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E-BOCHNER CURVATURE TENSOR ON (κ, μ)-CONTACT METRIC MANIFOLDS

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Dedicated to memory of Proffessor Franki Dillen

Abstract. We study E-Bochner curvature tensor B^e satisfying $\mathrm{R}\cdot\mathrm{B}^e = 0$, $\mathrm{B}^e\cdot\mathrm{R} = 0$, $\mathrm{B}^e\cdot\mathrm{B}^e = 0$ and $\mathrm{B}^e\cdot\mathrm{S} = 0$ in n-dimensional (κ, μ)-contact metric manifolds.

1. Introduction

In [3], Blair, Koufogiorgos and Papantoniou introduced (κ, μ)-contact metric manifolds. A class of contact metric manifold M with contact metric structure (φ, ξ, η, g) in which the curvature tensor R satisfies the condition

$$R(X,Y)\xi = (KI + \mu h)(\eta(Y)X - \eta(X)Y), \tag{1.1}$$

for all $X, Y \in TM$ is called (κ, μ)-manifolds. On the other hand, Bochner [5] introduced a Kahler analogue of Weyl conformal curvature tensor by purely formal consideration which is known as Bochner Curvature Tensor. A geometric meaning of Bochner Curvature Tensor was given by Blair [2]. By using Boothby-Wang's fibration [7], Matsumoto and Chuman [15] constructed C-Bochner curvature tensor. In [9], Endo defined E-Bochner curvature tensor as an extended C-Bochner curvature tensor. E-Bochner curvature tensor is defined as
$\mathrm{B}^e(X,Y)Z = \mathrm{B}(X,Y)Z - \eta(X)\mathrm{B}(\xi,Y)Z - \eta(Y)\mathrm{B}(X,\xi)Z - \eta(Z)\mathrm{B}(X,Y)\xi$, for all

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X, Y, Z belongs to TM, B is the C-Bochner curvature tensor defined by

$$\begin{aligned}B(X,Y)Z &= R(X,Y)Z + \frac{1}{n+3}\big[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY \\ &\quad -g(Y,Z)QX + S(\varphi X,Z)\varphi Y - S(\varphi Y,Z)\varphi X + g(\varphi X,Z) \\ &\quad Q\varphi Y - g(\varphi Y,Z)Q\varphi X + 2S(\varphi X,Y)\varphi Z + 2g(\varphi X,Y)Q\varphi Z \\ &\quad -S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY + \eta(Y) \\ &\quad \eta(Z)QX\big] - \frac{p+n-1}{n+3}\big[g(\varphi X,Y)\varphi Y - g(\varphi Y,Z)\varphi X + 2 \\ &\quad g(\varphi X,Y)\varphi Z\big] - \frac{p-4}{n+3}\big[g(X,Z)Y - g(Y,Z)X\big] + \frac{p}{n+3} \\ &\quad \big[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\big], \end{aligned} \tag{1.2}$$

where S is Ricci tensor of type of type $(0,2)$, Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and $p = \frac{n+r-1}{n+1}$, r is the scalar curvature of the manifold. E-Bochner curvature tensor is denoted by B^e.
A Riemannian manifold (M^{2m+1}, g) is said to be semisymmetric if its curvature tensor R satisfies the condition $R(X,Y) \cdot R = 0$, for all X, Y belongs to TM where $R(X,Y)$ acts on R as a derivation ([14], [18]).
In [20], Yildiz and De studied h-projectively semisymmetric on (κ, μ)-contact metric manifolds. Besides this, in [13] Kim, Tripathi and Choi proved that a (κ, μ)-contact metric manifold with vanishing E-Bochner curvature tensor is a Sasakian manifold. Motivated by the above studies, we characterize a (κ, μ)-contact metric manifold satisfying certain curvature conditions on E-Bochner curvature tensor. The present paper is organized as follows:
After preliminaries in section 3 and 4, we characterize (κ, μ)-contact metric manifolds satisfying $\mathrm{R} \cdot \mathrm{B}^e = 0$ and $\mathrm{B}^e \cdot \mathrm{R} = 0$ respectively. Besides these we prove that a (κ, μ)-contact metric manifold is Sasakian if and only if it satisfies $\mathrm{B}^e \cdot \mathrm{B}^e = 0$. Finally, we prove that a (κ, μ)-contact metric manifold satisfying $\mathrm{B}^e \cdot \mathrm{S} = 0$ is an η-Einstein manifold. Also we obtain some important corollaries.

