially null curves in Minkowski space-time to be normal curves. We prove that the null, pseudo null and partially null normal curves have a common property that their orthogonal projection onto non-degenerate hyperplane of E_1^4 or onto lightlike 2-plane of E_1^4 is the corresponding rectifying curve. Finally, we give some examples of such curves in E_1^4 .

1. INTRODUCTION.

In the Euclidean space \mathbb{E}^3 there exist three classes of curves, so-called *rectifying, normal and osculating curves* satisfying Cesaro's fixed point condition ([8]) meaning that rectifying, normal and osculating planes of such curves always contain a particular point. If all normal or osculating planes of a curve in \mathbb{E}^3 pass through a particular point, then the curve is spherical or planar, respectively. It is also known that if all rectifying planes of a non-planar curve in \mathbb{E}^3 pass through a particular point, then the ratio of its torsion and curvature is a non-constant linear function ([3]). Some characterizations of rectifying curves in Minkowski 3-space E_1^3 are given in [6]. In particular, there exists a simple relationship between rectifying curves and Darboux vectors (centrodes), which play some important roles in mechanics, kinematics as well as in differential geometry in defining the curves of constant precession ([4]).

Normal curves in Minkowski space-time E_1^4 are defined in [5] as the space curves whose position vector (with respect to some chosen origin) always lies in its normal space T^\perp , which represents the orthogonal complement of the tangent vector field of the curve. Spacelike and timelike normal curves in E_1^4 , whose Frenet frame contains only non-null vector fields, are characterized in [5]. Such curves always lie in some hyperquadric in E_1^4 .

In this paper, we firstly give the necessary and sufficient conditions for null, pseudo null and partially null curves in Minkowski space-time to be normal curves.

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It is a quite interesting problem to investigate if there exist some geometrical relations between null, pseudo null and partially null normal curves and rectifying curves in Minkowski space-time. In this paper we prove that the null, pseudo null and partially null normal curves have a common property that their orthogonal projection onto non-degenerate hyperplane of E_1^4 or onto lightlike 2-plane of E_1^4 is the corresponding rectifying curve. Finally, we give some examples of such null, pseudo null and partially null normal curves in E_1^4 .

2. PRELIMINARIES.

The Minkowski space-time E_1^4 is the Euclidean 4-space E^4 equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of E_1^4 . Recall that a vector $v \in E_1^4 \setminus \{0\}$ can be *spacelike* if $g(v, v) > 0$, *timelike* if $g(v, v) < 0$ and *null (lightlike)* if $g(v, v) = 0$. In particular, the vector $v = 0$ is a spacelike. The norm of a vector v is given by $\|v\| = \sqrt{|g(v, v)|}$. Two vectors v and w are said to be orthogonal, if $g(v, w) = 0$. An arbitrary curve $\alpha(s)$ in E_1^4 , can locally be *spacelike*, *timelike* or *null (lightlike)*, if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null ([7]). Spacelike curve in E_1^3 or E_1^4 is called respectively *pseudo null* or *partially null curve*, if its principal normal or first binormal vector is null ([2]).

A null curve α is parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$ ([1]). A non-null curve α has unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

Recall that the pseudosphere, pseudohyperbolic space and nullcone are the hyperquadrics in E_1^4 respectively defined by ([7])

$$\begin{aligned} S_1^3(r) &= \{X \in E_1^4 : g(X, X) = r^2, r \in R_0^+\}, \\ H_0^3(r) &= \{X \in E_1^4 : g(X, X) = -r^2, r \in R_0^+\}, \\ C^3(O) &= \{X \in E_1^4 : g(X, X) = 0, X \neq 0\}. \end{aligned}$$

Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along a curve α in E_1^4 , consisting of the tangent, the principal normal, the first binormal and the second binormal vector field respectively. Depending on the causal character of α , the Frenet equations have the following forms.

