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Direct Estimate of Accumulated Errors for a General Iteration Method

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Abstract: We study error analysis for an SP iteration mehod with mixed errors defined by Chugh and Kumar in [Convergence of SP iterative scheme with mixed errors for accretive Lipschitzian and strongly accretive Lipschitzian operators in Banach spaces, Int. J. Comput. Math. 90 (2013), pp. 1865-1880]; in particular, controllability and accumulation of errors for such an iteration method are investigated.

Throughout this paper, we suppose that $(X, \|\cdot\|)$ is an arbitrary real Banach space, S a nonempty closed and convex subset of X, $T : S \to S$ an operator, and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty} \subseteq [0,1]$ are parameter sequences satisfying certain control condition(s).
In 2011, Phuengrattana and Suantai [1] defined the SP iteration method on S as follows:

$$\begin{cases} x_0 \in S, \\ x_{n+1} = (1-\alpha_n) y_n + \alpha_n T y_n \\ y_n = (1-\beta_n) z_n + \beta_n T z_n \\ z_n = (1-\gamma_n) x_n + \gamma_n T x_n, \text{ for all } n \in \mathbb{N}. \end{cases} \tag{1}$$

Remark 0.1. The iteration method (1) reduces to:
(i) Mann iteration method [3] if $\beta_n = \gamma_n = 0$ for all $n \in \mathbb{N}$;
(ii) Picard iteration method [2] if $\alpha_n = 1$, $\beta_n = \gamma_n = 0$ for all $n \in \mathbb{N}$.
However, the iteration method (1) is independent of Ishikawa [4] and Noor [5] iteration methods.

Recently, Chugh and Kumar [6] introduced the SP iteration method with mixed errors as

$$\begin{cases} x_0 \in S, \\ x_{n+1} = (1-\alpha_n) y_n + \alpha_n T y_n + u_n \\ y_n = (1-\beta_n) z_n + \beta_n T z_n + v_n \\ z_n = (1-\gamma_n) x_n + \gamma_n T x_n + w_n, \text{ for all } n \in \mathbb{N}, \end{cases} \tag{2}$$

where $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are sequences in S satisfying the following conditions:
(i) $u_n = u_n' + u_n''$, $\|u_n'\| = o(\alpha_n)$ $(n \geq 0)$ and $\sum_{n=0}^{\infty} \|u_n''\| < \infty$,
(ii) $\sum_{n=0}^{\infty} \|v_n\| < \infty$, $\sum_{n=0}^{\infty} \|w_n\| < \infty$.

Remark 0.2. The iteration method (2) reduces to:
(i) Mann iteration method [3] if $\beta_n = \gamma_n = w_n = v_n = u_n = 0$ for all $n \in \mathbb{N}$;
(ii) Picard iteration method [2] if $\alpha_n = 1$, $\beta_n = \gamma_n = w_n = v_n = u_n = 0$ for all $n \in \mathbb{N}$;
(iii) SP iteration method (1) if $w_n = v_n = u_n = 0$ for all $n \in \mathbb{N}$.
However, the iteration method (2) is independent of Ishikawa [4] and Noor [5] iteration methods.

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We are primarily interested in evaluating the error estimate for the SP iteration method (1) of an operator on a real Banach space. Many researchers have achieved this in an indirect way. As regards, their direct calculations (estimation), recently some papers have appeared in the literature (see, e.g., [7, 8]). In this paper, we develop new ideas for the direct error estimation of the SP iteration(1) in respect of accumulation and control of random errors. It is remarked that direct calculations of errors for this method are much more complicated as compared to those in the case of Mann and Ishikawa iteration methods (cf. [7, 8]).
Define the errors of Tx_n, Ty_n and Tz_n by

$$w_n = Tx_n - \overline{Tx_n},\ v_n = Tz_n - \overline{Tz_n} \text{ and } u_n = Ty_n - \overline{Ty_n} \tag{3}$$