2. Preliminaries

An $n(= 2m+1)$ dimensional differentiable manifold M is called an almost contact manifold if there is an almost contact structure (φ, ξ, η) consisting of a $(1,1)$ tensor field φ, a vector field ξ, a 1-form η satisfying

$$\varphi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \varphi\xi = 0,\ \eta \circ \varphi = 0. \tag{2.1}$$

An almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $\mathrm{M}^n \times \mathbb{R}$ defined by
$\mathrm{J}(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ is integrable where X is tangent to M, t is the coordinate of R and f is a smooth function on $\mathrm{M}^n \times \mathbb{R}$.
The condition for being normal is equivalent to vanishing of the torsion tensor $[\varphi, \varphi] + 2d\eta \otimes \xi$ where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ.

Let g be a compatible Reimannian metric with (φ,ξ,η), that is,

$$g(X,Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \tag{2.2}$$

or equivalently,

$$g(X,\xi)=\eta(X), g(\varphi X,Y)=-g(X,\varphi Y), \tag{2.3}$$

for all X,Y belongs to TM.

An almost contact metric structure becomes a contact metric structure if

$$g(X,\varphi Y)=d\eta(X,Y), \tag{2.4}$$

for all X,Y belongs to TM. Given a contact metric manifold $M^n(\varphi,\xi,\eta,g)$ we define a $(1,1)$ tensor field h by $h=\frac{1}{2}\mathrm{L}_\xi\varphi$ where L denotes the Lie differentiation. Then h is symmetric and satisfies

$$h\xi=0,\ h\varphi+\varphi h=0, \tag{2.5}$$

$$\nabla\xi=-\varphi-\varphi h,\ trace(h)=trace(\varphi h)=0, \tag{2.6}$$

where ∇ is the Levi-Civita connection.
A contact metric manifold is said to be an η-Einstein if

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y), \tag{2.7}$$

where a,b are smooth functions and S is the Ricci tensor.
A normal contact metric manifold is called a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X\varphi)Y=g(X,Y)\xi-\eta(Y)X. \tag{2.8}$$

On a Sasakian manifold the following relation holds

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y, \tag{2.9}$$

for all X,Y belongs to TM.Blair, Koufogiorgos and Papantoniou [3] considered the $(\kappa,\ \mu)$-nullity condition and gave several reasons for studying it. The $(\kappa,\ \mu)$-nullity distribution N$(\kappa,\ \mu)$ ([3], [17]) of a contact metric manifold M is defined by

$$N(\kappa,\mu):p\mapsto N_p(\kappa,\mu)=[U\in T_pM\mid R(X,Y)U=(\kappa I+\mu h)(g(Y,U)X-g(X,U)Y)]$$

for all X,Y belongs to TM, where $(\kappa,\mu)\in\mathbb{R}^2$.
A contact metric manifold M^n with $\xi\in N(\kappa,\mu)$ is called a $(\kappa,\ \mu)$- contact metric manifold. Then we have

$$\begin{aligned} R(X,Y)\xi &= \kappa[\eta(Y)X-\eta(X)Y]+ \\ &\quad \mu[\eta(Y)hX-\eta(X)hY], \end{aligned} \tag{2.10}$$

for all X,Y belongs to TM.For $(\kappa,\ \mu)$-metric manifolds, it follows that $h^2=(\kappa-1)\varphi^2$. This class contains Sasakian manifolds for $\kappa=1$ and $h=0$. In fact, for a $(\kappa,\ \mu)$-metric manifold, the condition of being Sasakian manifold, κ-contact manifold, $\kappa=1$ and $h=0$ are equivalent. If $\mu=0$, then the $(\kappa,\ \mu)$-nullity distribution N$(\kappa,\ \mu)$ is reduced to κ-nullity distribution N(κ) [19]. If $\xi\in N(\kappa)$, then we call contact metric manifold M an N(κ)- contact metric manifold.
$(\kappa,\ \mu)$-contact metric manifolds have been studied by several authors ([16], [1], [8], [10], [11], [12]) and many others.