Case I. If α is a null curve, the Frenet equations are given by ([1,10])

$$(2.1) \quad \begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ \kappa_2 & 0 & -\kappa_1 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ -\kappa_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where the first curvature $\kappa_1(s) = 0$, if $\alpha(s)$ is straight line, or $\kappa_1(s) = 1$ in all other cases. Therefore, such curve has two curvatures $\kappa_2(s)$ and $\kappa_3(s)$ and the following equations hold:

$$g(T, T) = g(B_1, B_1) = 0, \quad g(N, N) = g(B_2, B_2) = 1,$$

$$g(T, N) = g(T, B_2) = g(N, B_1) = g(N, B_2) = g(B_1, B_2) = 0, \quad g(T, B_1) = 1.$$

Case II. If α is pseudo null curve, the Frenet formulas are ([2,10])

$$(2.2) \quad \begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ 0 & 0 & \kappa_2 & 0 \\ 0 & \kappa_3 & 0 & -\kappa_2 \\ -\kappa_1 & 0 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where the first curvature $\kappa_1(s) = 0$, if α is straight line, or $\kappa_1(s) = 1$ in all other cases. Such curve has two curvatures $\kappa_2(s)$ and $\kappa_3(s)$ and the following conditions are satisfied:

$$g(T, T) = g(B_1, B_1) = 1, \quad g(N, N) = g(B_2, B_2) = 0,$$

$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(B_1, B_2) = 0, \quad g(N, B_2) = 1.$$

Case III. If α is partially null curve, the Frenet formulas read ([2,10])

$$(2.3) \quad \begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 & 0 \\ 0 & -\kappa_2 & 0 & -\kappa_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where the third curvature $\kappa_3(s) = 0$ for each s . Such curve has two curvatures $\kappa_1(s)$ and $\kappa_2(s)$ and lies fully in a lightlike hyperplane of E_1^4 . In particular, the following equations hold:

$$g(T, T) = g(N, N) = 1, \quad g(B_1, B_1) = g(B_2, B_2) = 0,$$

$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(N, B_2) = 0, \quad g(B_1, B_2) = 1.$$

Recall that a rectifying curve in E_1^3 is defined in [6] as a space curve whose the position vector (with respect to some chosen origin) always lies in its rectifying plane

$$N^\perp = \{W \in E_1^3 \mid g(W, N) = 0\}.$$

In particular, the normal curve in E_1^4 is defined in [5] as a curve whose the position vector (with respect to some chosen origin) always lies in its normal space given by

$$T^\perp = \{W \in E_1^4 \mid g(W, T) = 0\}.$$

Consequently, the position vector of null, pseudo null and partially null normal curve in E_1^4 , satisfies respectively the equations

$$(2.4) \quad \alpha(s) = a(s)T(s) + b(s)N(s) + c(s)B_2(s),$$

$$(2.5) \quad \alpha(s) = a(s)N(s) + b(s)B_1(s) + c(s)B_2(s),$$

$$(2.6) \quad \alpha(s) = a(s)N(s) + b(s)B_1(s),$$

for some differentiable functions $a(s)$, $b(s)$ and $c(s)$.

3. NULL, PSEUDO NULL AND PARTIALLY NULL NORMAL CURVES IN E_1^4

In this section, we firstly give the necessary and sufficient conditions for null, pseudo null and partially null curves in E_1^4 to be normal curves. One of the sufficient conditions is that such curves (with the first and the second curvature different from zero) always lie in some hyperquadric of the ambient space. It can be easily verified that every null straight line in E_1^4 is a normal curve. For the null normal curves in E_1^4 with non-zero first curvature we have the following theorem.

Theorem 3.1. *Let $\alpha(s)$ be a null curve in E_1^4 parameterized by pseudo-arc s and with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$, $k_3(s) \neq 0$. Then α is a normal curve if and only if one of the following statements hold:*

- (i) α lies in pseudosphere $S_1^3(r)$, $r \in R_0^+$;
- (ii) the third curvature $\kappa_3(s)$ is non-zero constant;
- (iii) the second binormal component of the position vector α is non-zero constant, i.e. $g(\alpha, B_2) = c_0$, $c_0 \in R_0$.

Proof. First assume that α is a normal curve. By taking the derivative of the relation (2.4) with respect to s and applying (2.1), we obtain the system of equations

$$a'(s) + b(s)\kappa_2(s) - c(s)\kappa_3(s) = 1, \quad a(s) + b'(s) = 0, \quad b(s) = 0, \quad c'(s) = 0.$$

It follows that

$$(3.1) \quad a(s) = 0, \quad b(s) = 0, \quad c(s) = c_0, \quad k_3(s) = -1/c_0, \quad c_0 \in R_0.$$

Therefore, statement (ii) holds. Substituting (3.1) in (2.4), we easily obtain that statements (i) and (iii) hold.

