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Second Order Parallel Tensor and Ricci Solitons on Generalized (k, μ)-Space forms

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Abstract: The object of the present paper is to study the symmetric and skew-symmetric properties of a second order parallel tensor and it is shown that a symmetric parallel second order covariant tensor in a generalized (k, μ)-Space forms is a constant multiple of the metric tensor g. Further we shown that there is no nonzero second order skew-symmetric parallel tensor provided that $(f_1 - f_3)^2 + (k-1)(f_4 - f_6)^2 \neq 0$. Also we studied Ricci solitons on generalized (k, μ)-Space forms and obtained some interesting results.

1. Introduction

A generalized Sasakian space form was first introduced by Carriazo et al. in [1] as that almost contact metric manifold (M, ϕ, ξ, η, g) whose curvature tensor R is given by

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3, \tag{1}$$

where f_1, f_2, f_3 are some differentiable functions on M and

$$\begin{aligned} R_1(X,Y)Z &= g(Y,Z)X - g(X,Z)Y, \\ R_2(X,Y)Z &= g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z, \\ R_3(X,Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \end{aligned}$$

for any vector fields X, Y, Z on M.
By motivating the works on generalized Sasakian-space forms and (k, μ)-space forms, the authors [4] introduced the thought of generalized (k, μ)-space forms. A generalized (k, μ)-space form as an almost contact metric manifold (M, ϕ, ξ, η, g) whose curvature tensor are often written as

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3 + f_4 R_4 + f_5 R_5 + f_6 R_6, \tag{2}$$

where $f_1, f_2, f_3, f_4, f_5, f_6$ are some differentiable functions on M, R_1, R_2, R_3 are the tensors defined above and

$$\begin{aligned} R_4(X,Y)Z &= g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y, \\ R_5(X,Y)Z &= g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX, \\ R_6(X,Y)Z &= \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi, \end{aligned}$$

where $2h = L_\xi \phi$ and L is the usual Lie derivative and we will denote such a manifold by $M(f_1, f_2, f_3, f_4, f_5, f_6)$.
Natural examples of generalized (k, μ)-space forms are (k, μ)-space forms and generalized Sasakian space forms. The authors in [1] established that contact metric generalized (k, μ)-space forms are generalized (k, μ) spaces and if dimension is greater than or equal to 5, then they are (k, μ) spaces with constant ϕ-sectional curvature $2f_6 - 1$. They gave a technique of constructing examples of generalized (k, μ)-space forms and established that generalized (k, μ)-space forms with trans-Sasakian structure reduces to generalized Sasakian space forms. More in [3], it is

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proved that under D_a-homothetic deformation, generalized (k,μ)-space form structure is preserved for dimension 3, but not in general. (k,μ)-space form have been studied widely by several authors like [6, 10, 15, 16] and various others.

Ricci soliton, introduced by Hamilton [7] are natural generalizations of the Einstein metrics and is defined on a Riemannian manifold (M,g). A Ricci soliton (g,V,λ) defined on (M,g) as

$$(L_V g)(X,Y)+2S(X,Y)+2\lambda g(X,Y)=0, \tag{3}$$

where L_V denotes the Lie-derivative of Riemannian metric g along a vector field V, λ be a consant and X,Y are arbitrary vector fields on M. A Ricci soliton is said to shrinking or steady or expanding to the extent that λ is negative, zero or positive respectively. Ricci solitons have been considered broadly with regards to contact geometry; we may refer to [5, 8, 14, 17, 18] and references therein.

The paper is organized as follows: The section 2 contains some basic results on almost contact geometry and generalized (k,μ)-space forms. In section 3, it is shown that if a generalized (k,μ)-Space form admits a second order symmetric parallel tensor is a constant multiple of the associated metric tensor. We also obtain that on a generalized (k,μ)-Space form with $k\neq 0$, there is no nonzero second order skew-symmetric parallel tensor provided that $(f_1-f_3)^2+(k-1)(f_4-f_6)^2\neq 0$. Finally we studied Ricci solitons in generalized (k,μ)-Space form and obtained some interesting results.