In a $(\kappa,\ \mu)$-contact metric manifold the following relations hold [3]:

$$h^2=(\kappa-1)\varphi^2, \tag{2.11}$$

$$(\nabla_X\varphi)Y=g(X+hX,Y)\xi-\eta(Y)(X+hX), \tag{2.12}$$

$$R(\xi, X)Y = \kappa[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.13}$$

$$S(X,\xi) = (n-1)\kappa\eta(X), \tag{2.14}$$

$$\begin{aligned} S(X,Y) &= [(n-3) - \frac{n-1}{2}\mu]g(X,Y) + \\ & [(n-3)+\mu]g(hX,Y) + \big[(3-n) + \frac{n-1}{2} \\ & (2\kappa+\mu)\big]\eta(X)\eta(Y), \end{aligned} \tag{2.15}$$

$$r = (n-1)\big(n-3+\kappa - \frac{n-1}{2}\mu\big), \tag{2.16}$$

$$\begin{aligned} S(X,hY) &= \big[(n-3) - \frac{(n-1)}{2}\mu\big] \\ & g(X,hY) - (\kappa-1)[(n-3)+\mu]g(X,Y) \\ & +(\kappa-1)[(n-3)+\mu)]\eta(X)\eta(Y), \end{aligned} \tag{2.17}$$

$$Q\varphi - \varphi Q = 2[(n-3)+\mu]h\varphi,$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$. Let us recall the following result:

Lemma 2.1. [4] *A contact metric manifold M^{2m+1} satisfying $R(X,Y)\xi = 0$ is locally isometric to the Riemannian product $E^{m+1} \times S^m(4)$ for $m > 1$.*

Besides these, it can be easily verified that in a (κ, μ)-contact metric manifold M^n, $n \geq 5$, the E-Bochner curvature tensor satisfies the following conditions

$$B^e(X,Y)\xi = \frac{4(\kappa-1)}{n+3}(\eta(X)Y - \eta(Y)X) + \mu(\eta(X)hY - \eta(Y)hX), \tag{2.18}$$

$$B^e(\xi,X)Z = \eta(Z)\frac{4(\kappa-1)}{n+3}(X - \eta(X)\xi) + \eta(Z)\mu hX, \tag{2.19}$$

$$B^e(X,\xi)Z = \eta(Z)\frac{4(\kappa-1)}{n+3}(\eta(X)\xi - X) - \eta(Z)\mu hX, \tag{2.20}$$

$$B^e(X,\xi)\xi = \frac{4(\kappa-1)}{n+3}[\eta(X)\xi - X] - \mu hX, \tag{2.21}$$

$$B^e(\xi,X)\xi = \frac{4(\kappa-1)}{n+3}[X - \eta(X)\xi] + \mu hX, \tag{2.22}$$

$$B^e(\xi,\xi)\xi = 0, \tag{2.23}$$

$$\begin{aligned} \sum_{i=1}^{n} \widetilde{B^e}(e_i,Y,Z,e_i) &= \frac{6(n-3)+\mu)}{n+3}g(hY,Z) \\ & -\frac{4(\kappa-1)}{(n+3)}g(Y,Z) - \\ & [\frac{(n-1)\kappa + (2-n)p + r}{n+3} \\ & -\frac{4(\kappa-1)}{n+3}]\eta(Y)\eta(Z), \end{aligned} \tag{2.24}$$

where $p = \frac{n+r-1}{n+1}$, r is the scalar curvature of M, $\{e_1, e_2, ..., e_m, e_{m+1} = \varphi e_1, ..e_{2m} = \varphi e_m, e_{2m+1} = \xi\}$ is a φ basis of M and $\widetilde{B^e}(X, Y, Z, W) = g((B^e(X, Y)Z, W)$.

3. E-Bochner Semisymmetric (κ, μ)-Contact Metric Manifolds

Definition 3.1: An n-dimensional (κ, μ)-contact metric manifold is said to be E-Bochner semisymmetric if it satisfies the following equation

$$R(X, Y) \cdot B^e = 0, \tag{3.1}$$

for all X, Y belongs to TM and B^e is E-Bochner curvature tensor.

Let us consider that M be an $n(= 2m + 1)$ dimensional (κ, μ)-contact metric manifold and M is E-Bochner semisymmetric. From (3.1) we have $(R(X, Y) \cdot B^e)(U, V)W = 0$, which implies that

$$\begin{aligned}&R(X, Y)B^e(U, V)W - B^e(R(X, Y)U, V)W - B^e(U, R(X, Y)V)W -\\&B^e(U, V)R(X, Y)W = 0.\end{aligned} \tag{3.2}$$