Conversely, assume that statement (i) holds. Differentiating the equation $g(\alpha, \alpha) = r^2$ with respect to s , we find $g(\alpha, T) = 0$ which means that α is the normal curve. If statement (ii) holds, putting $k_3(s) = -1/c_0$, $c_0 \in R_0$ and applying (2.1) we obtain $d/ds[\alpha(s) - c_0 B_2(s)] = 0$. Thus α is congruent to a normal curve. If statement (iii) holds, differentiating the equation $g(\alpha, B_2) = c_0$ with respect to s , we easily obtain that α is a normal curve which completes the proof. \square

Theorem 3.2. *Let $\alpha(s)$ be a unit speed pseudo null curve in E_1^4 with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$ and $k_3(s) \neq 0$. Then α is a normal curve if and only if one of the following statements hold:*

- (i) α lies in pseudosphere or pseudohyperbolic space in E_1^4 ;
- (ii) the first binormal and the second binormal component of the position vector α are respectively given by $g(\alpha, B_1) = 0$, $g(\alpha, B_2) = c$, $c \in R$.

Proof. First assume that α is a normal curve. Then its position vector satisfies relation (2.5). By taking the derivative of (2.5) with respect to s and applying (2.2), we obtain the system of equations

$$(3.2) \quad \begin{aligned} c(s) = -1, \quad c'(s) - b(s)\kappa_2(s) &= 0, \\ a(s)\kappa_2(s) + b'(s) - c(s)\kappa_3(s) &= 0, \quad a'(s) + b(s)\kappa_3(s) = 0. \end{aligned}$$

From the first and second equation of (3.2), we get $b(s) = 0$. Therefore, the system of equations (3.2) reduces to the system

$$(3.3) \quad a(s) = -\kappa_3(s)/\kappa_2(s), \quad b(s) = 0, \quad c(s) = -1, \quad a'(s) = 0.$$

From the first and the last equation of (3.3) we get $a(s) = a_0 \in R_0$. Consequently, by using the last relation and (3.3), relation (2.5) becomes

$$\alpha(s) = a_0 N(s) - B_2(s), \quad a_0 \in R_0.$$

The last relation implies $g(\alpha, \alpha) = -2a_0$, $g(\alpha, B_1) = 0$, $g(\alpha, B_2) = a_0$, $a_0 \in R_0$, which proves statements (i) and (ii).

Conversely, suppose that statement (i) holds. Differentiating the equation $g(\alpha, \alpha) = c$, $c \in R$ with respect to s we easily get that α is a normal curve. If statement (ii) holds, by taking the derivative of the equation $g(\alpha, B_2) = c$ with respect to s , it follows that α is a normal curve, which proves the theorem. \square

Remark 3.1. *If pseudo null normal curve has the third curvature $\kappa_3(s) = 0$, then its parameter equation reads $\alpha(s) = -B_2(s)$, so it lies in the lightcone $C^3(O)$.*

The following theorem can be proved in a similar way as theorems 3.1 and 3.2., so we omit its proof.

Theorem 3.3. *A unit speed partially null curve $\alpha(s)$ in E_1^4 with the curvatures $\kappa_1(s) \neq 0$, $\kappa_2(s) \neq 0$ and $\kappa_3(s) = 0$ is a normal curve if and only if one of the following statements hold:*

- (i) *the first curvature $\kappa_1(s)$ is non-zero constant;*
- (ii) *α lies in a circular cylinder with null axis;*
- (iii) *the principal normal and the first binormal component of the position vector α are respectively given by $g(\alpha, N) = a_0$, $a_0 \in R_0$, $g(\alpha, B_1) = 0$.*
- (iv) *the first binormal and the second binormal component of the position vector α are respectively given by $g(\alpha, B_1) = 0$, $g(\alpha, B_2) = -a_0 \int_0^s \kappa_2(s) ds$, $a_0 \in R_0$.*

Recall that arbitrary curve in E_1^4 is called a *helix* or a *W-curve*, if all its curvature functions are constant. It is known that every pseudo null helix lies in some hyperquadric in E_1^4 ([9]).

Corollary 3.1. *Every null, pseudo null and partially null helix in E_1^4 is a normal curve.*

Next we obtain a simple relationship between null normal curves and non-null rectifying curves in Minkowski space-time.

Theorem 3.4. *Let α be a null curve in E_1^4 and let β be the orthogonal projection of α onto the spacelike (timelike) hyperplane Σ of E_1^4 . Then α is a normal curve if and only if β is a spacelike (timelike) rectifying curve.*

Proof. We give the proof when Σ is a spacelike hyperplane. When Σ is a timelike hyperplane, the proof is analogous.