2. Preliminaries

In this section, we recall some general definitions and fundamental equations are presented which will be utilized later. A $(2n+1)$-dimensional smooth manifold M is said to be contact if it has a global 1-form η such that $\eta\wedge(d\eta)^n\neq 0$ on M. Given a contact 1-form η there always exists a unique vector field ξ such that $(d\eta)(\xi,X)=0$. Polarization of $d\eta$ on the contact subbundle D (defined by $D=0$), yields a Riemannian metric g and a $(1,1)$-tensor field ϕ such that

$$\phi^2X=-X+\eta(X)\xi,\quad \phi\xi=0,\quad g(X,\xi)=\eta(X),\quad \eta(\xi)=1,\quad \eta\circ\phi=0, \tag{4}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{5}$$

$$g(X,\phi Y)=d\eta(X,Y),\qquad g(X,\phi Y)=-g(Y,\phi X). \tag{6}$$

for all vector fields X,Y on M. In a contact metric manifold, we characterize a $(1,1)$ tensor field h by $h=\frac{1}{2}L_\xi\phi$, where L signifies the Lie differentiation. At this point h is symmetric and satisifies $h\phi=-\phi h$. Likewise we have $Tr\cdot h=Tr\cdot\phi h=0$ and $h\xi=0$.

Moreover, if ∇ signifies the Riemannian connection of g, then the following relation holds:

$$\nabla_X\xi=-\phi X-\phi hX. \tag{7}$$

In a (k,μ)-contact metric manifold the following relations hold [2, 9]:

$$\begin{aligned} h^2 &= (k-1)\phi^2,\quad k\leq 1, && (8)\\ (\nabla_X\phi)Y &= g(X+hX,Y)\xi-\eta(Y)(X+hX), && (9)\\ (\nabla_X h)Y &= [(1-k)g(X,\phi Y)-g(X,\phi hY)]\xi \\ &- \eta(Y)[(1-k)\phi X+\phi hX]-\mu\eta(X)\phi hY. && (10)\end{aligned}$$

Also in a $(2n+1)$-dimensional generalized (k,μ)-space form, the following relations hold :

$$\begin{aligned} R(X,Y)\xi &= (f_1-f_3)\{\eta(Y)X-\eta(X)Y\} \\ &+ (f_4-f_6)\{\eta(Y)hX-\eta(X)hY\}, && (11)\\ R(\xi,X)Z &= (f_1-f_3)\{g(X,Z)\xi-\eta(Z)X\} \\ &+ (f_4-f_6)\{g(hX,Z)\xi-\eta(Z)hX\}, && (12)\\ QX &= \{2nf_1+3f_2-f_3\}X+\{(2n-1)f_4-f_6\}hX \\ &- \{3f_2+(2n-1)f_3\}\eta(X)\xi, && (13)\\ S(X,Y) &= \{2nf_1+3f_2-f_3\}g(X,Y)+\{(2n-1)f_4-f_6\}g(hX,Y) \\ &- \{3f_2+(2n-1)f_3\}\eta(X)\eta(Y), && (14)\\ S(X,\xi) &= 2n(f_1-f_3)\eta(X), && (15)\\ r &= 2n\{(2n+1)f_1+3f_2-2f_3\}, && (16)\end{aligned}$$

where Q is the Ricci operator, S is the Ricci tensor and r is the scalar curvature of $M(f_1.....,f_6)$.

3. Second Order Parallel Tensor and Ricci Solitons

In this section, we consider a second order symmetric parallel tensor on generalized (k,μ)-contact metric manifolds. Mondal et al. [13], De et al. [5] obtained some classification results on second order parallel tensors in (k,μ)-contact metric manifolds.

Definition 3.1. (see [11, 19]) Let M be a Riemannian manifold with metric g, ξ an unitary vector field, η be the 1-form dual to ξ. Further, let ρ be a symmetric tensor field of $(0,2)$-type on M which we suppose to be parallel with respect to ∇ that is $\nabla\rho = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

Suppose ρ be a second order symmetric tensor field, that is, $\rho(X,Y) = \rho(Y,X)$ on a generalized (k,μ)-space form $M(f_1.....,f_6)$, such that $\nabla\rho = 0$. Then it follows that

$$\nabla^2\rho(X,Y;Z,W) - \nabla^2\rho(X,Y;W,Z) = 0. \tag{17}$$

From (17), we obtain the relation:

$$\rho(R(X,Y)Z,W) + \rho(R(X,Y)W,Z) = 0, \tag{18}$$

for arbitrary vector fields X,Y,Z on M.
Substitution of $X = Z = W = \xi$ in (18) gives us

$$\rho(\xi,R(\xi,Y)\xi) = 0. \tag{19}$$

Using (11) in (19), we get

$$(f_1 - f_3)\{\eta(Y)\rho(\xi,\xi) - \rho(\xi,Y)\} = 0. \tag{20}$$

Supposing $(f_1 - f_3) \neq 0$, (20) reduces to

$$\eta(Y)\rho(\xi,\xi) - \rho(\xi,Y) = 0. \tag{21}$$

Taking the covariant differentiation of (21) with respect to X, we get

$$\begin{aligned} &g(\nabla_X Y,\xi)\rho(\xi,\xi) + g(Y,\nabla_X\xi)\rho(\xi,\xi) + 2g(Y,\xi)\rho(\nabla_X\xi,\xi) \\ &-\rho(\nabla_X\xi,Y) - \rho(\xi,\nabla_X Y) = 0. \end{aligned} \tag{22}$$