for all $n \in \mathbb{N}$, where $\overline{Tx_n}$, $\overline{Ty_n}$ and $\overline{Tz_n}$ are the exact values of Tx_n, Ty_n and Tz_n, respectively, that is, Tx_n, Ty_n and Tz_n are approximate values of $\overline{Tx_n}$, $\overline{Ty_n}$ and $\overline{Tz_n}$, respectively. The theory of errors implies that $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are bounded. Set

$$B = \max\{B_u, B_v, B_w\} \tag{4}$$

where $B_w = \sup_{n\in\mathbb{N}} \|w_n\|$, $B_v = \sup_{n\in\mathbb{N}} \|v_n\|$ and $B_u = \sup_{n\in\mathbb{N}} \|u_n\|$ are the bounds on the absolute errors of $\{Tx_n\}_{n=0}^{\infty}$, $\{Tz_n\}_{n=0}^{\infty}$ and $\{Ty_n\}_{n=0}^{\infty}$, respectively.
The accumulated errors in (1) comes from u_n, v_n and w_n; hence we can set

$$\begin{cases} \overline{x_0} \in S, \\ \overline{x_{n+1}} = (1-\alpha_n)\overline{y_n} + \alpha_n \overline{Ty_n} \\ \overline{y_n} = (1-\beta_n)\overline{z_n} + \beta_n \overline{Tz_n} \\ \overline{z_n} = (1-\gamma_n)\overline{x_n} + \gamma_n \overline{Tx_n}, \text{ for all } n \in \mathbb{N}. \end{cases} \tag{5}$$

where $\overline{x_n}$, $\overline{y_n}$ and $\overline{z_n}$ are exact values of x_n, y_n and z_n, respectively. Obviously, error of an iteration will affect the next $(n+1)$ steps. So, utilizing (1), (3) and (5), we have

$$\begin{aligned}
x_0 &= \overline{x_0};\\
x_1 &= \overline{x_1} + (1-\alpha_0)[(1-\beta_0)\gamma_0 w_0 + \beta_0 v_0] + \alpha_0 u_0;\\
x_2 &= \overline{x_2} + (1-\alpha_1)(1-\beta_1)(1-\gamma_1)[(1-\alpha_0)[(1-\beta_0)\gamma_0 w_0 + \beta_0 v_0] + \alpha_0 u_0]\\
&\quad + (1-\alpha_1)[(1-\beta_1)\gamma_1 w_1 + \beta_1 v_1] + \alpha_1 u_1;\\
x_3 &= \overline{x_3} + (1-\alpha_2)(1-\beta_2)(1-\gamma_2)(1-\alpha_1)(1-\beta_1)(1-\gamma_1)[(1-\alpha_0)[(1-\beta_0)\gamma_0 w_0 + \beta_0 v_0] + \alpha_0 u_0]\\
&\quad + (1-\alpha_2)(1-\beta_2)(1-\gamma_2)[(1-\alpha_1)[(1-\beta_1)\gamma_1 w_1 + \beta_1 v_1] + \alpha_1 u_1]\\
&\quad + (1-\alpha_2)[(1-\beta_2)\gamma_2 w_2 + \beta_2 v_2] + \alpha_2 u_2;
\end{aligned}$$