Assume that α is a normal curve. The equation of α in terms of β can be written as

$$(3.4) \quad \alpha(s) = \beta(s) + \mu(s)V,$$

where s is arclength parameter of β , $\mu(s)$ is a nonconstant differentiable function and $V \in \Sigma^\perp$ is a constant unit timelike vector. Taking the derivative of (3.4) with respect to s and using the condition $g(\alpha'(s), \alpha'(s)) = 0$ we get $g(\beta'(s), \beta'(s)) = \mu'^2(s) = 1$. Up to a translations of E_1^4 , we may take

$$(3.5) \quad \mu(s) = s.$$

From the condition $g(\alpha(s), \alpha'(s)) = 0$, we obtain

$$g(\beta(s), \beta'(s)) = \mu(s)\mu'(s).$$

Differentiating the previous equation with respect to s and using (3.5) we find $g(\beta(s), \beta''(s)) = 0$, which means that β is a spacelike rectifying curve.

Conversely, assume that the projected curve β is a spacelike rectifying curve lying fully in the spacelike hyperplane Σ of E_1^4 . Then β is given by ([3])

$$\beta(t) = \frac{a}{\cos t}y(t), \quad a \in R_0^+,$$

where $y(t)$ is a unit speed spacelike curve lying in the unit sphere $S^2(1)$. Hence the equation of α can be written as

$$(3.6) \quad \alpha(t) = \frac{a}{\cos t}y(t) + \mu(t)V, \quad a \in R_0^+,$$

where $\mu(t)$ is a nonconstant differentiable function and $V \in \Sigma^\perp$ is a constant unit timelike vector. By using the condition $g(\alpha'(t), \alpha'(t)) = 0$, we obtain $\mu(t) = a \tan t$. Substituting this in (3.6), it follows that $g(\alpha(t), \alpha'(t)) = 0$, which means that α is a normal curve. \square

It can be verified that the orthogonal projection of a pseudo null normal curve α in E_1^4 onto non-degenerate hyperplane of E_1^4 is not necessarily a rectifying curve. However, the mentioned projection can be a rectifying curve under some additional conditions given in the next theorem.

Theorem 3.5. *Let α be non-null normal curve in E_1^4 and β a unit speed timelike orthogonal projection of α onto timelike hyperplane Σ of E_1^4 . Then α is pseudo null curve and holds*

$$(3.7) \quad g(\alpha(s), V) = \sqrt{s^2 + a}, \quad a \in R_0^-,$$

where s is arclength parameter of β and $V \in \Sigma^\perp$ is a constant unit spacelike vector, if and only if β is a rectifying curve with the first curvature

$$(3.8) \quad k_\beta(s) = \frac{\sqrt{|a|}}{s^2 + a}, \quad a \in R_0^-.$$

Proof. First assume that α is pseudo null normal curve and holds $g(\alpha(s), V) = \sqrt{s^2 + a}$, $a \in R_0^-$. The equation of α , in terms of its orthogonal projection β , can be written as

$$(3.9) \quad \alpha(s) = \beta(s) + \mu(s)V,$$

where $\mu(s) = g(\alpha(s), V)$. Differentiating the previous equation with respect to s we obtain

$$(3.10) \quad \alpha'(s) = \beta'(s) + \mu'(s)V.$$

Since α is a normal curve, it follows that $g(\alpha(s), \alpha'(s)) = 0$. Substituting (3.9) and (3.10) into the last equation, we get

$$g(\beta(s), \beta'(s)) + \mu(s)\mu'(s) = 0.$$

Differentiating the previous equation with respect to s and using the conditions $g(\beta'(s), \beta'(s)) = -1$, $\mu(s) = \sqrt{s^2 + c}$, $c \in R_0^-$, we obtain $g(\beta(s), N_\beta(s)) = 0$. Hence β is a rectifying curve. Denote by t arclength parameter of α . Since $\alpha(s) = \alpha(t(s))$,

differentiating the last equation two times with respect to s and using the condition $g(\alpha''(t), \alpha''(t)) = 0$, we obtain

$$(3.11) \quad g(\alpha''(s), \alpha''(s)) = t''^2(s).$$

In particular, differentiating relation (3.10) with respect to s yields

$$(3.12) \quad g(\alpha''(s), \alpha''(s)) = \kappa_\beta^2(s) + \mu''^2(s).$$

The arclength parameter t of α is given by

$$(3.13) \quad t(s) = \int_0^s \|\alpha'(u)\| du = \int_0^s \sqrt{\mu'^2(u) - 1} du.$$

Differentiating relation (3.13) two times with respect to s and using (3.11) and (3.12) we obtain that the first curvature κ_β is given by (3.8).