Replacing Y by $\nabla_X Y$ in (21), we obtain

$$g(\nabla_X Y,\xi)\rho(\xi,\xi) - \rho(\xi,\nabla_X Y) = 0. \tag{23}$$

In view of (23), it follows from (22) that

$$g(Y,\nabla_X\xi)\rho(\xi,\xi) + 2g(Y,\xi)\rho(\nabla_X\xi,\xi) - \rho(\nabla_X\xi,Y) = 0. \tag{24}$$

Using (7) in (24), we get

$$\rho(Y,\phi X) - \rho(Y,h\phi X) - \rho(\xi,\xi)g(Y,\phi X) + \rho(\xi,\xi)g(Y,h\phi X) = 0. \tag{25}$$

Replacing X by ϕX in (25) and then using (4), we obtain

$$\rho(Y,X) - \rho(\xi,\xi)g(X,Y) - \rho(Y,hX) + \rho(\xi,\xi)g(Y,hX) - \eta(X)\rho(Y,\xi) = 0. \tag{26}$$

Replacing X by hX in (26) and using (4) and (8), we get

$$\rho(Y,hX) - \rho(\xi,\xi)g(Y,hX) + (k-1)\{\rho(Y,X) - \rho(\xi,\xi)g(X,Y)\}. \tag{27}$$

Using (26) in (27), we obtain

$$k\{\rho(Y,X) - \rho(\xi,\xi)g(X,Y)\} = 0. \tag{28}$$

Since $k \neq 0$, it follows that

$$\rho(Y,X) = \rho(\xi,\xi)g(X,Y). \tag{29}$$

Thus, we can state the following:

Theorem 3.2. *A symmetric parallel second order covariant tensor in a generalized (k,μ)-space form $M(f_1.....,f_6)$, with $f_1 \neq f_3$ is a constant multiple of the metric tensor.*

As an immediate corollary of theorem 3.1 we have the following result.

Corollary 3.3. *A locally Ricci symmetric ($\nabla S = 0$) generalized (k,μ)-space form $M(f_1.....,f_6)$, with $f_1 \neq f_3$ is an Einstein manifold.*

Next, we consider, let $M(f_1.....,f_6)$ be a generalized (k,μ)-space form admitting second order skew-symmetric parallel tensor ρ [12]. Putting $Y = W = \xi$ in (18) and using (12), we get

$$(f_1 - f_3)\{\eta(X)\rho(\xi,Z) - \rho(X,Z) - \eta(Z)\rho(\xi,X)\} \\ = (f_4 - f_6)\{\rho(hX,Z) + \eta(Z)\rho(\xi,hX)\}. \tag{30}$$

Replacing X by hX in (30) and using (8), we get

$$(f_1 - f_3)\{\rho(hX,Z) + \eta(Z)\rho(\xi,hX)\} \\ = (f_4 - f_6)(k-1)\{\rho(X,Z) - \eta(X)\rho(\xi,Z) + \eta(Z)\rho(\xi,X)\}. \tag{31}$$

Using (30) and (31), we obtain

$$\{(f_1 - f_3)^2 + (k-1)(f_4 - f_6)^2\}\{\eta(X)\rho(\xi,Z) \\ -\rho(X,Z) + \eta(Z)\rho(\xi,X)\} = 0. \tag{32}$$

Consider a non-empty open subset U of M such that $\{(f_1 - f_3)^2 + (k-1)(f_4 - f_6)^2\} \neq 0$, then we have

$$\rho(X,Z) - \eta(X)\rho(\xi,Z) + \eta(Z)\rho(\xi,X) = 0. \tag{33}$$

Now, let A be a $(1,1)$-type tensor field which is metrically equivalent to ρ, that is, $\rho(X,Y) = g(AX,Y)$, Then from (33), we have

$$g(AX,Z) = \eta(X)g(A\xi,Z) - \eta(Z)g(A\xi,X), \tag{34}$$

and thus

$$AX = \eta(X)A\xi - g(A\xi,X)\xi. \tag{35}$$

From (35), we can see if $A\xi = 0$, then $AX = 0$, and hence $\rho = 0$.
Now, we suppose that $A\xi \neq 0$, let (35) take the inner product with $A\xi$, we obtain $g(A\xi,AX) = \eta(X)g(A\xi,A\xi)$. So it holds