Putting $Y = \xi$ in (3.2) and using (2.13), we obtain

$$\begin{aligned}&\kappa[-g(X, B^e(U, V)W)\xi + \eta(B^e(U, V)W)X + g(X, U)B^e(\xi, V)W\\&-\eta(U)B^e(X, V)W + g(X, V)B^e(U, \xi)W - \eta(V)B^e(U, X)W\\&+g(X, W)B^e(U, V)\xi - \eta(W)B^e(U, V)X] +\\&\mu[-g(hX, B^e(U, V)W)\xi + \eta(B^e(U, V)W)hX + g(hX, U)B^e(\xi, V)W\\&-\eta(U)B^e(hX, V)W + g(hX, V)B^e(\xi, V)W - \eta(V)B^e(hX, V)W +\\&g(hX, W)B^e(U, V)\xi - \eta(W)B^e(U, V)hX] = 0.\end{aligned} \tag{3.3}$$

Again putting $W = \xi$ in (3.3) and using (2.18), (2.20), (2.21) and (2.22) we have

$$\begin{aligned}&\frac{4\kappa(\kappa-1)}{n+3}[g(X, U)V - g(X, V)U] + \frac{4(\kappa-1)\mu}{n+3}[g(hX, U) - g(hX, V)U]\\&+\mu\kappa[\eta(V)g(X, hU)\xi - \eta(U)g(X, hV)\xi + g(X, U)hV - g(X, V)hU]\\&+\mu^2[\eta(V)g(hX, hU)\xi - \eta(U)g(hX, hV)\xi + g(hX, U)hV -\\&g(hX, V)hU] - \kappa B^e(U, V)X - \mu B^e(U, V)hX = 0.\end{aligned} \tag{3.4}$$

Taking inner product of (3.4) with Z we obtain

$$\begin{aligned}&\frac{4\kappa(\kappa-1)}{n+3}[g(X, U)g(V, Z) - g(X, V)g(U, Z)] + \frac{4(\kappa-1)\mu}{n+3}[g(hX, U)g(V, Z)\\&-g(hX, V)g(U, Z)] + \mu\kappa[\eta(V)\eta(Z)g(X, hU) - \eta(U)\eta(Z)g(X, hV)\\&+g(X, U)g(hV, Z) - g(X, V)g(hU, Z)] + \mu^2[\eta(V)g(hX, hU)\eta(Z) - \eta(U)\eta(Z)\\&g(hX, hV) + g(hX, U)g(hV, Z) + g(hX, U)g(hV, Z)] - \kappa\widetilde{B^e}(U, V, X, Z) -\\&\mu\widetilde{B^e}(U, V, hX, Z) = 0.\end{aligned}$$

Let $\{e_i\}_{i=1}^n$, be an orthonormal basis of the tangent space. Putting $U = Z = e_i$ and summing up over 1 to n, we have

$$g(hX, V) = a_1 g(X, V) + b_1\eta(X)\eta(V), \tag{3.5}$$

where

$$a_1 = \Big[\frac{-4\kappa(\kappa-1)(2-n) - 6\mu(n-3+\mu)(\kappa-1) + \mu^2(n+3)}{4\mu(\kappa-1)(1-n) - (n-1)\mu^2 - 6\kappa(n-3+\mu) + \mu\kappa(n+3) + 4\mu(\kappa-1)}\Big],$$

and

$$b_1 = \Big[\frac{\kappa^2(n-1) + \kappa p(2-n) + r\kappa + 4\kappa(\kappa-1) + 6\mu(n-3+\mu)(\kappa-1) + \mu^2(\kappa-1)(n+3)}{4\mu(\kappa-1)(1-n) - (n-1)\mu^2 - 6\kappa(n-3+\mu) + \mu\kappa(n+3) + 4\mu(\kappa-1)}\Big],$$

where $p = \frac{n+r-1}{n+1}$, r being the scalar curvature of M. From (2.15) and (3.5) we obtain

$$S(X,V) = ag(X,V) + b\eta(X)\eta(V), \tag{3.6}$$

where

$$\begin{aligned} a &= \big[(n-3) - \frac{n-1}{2}\mu\big] + [(n-3)+\mu] \\ &\quad \Big[\frac{-4\kappa(\kappa-1)(2-n) - 6\mu(n-3+\mu)(\kappa-1) + \mu^2(n+3)}{4\mu(\kappa-1)(1-n) - (n-1)\mu^2 - 6\kappa(n-3+\mu) + \mu\kappa(n+3) + 4\mu(\kappa-1)}\Big], \end{aligned}$$

and

$$\begin{aligned} b &= \big[(3-n) + \frac{n-1}{2}(2\kappa+\mu)\big] + [(n-3)+\mu] \\ &\quad \Big[\frac{\kappa^2(n-1) + \kappa p(2-n) + r\kappa + 4\kappa(\kappa-1) + 6\mu(n-3+\mu)(\kappa-1) + \mu^2(\kappa-1)(n+3)}{4\mu(\kappa-1)(1-n) - (n-1)\mu^2 - 6\kappa(n-3+\mu) + \mu\kappa(n+3) + 4\mu(\kappa-1)}\Big]. \end{aligned}$$