$$\begin{aligned}
y_0 &= \overline{y_0} + (1-\beta_0)\gamma_0 w_0 + \beta_0 v_0;\\
y_1 &= \overline{y_1} + (1-\beta_1)\gamma_1 w_1 + \beta_1 v_1 + (1-\beta_1)(1-\gamma_1)[(1-\alpha_0)[(1-\beta_0)\gamma_0 w_0 + \beta_0 v_0] + \alpha_0 u_0];\\
y_2 &= \overline{y_2} + (1-\beta_2)\gamma_2 w_2 + \beta_2 v_2\\
&\quad + (1-\beta_2)(1-\gamma_2)[(1-\alpha_1)(1-\beta_1)(1-\gamma_1)[(1-\alpha_0)[(1-\beta_0)\gamma_0 w_0 + \beta_0 v_0] + \alpha_0 u_0]\\
&\quad + (1-\alpha_1)[(1-\beta_1)\gamma_1 w_1 + \beta_1 v_1] + \alpha_1 u_1];\\
y_3 &= \overline{y_3} + (1-\beta_3)\gamma_3 w_3 + \beta_3 v_3 + (1-\beta_3)(1-\gamma_3)\times\\
&\quad [(1-\alpha_2)(1-\beta_2)(1-\gamma_2)(1-\alpha_1)(1-\beta_1)(1-\gamma_1)[(1-\alpha_0)[(1-\beta_0)\gamma_0 w_0 + \beta_0 v_0] + \alpha_0 u_0]\\
&\quad + (1-\alpha_2)(1-\beta_2)(1-\gamma_2)[(1-\alpha_1)[(1-\beta_1)\gamma_1 w_1 + \beta_1 v_1] + \alpha_1 u_1]\\
&\quad + (1-\alpha_2)[(1-\beta_2)\gamma_2 w_2 + \beta_2 v_2] + \alpha_2 u_2];
\end{aligned}$$

$$\begin{aligned}
z_0 &= \overline{z_0} + \gamma_0 w_0;\\
z_1 &= \overline{z_1} + \gamma_1 w_1 + (1-\gamma_1)[(1-\alpha_0)[(1-\beta_0)\gamma_0 w_0 + \beta_0 v_0] + \alpha_0 u_0];\\
z_2 &= \overline{z_2} + \gamma_2 w_2 + (1-\gamma_2)[(1-\alpha_1)(1-\beta_1)(1-\gamma_1)[(1-\alpha_0)[(1-\beta_0)\gamma_0 w_0 + \beta_0 v_0] + \alpha_0 u_0]\\
&\quad + (1-\alpha_1)[(1-\beta_1)\gamma_1 w_1 + \beta_1 v_1] + \alpha_1 u_1];\\
z_3 &= \overline{z_3} + \gamma_3 w_3 + (1-\gamma_3)\times\\
&\quad [(1-\alpha_2)(1-\beta_2)(1-\gamma_2)(1-\alpha_1)(1-\beta_1)(1-\gamma_1)[(1-\alpha_0)[(1-\beta_0)\gamma_0 w_0 + \beta_0 v_0] + \alpha_0 u_0]\\
&\quad + (1-\alpha_2)(1-\beta_2)(1-\gamma_2)[(1-\alpha_1)[(1-\beta_1)\gamma_1 w_1 + \beta_1 v_1] + \alpha_1 u_1]\\
&\quad + (1-\alpha_2)[(1-\beta_2)\gamma_2 w_2 + \beta_2 v_2] + \alpha_2 u_2];
\end{aligned}$$

A repetition of the above arguments, gives:

$$x_{n+1} = \overline{x_{n+1}} + \sum_{k=0}^{n} \left[(1-\alpha_k)\left[(1-\beta_k)\gamma_k w_k + \beta_k v_k\right] + \alpha_k u_k\right] \left[\prod_{i=k+1}^{n} (1-\alpha_i)(1-\beta_i)(1-\gamma_i)\right],$$

$$\begin{aligned} y_n &= \overline{y_n} + (1-\beta_n)\gamma_n w_n + \beta_n v_n + (1-\beta_n)(1-\gamma_n) \times \\ &\quad \sum_{k=0}^{n-1} \left[(1-\alpha_k)\left[(1-\beta_k)\gamma_k w_k + \beta_k v_k\right] + \alpha_k u_k\right] \left[\prod_{i=k+1}^{n-1} (1-\alpha_i)(1-\beta_i)(1-\gamma_i)\right] \\ &= (1-\beta_n)\gamma_n w_n + \beta_n v_n + (1-\beta_n)(1-\gamma_n)(x_n - \overline{x_n}), \end{aligned}$$