Conversely, assume that β is a timelike rectifying curve with the first curvature given by (3.8). Then β has parameter equation ([6])

$$\beta(u) = \frac{a}{\sinh u} y(u), \quad a \in R_0^+,$$

where $y(u)$ is the unit speed spacelike curve lying in pseudohyperbolic space $H_0^2(1)$. The last equation in terms of arclength parameter s of β , reads

$$\beta(s) = y(s)\sqrt{s^2 - a^2}, \quad a \in R_0^+.$$

Substituting the last relation in (3.9) we get

$$(3.14) \quad \alpha(s) = y(s)\sqrt{s^2 - a^2} + \mu(s)V.$$

Differentiating the previous equation with respect to s we find

$$(3.15) \quad \alpha'(s) = y'(s)\sqrt{s^2 - a^2} + y(s)\frac{s}{\sqrt{s^2 - a^2}} + \mu'(s)V.$$

Since α is a normal curve, there holds $g(\alpha(s), \alpha'(s)) = 0$. Substituting (3.14) and (3.21) into the last equation we find

$$(3.16) \quad \mu(s) = g(\alpha(s), V) = \sqrt{s^2 + d}, \quad d \in R,$$

and therefore

$$(3.17) \quad \mu''(s) = \frac{d}{(s^2 + d)^{\frac{3}{2}}}, \quad d \in R.$$

Differentiating relation (3.9) two times with respect to s yields

$$\alpha''(s) = \beta''(s) + \mu''(s)V.$$

It follows that

$$g(\alpha''(s), \alpha''(s)) = \kappa_\beta^2(s) + \mu''^2(s).$$

In particular, substituting (3.8) and (3.17) into the last equation, we find

$$g(\alpha''(s), \alpha''(s)) = -\frac{a}{(s^2 + a)^2} + \frac{d^2}{(s^2 + d)^3}, \quad a \in R_0^-, d \in R.$$

Without loss of generality, we may take $d = a$. Then the last equation becomes

$$(3.18) \quad g(\alpha''(s), \alpha''(s)) = -\frac{as^2}{(s^2 + a)^3}, \quad a \in R_0^-.$$

In particular, relation (3.16) becomes

$$(3.19) \quad \mu(s) = g(\alpha(s), V) = \sqrt{s^2 + a}, \quad a \in R_0^-.$$

Denote by t arclength parameter of α . Differentiating the equation $\alpha(s) = \alpha(t(s))$ two times with respect to s we get $\alpha''(s) = \alpha''(t)t'^2 + \alpha'(t)t''$. The last relation implies

$$(3.20) \quad g(\alpha''(s), \alpha''(s)) = g(\alpha''(t), \alpha''(t))t'^4 + g(\alpha'(t), \alpha'(t))t''^2.$$

On the other hand, from (3.13) we have

$$t' = \sqrt{\mu'^2 - 1}, \quad t'' = \frac{\mu'\mu''}{\sqrt{\mu'^2 - 1}}.$$

Substituting this into (3.20) we obtain

$$g(\alpha''(s), \alpha''(s)) = \kappa_\alpha^2(t)g(N_\alpha(t), N_\alpha(t))(\mu'^2 - 1)^2 + g(\alpha'(t), \alpha'(t))\frac{\mu'^2\mu''^2}{\mu'^2 - 1}.$$

By using (3.19), the last equation becomes

$$g(\alpha''(s), \alpha''(s)) = \kappa_\alpha^2(t)g(N_\alpha(t), N_\alpha(t))\frac{a^2}{(s^2 + a)^2} - g(\alpha'(t), \alpha'(t))\frac{as^2}{(s^2 + a)^3}.$$

The last equation and (3.18) imply

$$(3.21) \quad \frac{s^2}{s^2 + a}(g(\alpha'(t), \alpha'(t)) - 1) = a\kappa_\alpha^2(t)g(N_\alpha(t), N_\alpha(t)), \quad a \in R_0^-.$$