Thus, from (3.6) we can state the following:

Theorem 3.1. *Let M be an n-dimensional ($n \geq 5$) E-Bochner semisymmetric (κ, μ)-contact metric manifold. Then the manifold is an η-Einstein manifold.*

Putting the value $\kappa = 1$ and $h = 0$ in the equation (3.5) we have the following:

Corollary 3.1. *An E-Bochner semisymmetric Sasakian manifold $M^n (n \geq 5)$ is E-Bochner flat.*

In general, in a (κ, μ)-contact metric manifold the Ricci operator Q does not commute with φ. However, Yildiz and De [20] proved the following:

Lemma 3.1. *In a non-Sasakian (κ, μ)-contact metric manifold the following conditions are equivalent:*
(a) η-Einstein manifold,
(b) $Q\varphi = \varphi Q$.

From Lemma 3.1 we can state that

Corollary 3.2. *Let M^n be an n-dimensional ($n \geq 5$) E-Bochner semisymmetric non-Sasakian (κ, μ)-contact metric manifold. Then the Ricci operator Q commutes with φ.*

4. (κ, μ)-Contact Metric Manifolds Satisfying $B^e(\xi, U) \cdot R = 0$

In this section,we consider an n-dimensional (κ, μ)-contact metric manifold satisfying $\big(B^e(\xi, U) \cdot R\big)(X,Y)Z = 0$. Therefore, we have

$$\begin{aligned} &B^e(\xi,U)R(X,Y)Z - R(B^e(\xi,U)X,Y)Z - R(X,B^e(\xi,U)Y)Z - \\ &R(X,Y)B^e(\xi,U)Z = 0. \end{aligned} \tag{4.1}$$

Using (2.19) in (4.1), we get

$$\begin{aligned} \frac{4(\kappa-1)}{n+3} \Big[& -\eta(U)\eta(R(X,Y)Z)\xi + \eta(R(X,Y)Z)U + \\ & \eta(X)\eta(U)R(\xi,Y)Z - \eta(X)R(U,Y)Z + \eta(Y)\eta(U)R(X,\xi)Z \\ & -\eta(Y)R(X,U)Z + \eta(Z)\eta(U)R(X,Y)\xi - \eta(Z)R(X,Y)U] \\ & +\mu[\eta(R(X,Y)Z)hU - \eta(X)R(hU,Y)Z - \\ & \eta(Y)R(X,hU)Z - \eta(Z)R(X,Y)hU\Big] = 0, \end{aligned} \tag{4.2}$$

Taking inner product with ξ of (4.2) and using $h\xi = 0$, $g(R(X,Y)\xi,\xi) = 0$ we get

$$\begin{aligned} \frac{4(\kappa-1)}{n+3} \Big[& \eta(X)\eta(U)g(R(\xi,Y)Z,\xi) - \eta(X)g((U,Y)Z,\xi) + \\ & \eta(Y)\eta(U)g(R(X,\xi)Z,\xi) - \eta(Y)g(R(X,U)Z,\xi) - \\ & \eta(Z)g(R(X,Y)U,\xi] + \\ & \mu[-\eta(X)g(R(hU,Y)Z,\xi) - \eta(Y)g(R(X,hU)Z,\xi) - \\ & \eta(Z)g(R(X,Y)hU,\xi)\Big] = 0, \end{aligned} \tag{4.3}$$

Let us consider the following cases:
CASE 1. $\kappa = 0 = \mu$.
CASE 2. $\kappa = 0$, $\mu \neq 0$.
CASE 3. $\kappa \neq 0$, $\mu = 0$.
CASE 4. $\kappa \neq 0$, $\mu \neq 0$
For Case 1, we observe that $R(X,Y)\xi = 0$, for all X,Y. Hence, by Lemma 2.1, M is locally the Remannian product $E^{m+1} \times S^m(4)$.
For Case 2, from (4.3) we get

$$\begin{aligned} \frac{4(-1)}{n+3} \Big[& \eta(X)\eta(U)g(R(\xi,Y)Z,\xi) - \eta(X)g((U,Y)Z,\xi) + \\ & \eta(Y)\eta(U)g(R(X,\xi)Z,\xi) - \eta(Y)g(R(X,U)Z,\xi) - \\ & \eta(Z)g(R(X,Y)U,\xi] + \\ & \mu[-\eta(X)g(R(hU,Y)Z,\xi) - \eta(Y)g(R(X,hU)Z,\xi) \\ & -\eta(Z)g(R(X,Y)hU,\xi)\Big] = 0. \end{aligned} \tag{4.4}$$