and

$$\begin{aligned} z_n &= \overline{z_n} + \gamma_n w_n + (1-\gamma_n) \times \\ &\quad \sum_{k=0}^{n-1} \left[(1-\alpha_k)\left[(1-\beta_k)\gamma_k w_k + \beta_k v_k\right] + \alpha_k u_k\right] \left[\prod_{i=k+1}^{n-1} (1-\alpha_i)(1-\beta_i)(1-\gamma_i)\right] \\ &= \gamma_n w_n + (1-\gamma_n)(x_n - \overline{x_n}) \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Define

$$E_n^{(1)} := x_{n+1} - \overline{x_{n+1}} = \sum_{k=0}^{n} \left[(1-\alpha_k)\left[(1-\beta_k)\gamma_k w_k + \beta_k v_k\right] + \alpha_k u_k\right] \left[\prod_{i=k+1}^{n} (1-\alpha_i)(1-\beta_i)(1-\gamma_i)\right], \tag{6}$$

$$E_n^{(2)} := y_n - \overline{y_n} = (1-\beta_n)\gamma_n w_n + \beta_n v_n + (1-\beta_n)(1-\gamma_n) E_{n-1}^{(1)}, \tag{7}$$

and

$$E_n^{(3)} := z_n - \overline{z_n} = \gamma_n w_n + (1-\gamma_n) E_{n-1}^{(1)} \text{ for all } n \in \mathbb{N}. \tag{8}$$

We note that the errors of the iteration method, after $(n+1)$ times iterations, are added up to $E_n^{(1)}$, $E_n^{(2)}$ and $E_n^{(3)}$. Now, we are in a position to give the following result.

Theorem 0.3. *Let S, T, B, $E_n^{(1)}$, $E_n^{(2)}$ and $E_n^{(3)}$ be as above.*
(i) If $\sum_{i=0}^{\infty} \alpha_i = +\infty$ (or $\sum_{i=0}^{\infty} \beta_i = +\infty$, or $\sum_{i=0}^{\infty} \gamma_i = +\infty$), then the accumulation of errors in (1) is bounded and does not exceed the number B;
(ii) If $\sum_{i=0}^{\infty} \left[(1-\alpha_i)\left[(1-\beta_i)\gamma_i + \beta_i\right] + \alpha_i\right] < +\infty$, $\lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} \gamma_n = 0$, then random errors of (1) are controllable.

Proof. (i) It is well known that $\sum_{i=0}^{\infty} \alpha_i = +\infty$ implies $\prod_{i=0}^{\infty} (1-\alpha_i) = 0$ (see, e.g., (Remark 2.1 of [9])). From (4), (6)-(8) we have

$$\begin{aligned} \left\| E_n^{(1)} \right\| &= \Bigg\| \left[(1-\alpha_0)\left[(1-\beta_0)\gamma_0 w_0 + \beta_0 v_0\right] + \alpha_0 u_0\right] \prod_{i=1}^{n} (1-\alpha_i)(1-\beta_i)(1-\gamma_i) \\ &\quad + \left[(1-\alpha_1)\left[(1-\beta_1)\gamma_1 w_1 + \beta_1 v_1\right] + \alpha_1 u_1\right] \prod_{i=2}^{n} (1-\alpha_i)(1-\beta_i)(1-\gamma_i) \\ &\quad + \cdots + (1-\alpha_n)\left[(1-\beta_n)\gamma_n w_n + \beta_n v_n\right] + \alpha_n u_n \Bigg\| \\ &\le \left\| \left[(1-\alpha_0)\left[(1-\beta_0)\gamma_0 w_0 + \beta_0 v_0\right] + \alpha_0 u_0\right] \prod_{i=1}^{n} (1-\alpha_i)(1-\beta_i)(1-\gamma_i) \right\| \\ &\quad + \left\| \left[(1-\alpha_1)\left[(1-\beta_1)\gamma_1 w_1 + \beta_1 v_1\right] + \alpha_1 u_1\right] \prod_{i=2}^{n} (1-\alpha_i)(1-\beta_i)(1-\gamma_i) \right\| \\ &\quad + \cdots + \left\| (1-\alpha_n)\left[(1-\beta_n)\gamma_n w_n + \beta_n v_n\right] + \alpha_n u_n \right\| \\ &\le \left[(1-\alpha_0)\left[(1-\beta_0)\gamma_0 \|w_0\| + \beta_0 \|v_0\|\right] + \alpha_0 \|u_0\|\right] \prod_{i=1}^{n} (1-\alpha_i)(1-\beta_i)(1-\gamma_i) \\ &\quad + \left[(1-\alpha_1)\left[(1-\beta_1)\gamma_1 \|w_1\| + \beta_1 \|v_1\|\right] + \alpha_1 \|u_1\|\right] \prod_{i=2}^{n} (1-\alpha_i)(1-\beta_i)(1-\gamma_i) \\ &\quad + \cdots + (1-\alpha_n)\left[(1-\beta_n)\gamma_n \|w_n\| + \beta_n \|v_n\|\right] + \alpha_n \|u_n\|, \end{aligned}$$