$$A^2\xi = -g(A\xi,A\xi)\xi. \tag{36}$$

Differentiating the above equation covariantly along X, we obtain

$$\nabla_X A^2\xi = A^2\nabla_X\xi = A^2(-\phi X - \phi hX), \tag{37}$$

$$\begin{aligned} \nabla_X A^2\xi &= 2g(A^2\xi,\nabla_X\xi)\xi + g(A^2\xi,\xi)\nabla_X\xi, \\ &= g(A\xi,A\xi)(\phi X + \phi hX). \end{aligned} \tag{38}$$

Combining (37) with (38), it follows that

$$A^2\phi X + A^2\phi hX + g(A\xi,A\xi)(\phi X + \phi hX) = 0. \tag{39}$$

Replacing X by hX and using (8), we obtain

$$A^2\phi hX - (k-1)A^2\phi X + g(A\xi,A\xi)(\phi hX - (k-1)\phi X) = 0. \tag{40}$$

From (39) and (40), we have

$$k\{A^2\phi X + g(A\xi,A\xi)\phi X\} = 0. \tag{41}$$

Replacing ϕX by X in (41) to get

$$k\{A^2X + g(A\xi,A\xi)X\} = 0. \tag{42}$$

If $k \neq 0$ implies

$$A^2X = -g(A\xi,A\xi)X = -\|A\xi\|^2X. \tag{43}$$

Now, if $\|A\xi\| \neq 0$, then $J = \frac{1}{\|A\xi\|}A$ is an almost complex structure on U. In fact, (J,g) is a Kaehler structure on U. The fundamental second order skew-symmetric parallel tensor is $g(JX,Y) = \frac{1}{\|A\xi\|}g(AX,Y) = \frac{1}{\|A\xi\|}\rho(X,Y)$ with $\frac{1}{\|A\xi\|} = constant$. But (34) implies ρ is degenerate, which is a contradition. So $\|A\xi\| = 0$ and hence $\rho = 0$. Thus we state the following:

Theorem 3.4. *In a generalized (k,μ)-space form $M(f_1.....,f_6)$ with $k\neq 0$, there is no nonzero second order skew-symmetric parallel tensor provided that $\{(f_1-f_3)^2+(k-1)(f_4-f_6)^2\}\neq 0$.*

A straightforward computation gives

$$(L_\xi g)(X,Y)=-2g(\phi hX,Y). \tag{44}$$

The metric g is called η-Einstein if there exists two real functions a and b such that the Ricci tensor S of g is given by

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y). \tag{45}$$

Let $\{e_1,e_2,e_3,.......e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Then by taking $X=Y=e_i$ in (45) and summing up with respect to i, we obtain

$$r=(2n+1)a+b. \tag{46}$$

Again by taking $X=Y=\xi$, in (45) and then using (4) and (15), we get

$$2n(f_1-f_3)=a+b. \tag{47}$$

From (46) and (47), we obtain

$$a=\frac{r}{2n}-(f_1-f_3) \qquad b=(2n+1)(f_1-f_3)-\frac{r}{2n}. \tag{48}$$

Substituting the values of a and b in (45), we get

$$\begin{aligned}S(X,Y)&=\{\frac{r}{2n}-(f_1-f_3)\}g(X,Y)\\&+\{(2n+1)(f_1-f_3)-\frac{r}{2n}\}\eta(X)\eta(Y).\end{aligned} \tag{49}$$

Suppose

$$\rho(X,Y)=(L_\xi g)(X,Y)+2S(X,Y). \tag{50}$$

Using (44) and (49) in (50), we obtain

$$\begin{aligned}\rho(X,Y)&=\{\frac{r}{n}-2(f_1-f_3)\}g(X,Y)\\&+\{2(2n+1)(f_1-f_3)-\frac{r}{n}\}\eta(X)\eta(Y)-2g(\phi hX,Y).\end{aligned} \tag{51}$$

Taking $X=Y=\xi$ in (51), we get

$$\rho(\xi,\xi)=4n(f_1-f_3). \tag{52}$$

If (g,ξ,λ) is a Ricci soliton on a generalized (k,μ)-space form $M(f_1.....,f_6)$, then from (3) and (50), we have

$$\rho(X,Y)=-2\lambda g(X,Y). \tag{53}$$

Setting $X=Y=\xi$ in (53), we get

$$\rho(\xi,\xi)=-2\lambda. \tag{54}$$

Hence from (52) and (54), we have

$$\lambda=-2n(f_1-f_3). \tag{55}$$

Thus we state the following:

Theorem 3.5. *If the tensor field $L_\xi g+2S$ on a generalized (k,μ)-space form $M(f_1.....,f_6)$ is parallel, then the Ricci soliton (g,ξ,λ) is shrinking if $f_1>f_3$ or expanding if $f_1<f_3$ or steady if $f_1=f_3$.*

Taking $V=\xi$ in (3), then we have

$$(L_\xi g)(X,Y)+2S(X,Y)+2\lambda g(X,Y)=0. \tag{56}$$

Making use of (14) and (44) in (56), we obtain

$$\begin{aligned}&-g(\phi hX,Y)+\{2nf_1+3f_2-f_3+\lambda\}g(X,Y)\\&+\{(2n-1)f_4-f_6\}g(hX,Y)-\{3f_2+(2n-1)f_3\}\eta(X)\eta(Y)=0.\end{aligned} \tag{57}$$

Replacing X by hX and using (4) and (8) in (57), we obtain

$$\begin{aligned}&(k-1)g(\phi X,Y)+\{2nf_1+3f_2-f_3+\lambda\}g(hX,Y)\\&+((2n-1)f_4-f_6)(k-1)\{-g(X,Y)+\eta(X)\eta(Y)\}=0.\end{aligned} \tag{58}$$

By taking $X=Y=e_i$, where $\{e_i : i=1,2,3,........,2n+1\}$ is an orthonormal basis, we get

$$-2n(k-1)\{(2n-1)f_4-f_6\}=0. \tag{59}$$

If $(2n-1)f_4\neq f_6$, then we must have $k=1$. Thus we state the following:

Theorem 3.6. *If a $(2n+1)$-dimensional generalized (k,μ)-space form $M(f_1.....,f_6)$ admitting a Ricci soliton with $(2n-1)f_4\neq f_6$, then $k=1$. i.e. M is Sasakian.*

A vector field V on a Kenmotsu manifold is said to be conformal Killing vector field [20] if

$$(L_V g)(X,Y)=2\sigma g(X,Y), \tag{60}$$

where σ is a function on the manifold.
Let (g,V,λ) be a Ricci soliton in a 3 dimensional generalized (k,μ)-space form $M(f_1.....,f_6)$. Then from (60) and (3), we have

$$S(X,Y)=-(\lambda+\sigma)g(X,Y), \tag{61}$$

which yields

$$QX=-(\lambda+\sigma)X, \tag{62}$$

$$S(X,\xi)=-(\lambda+\sigma)\eta(X), \tag{63}$$

$$r=-3(\lambda+\sigma). \tag{64}$$

Since in a three-dimensional Riemannian manifold the conformal curvature tensor C vanishes, we have

$$\begin{aligned}R(X,Y)Z=&\,g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y\\&-\frac{r}{2}[g(Y,Z)X-g(X,Z)Y],\end{aligned} \tag{65}$$

where R is Riemannian curvature tensor of type (1,3).
Using (62), (63) and (64) in (65) and by taking $Z=\xi$, we get

$$R(X,Y)\xi=\frac{(\lambda+\sigma)}{2}\{\eta(X)Y-\eta(Y)X\}. \tag{66}$$

By comparing (11) and (66), we obtain

$$\lambda=-\{2(f_1-f_3)+\sigma\} \quad \text{and} \quad f_4=f_6. \tag{67}$$

This leads to the following:

Theorem 3.7. *If the generating vector field V is a conformal Killing vector field with associated function σ, then the Ricci soliton in a three-dimensional generalized (k,μ)-space form $M(f_1.....,f_6)$ is shrinking if $f_1<f_3$ or expanding if $f_1>f_3$ or steady if $f_4=f_6$.*

Replacing Y by hY in (11) and (66), then by comparing and using (8), we get

$$\{\frac{\lambda+\sigma}{2}+f_1-f_3\}\eta(X)hY+(k-1)(f_4-f_6)\eta(Y)\phi^2X=0. \tag{68}$$

Taking $Y=\xi$ in (68), we get $k=1$ or $f_4=f_6$. Thus we state the following:

Theorem 3.8. *In a three-dimensional generalized (k,μ)-space form $M(f_1.....,f_6)$ admitting a Ricci soliton (g,V,λ), where V is a conformal Killing vector field with associated function σ, then $k=1$ or $f_4=f_6$.*

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