Let $\{e_i\}_{i=1}^n$, be an orthonormal basis of the tangent space. Putting $Y = Z = e_i$ in (4.3) and summing over 1 to n, we get

$$g(X,hU) = a_1 g(X,U) + b_1\eta(X)\eta(U), \tag{4.5}$$

where

$$a_1 = [\frac{\mu(n+3)}{4}],$$

and

$$b_1 = -[\frac{\mu(n+3)}{4}].$$

From (2.15) and (4.5) we get

$$S(X,U) = ag(X,U) + b\eta(X)\eta(U), \tag{4.6}$$

where

$$a = \big[(n-3) - \frac{n-1}{2}\mu\big] + [(n-3)+\mu][\frac{\mu(n+3)}{4}],$$

$$b = \big[(3-n) + \frac{n-1}{2}\mu\big] - [(n-3)+\mu]\big[\frac{\mu(n+3)}{4}\big].$$

For Case 2, M becomes an η-Einstein manifold.
For Case 3, from (4.3) we get

$$\begin{aligned}\frac{4(\kappa-1)}{n+3}\ [\ & \eta(X)\eta(U)g(R(\xi,Y)Z,\xi) - \eta(X)g((U,Y)Z,\xi) + \\ & \eta(Y)\eta(U)g(R(X,\xi)Z,\xi) - \eta(Y)g(R(X,U)Z,\xi) - \\ & \eta(Z)g(R(X,Y)U,\xi] = 0. \end{aligned} \tag{4.7}$$

Therefore, either $\kappa = 1$ or

$$\begin{aligned}& [\eta(X)\eta(U)g(R(\xi,Y)Z,\xi) - \eta(X)g((U,Y)Z,\xi) + \\ & \eta(Y)\eta(U)g(R(X,\xi)Z,\xi) - \eta(Y)g(R(X,U)Z,\xi) - \\ & \eta(Z)g(R(X,Y)U,\xi] = 0. \end{aligned} \tag{4.8}$$

Let $\{e_i\}_{i=1}^n$, be an orthonormal basis of the tangent space. Putting $Y = Z = e_i$ in (4.8) and summing over 1 to n, we get

$$g(X,U) = \eta(X)\eta(U), \tag{4.9}$$

which is not possible.
Thus, for Case 3, M is a Sasakian manifold.
For Case 4, putting $Y = Z = e_i$ in (4.3) we get

$$g(X,hU) = a_1 g(X,U) + b_1\eta(X)\eta(U), \tag{4.10}$$

where

$$a_1 = \big[\frac{\mu^2(\kappa-1)(n+3) - 4\kappa(\kappa-1)}{4\mu(\kappa-1)+\kappa\mu(n+3)}\big],$$

$$b_{1=} = -\big[\frac{\mu^2(\kappa-1)(n+3) - 4\kappa(\kappa-1)}{4\mu(\kappa-1)+\kappa\mu(n+3)}\big].$$

Form (2.15) and (4.10) we get

$$S(X,U) = ag(X,U) + b\eta(X)\eta(U), \tag{4.11}$$

where

$$a = \big[(n-3) - \frac{n-1}{2}\mu\big] + [(n-3)+\mu]\big[\frac{\mu^2(\kappa-1)(n+3) - 4\kappa(\kappa-1)}{4\mu(\kappa-1)+\kappa\mu(n+3)}\big],$$

and

$$b = \big[(3-n) + \frac{n-1}{2}(2\kappa+\mu)\big] - [(n-3)+\mu]\big[\frac{\mu^2(\kappa-1)(n+3) - 4\kappa(\kappa-1)}{4\mu(\kappa-1)+\kappa\mu(n+3)}\big].$$

Summing up we can state the following:

Theorem 4.1. *Let M^n be an $(n = 2m+1)$-dimensional $(\kappa,\ \mu)$-contact metric manifold satisfying $B^e(\xi,U)\cdot R = 0$. Then we have one of the following:*
(a) M^n is locally the Remannian product $E^{m+1}\times S^m(4)$.
(b) M^n is an η-Einstein manifold.
(c) M^n is a Sasakian manifold.