By assumption, α is non-null curve, so $g(\alpha'(t), \alpha'(t)) = \pm 1$. If $g(\alpha'(t), \alpha'(t)) = -1$, relations $\alpha'(s) = \beta'(s) + \mu'(s)V$ and $\alpha'(s) = \alpha'(t)t'(s)$ imply

$$g(\alpha'(s), \alpha'(s)) = -t'^2(s) = g(\beta'(s), \beta'(s)) + \mu'^2(s).$$

Relation (3.13) implies $t'^2 = \mu'^2 - 1$. Substituting this into the last equation, we find $g(\beta'(s), \beta'(s)) = 1 - 2\mu'^2(s) = -1$. It follows that $\mu(s) = \pm s + b$, $b \in R$, which is a contradiction with (3.19). Hence $g(\alpha'(t), \alpha'(t)) = 1$, so relation (3.21) implies $g(N_\alpha, N_\alpha) = 0$ which means that α is a pseudo null curve. This completes the proof of the theorem. \square

The relationship between partially null normal curves and rectifying curves is given in the last theorem.

Theorem 3.6. *Let α be a partially null curve in E_1^4 and β a spacelike orthogonal projection of α onto lightlike 2-plane of E_1^4 . If α is a normal curve, then β is a rectifying curve.*

Proof. Assume that α is partially null normal curve lying in the lightlike hyperplane Ω of E_1^4 . We may assume that up to isometries of E_1^4 , Ω has the equation $x_1 = x_2$. Then up to a parametrization, parameter equation of α reads

$$\alpha(s) = (\alpha_1(s), \alpha_1(s), s, \sqrt{r^2 - s^2}), \quad r \in R_0^+,$$

where α_1 is some differentiable function. Moreover, the orthogonal projection of α onto lightlike 2-plane has the equation

$$\beta(s) = (\alpha_1(s), \alpha_1(s), s, 0).$$

It follows that $g(\beta(s), \beta(s)) = s^2$. Differentiating the last equation two times with respect to s , we get $g(\beta(s), N_\beta(s)) = 0$, which means that β is a rectifying curve. \square

4. SOME EXAMPLES OF NULL, PSEUDO NULL AND PARTIALLY NULL NORMAL CURVES IN E_1^4

In this section, we give some examples of null, pseudo null and partially normal curves in terms of their orthogonal projections onto non-degenerate hyperplanes and lightlike 2-plane of E_1^4 . Denote by Σ non-degenerate hyperplane of E_1^4 .

Case (1) Σ is spacelike. By theorem 3.4 the null normal curve α is given by

$$\alpha(s) = \beta(s) + sV,$$

where the rectifying curve $\beta(s) = \sqrt{s^2 + c^2}y(s)$, $c \in R_0^+$, is the unit speed orthogonal projection of α onto Σ , $y(s)$ is a spacelike curve lying in the unit sphere $S^2(1)$ and $V \in \Sigma^\perp$ is a constant unit timelike vector.

Case (2) Σ is timelike. According to theorem 3.4, the null normal curve α has the equation

$$\alpha(s) = \beta(s) + sV, \quad c \in R_0^+,$$

where the rectifying curve $\beta(s) = (c - s)\sqrt{\frac{s+c}{s-c}}y(s)$ is the unit speed orthogonal projection of α onto Σ , $y(s)$ is a spacelike curve lying in pseudohyperbolic space $H_0^2(1)$ and $V \in \Sigma^\perp$ is a constant unit spacelike vector.

According to theorem 3.5, parameter equation of pseudo null normal curve is given by

$$\alpha(s) = \beta(s) + \sqrt{s^2 + a}V, \quad a \in R_0^-,$$

where the rectifying curve $\beta(s) = (c - s)\sqrt{\frac{s+c}{s-c}}y(s)$, $c \in R_0^+$, is the unit speed orthogonal projection of α onto timelike hyperplane Σ , $y(s)$ is a spacelike curve lying in pseudohyperbolic space $H_0^2(1)$ and $V \in \Sigma^\perp$ is a constant unit spacelike vector.

Finally, according to theorem 3.6, parameter equation of partially null normal curve can be written as

$$\alpha(s) = (s^2, s^2, s, \sqrt{r^2 - s^2}), \quad r \in R_0^+,$$

where the rectifying curve $\beta(s) = (s^2, s^2, s, 0)$ is the orthogonal projection of α onto lightlike 2-plane of E_1^4 . Since $\kappa_\beta(s) = 1$ and $\tau_\beta(s) = 0$, the projected curve β is a pseudo null circle.

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