5. (κ, μ)-Contact Metric Manifolds Satisfying $B^e(\xi, U) \cdot B^e = 0$

Let M^n be an n-dimensional (κ, μ)-contact metric manifold satisfying $B^e(\xi, U) \cdot B^e = 0$
Then we have

$$\begin{aligned} &B^e(\xi, U)B^e(X, Y)Z - B^e(B^e(\xi, U)X, Y)Z - B^e(X, B^e(\xi, U)Y)Z - \\ &B^e(X, Y)B^e(\xi, U)Z = 0, \end{aligned} \tag{5.1}$$

Using (2.19), we get

$$\begin{aligned} &\frac{4(\kappa-1)}{n+3}[\eta(B^e(X, Y)Z)U - \eta(B^e(X, Y)Z)\eta(U)\xi - \eta(X)B^e(U, Y)Z \\ &+\eta(U)\eta(X)B^e(\xi, Y)Z - \eta(Y)B^e(X, U)Z + \eta(Y)\eta(U)B^e(X, \xi)Z \\ &-\eta(Z)B^e(X, Y)U + \eta(Z)\eta(U)B^e(X, Y)\xi] + \mu[\eta(B^e(X, Y)Z)hU - \\ &\eta(X)B^e(hU, Y)Z - \eta(Y)B^e(X, hU)Z - \eta(Z)B^e(X, Y)hU] = 0. \end{aligned} \tag{5.2}$$

Putting $X = Z = \xi$ in (5.2) and using (2.22) and (2.23), we obtain

$$\frac{4(\kappa-1)}{n+3}\big[2(\eta(U)\frac{4(\kappa-1)}{n+3}(Y - \eta(Y)\xi) + \mu\eta(U)hY)\big] = 0. \tag{5.3}$$

From(5.3) and (2.19) it follows that either $\kappa = 1$, or $B^e(\xi, Y)U = 0$.
For $\kappa = 1$, the manifold is Sasakian.
If $B^e(\xi, Y)U = 0$, then $B^e(\xi, Y) \cdot B^e = 0$ and if the manifold is Sasakian, then from (2.19) we obtain $B^e(\xi, Y)U = 0$ and hence $B^e(\xi, Y) \cdot B^e = 0$

Lemma 5.1. [13] *A (κ, μ)-contact metric manifold with vanishing E-Bochner curvature tensor is a Sasakian manifold.*

From the above discussion and Lemma 5.1 we conclude that

Theorem 5.1. *A (κ, μ)-contact metric manifold $M^n (n \geq 5)$ satisfies $B^e(\xi, U) \cdot B^e = 0$ if and only if the manifold is Sasakian.*

6. (κ, μ)-Contact Metric Manifolds Satisfyng $B^e(\xi, X) \cdot S = 0$

Let M be an n-dimensional (κ, μ)-contact metric manifold satisfying $B^e(\xi, X) \cdot S = 0$.
Therefore, $\big(B^e(\xi, X) \cdot S\big)(U, V) = 0$ implies

$$S(B^e(\xi, X)U, V) + S(U, B^e(\xi, X)V) = 0. \tag{6.1}$$

Using (2.19), we get

$$\begin{aligned} &\frac{4(\kappa-1)}{n+3}[\eta(U)S(X, V) + \eta(V)S(U, X)] + \mu[\eta(U)S(hX, V) + \\ &\eta(V)S(U, hX)] = 0. \end{aligned} \tag{6.2}$$

Putting $V = \xi$ in (6.2), we get

$$\frac{4(\kappa-1)}{n+3}[2\kappa(n-1)\eta(U)\eta(X) + S(U, X)] + \mu S(U, hX) = 0. \tag{6.3}$$

Using (2.15) and (2.17) in (6.3) we have

$$S(U, X) = ag(U, X) + b\eta(U)\eta(X), \tag{6.4}$$

where

$$a = \big[(n-3) - \frac{(n-1)}{2}\mu\big] + [(n-3)+\mu] \frac{[-4(\kappa-1)((n-3) - \frac{(n-1)}{2}\mu) + \mu((\kappa-1)(n^2-9) - \frac{(n+3)(n-1)}{2}\mu)]}{[4(\kappa-1)((n-3)+\mu) + \mu((n^2-9) - \frac{n+3)(n-1)}{2}\mu)]},$$

and

$$b = \big[(3-n) + \frac{(n-1)}{2}(2\kappa+\mu)\big] - [(n-3)+\mu] \frac{[4(\kappa-1)(3\kappa(n-1) + (3-n) + \frac{(n-1)}{2}\mu) + \mu((\kappa-1)(n^2-9) + \mu(\kappa-1)(n+3))]}{[4(\kappa-1)((n-3)+\mu) + \mu((n^2-9) - \frac{n+3)(n-1)}{2}\mu)]}.$$

From (6.4) we conclude that

Theorem 6.1. *Let $M^n(n \geq 5)$ be a $(\kappa,\ \mu)$-contact metric manifold satisfying $B^e(\xi, X) \cdot S = 0$. Then the manifold is an η-Einstein Manifold.*

From Lemma 3.1 we can state that

Corollary 6.1. *LetM^n be an n-dimensional $(n \geq 5)$ non-Sasakian $(\kappa,\ \mu)$-contact metric manifold satisfying $B^e(\xi, X) \cdot S = 0$. Then the Ricci operator Q commutes with φ.*

References

[1] Arslan, K., Ezentas, R., Mihai, I., Murthan, C. and Özgür, C., Certain inequalities for submanifolds in $(\kappa,\ \mu)$-contact space forms, Bull. Austral. Soc., **64**(2001), 201-212.

[2] Blair, D.E, Contact manifolds in Reimannian geometry, Lecture notes in Math., 509, Springer-verlag., 1976.

[3] Blair, D.E, Koufogiorgos T. and Papantoniou B.J., Contact metric manifold satisfying a nullity condition, Israel J.Math. **91**(1995), 189-214.

[4] Blair, D.E, Two remarks on contact metric structures, Tohoku Math. J. **29**(1977), 319-324.

[5] Bochner, S., Curvature and Betti number, Ann. of Math., **50**(1949), 77-93 .

[6] Boeckx, E., A full classificstion of contact metric $(\kappa,\ \mu)$-spaces, Illinois J.Math. **44**(2000), 212-219 .

[7] Boothby, W.M. and Wang, H.C.,On contact manifolds, Annals of Math., **68**(1958), 721-734 .

[8] De, U.C and Sarkar, A., On Quasi-conformal curvature tensor of $(\kappa,\ \mu)$-contact metric manifold, Math. Reports **14(64)**, (2012), 115-129.

[9] Endo, H., On K-contact Reimannian manifolds with vanishing E-contact Bochner curvature tensor, Colloq.Math. **62**(1991), 293-297.

[10] Ghosh,S. and De, U.C, On a class of $(\kappa,\ \mu)$-contact metric manifolds, Analele Universităii Oradea Fasc. Mathematica, Tom. **19**(2012), 231-242.

[11] Ghosh, S. and De, U.C., On ϕ-Quasiconformally symmetric $(\kappa,\ \mu)$-contact metric manifolds, Lobachevskii Journal of Mathematics, **31**(2010), 367-375.

[12] Jun, J.B., Yildiz, A. and De, U.C.,On ϕ-recurrent (κ, μ)-contact metric manifolds, Bull. Korean Math. Soc. **45(4)**, 689(2008).

[13] Kim, J.S, Tripathi, M.M. and Choi, J.D, On C-Bochner curvature tensor of a contact metric manifold, Bull. Korean Math. Soc. **42**(2005), 713-724.

[14] Kowalski, O., An explicit classification of 3-dimensional Reimannian spaces satisfying $R(X,Y) \cdot R = 0$, Czecchoslovak Math. J. **46(121)** (1996), 427-474.

[15] Matsumoto, M. and Chuman, G., On C-Bochner curvature tensor, TRU Math., **5**(1969), 21-30.

[16] Özgür, C., Contact metric manifolds with cyclic-parallel Ricci tensor, Diff. Geom. Dynamical systems, **4**(2000), 21-25.

[17] Papantoniou, B.J., Contact Remannian manifolds satisfying $R(\xi, X) \cdot R = 0$ and $\xi \in (\kappa,\ \mu)$-nullity distribution, Yokohama Math.J., **40**(1993), 149-161.

[18] Szabo, Z.I., Structure theorems on Reimannian spaces satisfying $R(X,Y) \cdot R = 0$. I. The local version, J. Differential Geom. **17**(1982), 531-582.

[19] Tanno, S., Ricci curvatures of contact Reimannian manifolds, Tôhoku Math. J., **40**(1988), 441-448.

[20] Yildiz, A., De, U.C, A classification of (κ, μ)-contact matric manifold, Commun. Korean Math.Soc. **27**(2012), 327-339.

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