which implies

$$\begin{aligned}
\left\|E_n^{(1)}\right\| &\leq B\left\{\prod_{i=0}^{n}(1-\alpha_i)(1-\beta_i)(1-\gamma_i)\right.\\
&\quad+[(1-\alpha_0)[(1-\beta_0)\gamma_0+\beta_0]+\alpha_0]\prod_{i=1}^{n}(1-\alpha_i)(1-\beta_i)(1-\gamma_i)\\
&\quad+[(1-\alpha_1)[(1-\beta_1)\gamma_1+\beta_1]+\alpha_1]\prod_{i=2}^{n}(1-\alpha_i)(1-\beta_i)(1-\gamma_i)\\
&\quad+\cdots+[(1-\alpha_n)[(1-\beta_n)\gamma_n+\beta_n]+\alpha_n]\\
&\quad\left.-\prod_{i=0}^{n}(1-\alpha_i)(1-\beta_i)(1-\gamma_i)\right\}\\
&= B\left[1-\prod_{i=0}^{n}(1-\alpha_i)(1-\beta_i)(1-\gamma_i)\right]\\
&= B\left[1-\prod_{i=0}^{n}(1-\alpha_i)\prod_{i=0}^{n}(1-\beta_i)\prod_{i=0}^{n}(1-\gamma_i)\right]\\
&\leq B\left[1-\prod_{i=0}^{\infty}(1-\alpha_i)\prod_{i=0}^{\infty}(1-\beta_i)\prod_{i=0}^{\infty}(1-\gamma_i)\right]=B, \qquad (9)
\end{aligned}$$

$$\begin{aligned}
\left\|E_n^{(2)}\right\| &= \left\|(1-\beta_n)\gamma_n w_n+\beta_n v_n+(1-\beta_n)(1-\gamma_n)E_{n-1}^{(1)}\right\|\\
&\leq (1-\beta_n)\gamma_n\|w_n\|+\beta_n\|v_n\|+(1-\beta_n)(1-\gamma_n)\left\|E_{n-1}^{(1)}\right\|\\
&\leq B[(1-\beta_n)\gamma_n+\beta_n+(1-\beta_n)(1-\gamma_n)]=B, \qquad (10)
\end{aligned}$$

and

$$\begin{aligned}
\left\|E_n^{(3)}\right\| &= \left\|\gamma_n w_n+(1-\gamma_n)E_{n-1}^{(1)}\right\|\\
&\leq \gamma_n\|w_n\|+(1-\gamma_n)\left\|E_{n-1}^{(1)}\right\|\\
&\leq \gamma_n B+(1-\gamma_n)B=B \text{ for all } n\in\mathbb{N}. \qquad (11)
\end{aligned}$$

Hence, we have $\max_{n\in\mathbb{N}}\left\{\left\|E_n^{(1)}\right\|,\left\|E_n^{(2)}\right\|,\left\|E_n^{(3)}\right\|\right\}\leq B$.

(ii) Indeed, $\sum_{i=0}^{\infty}[(1-\alpha_i)[(1-\beta_i)\gamma_i+\beta_i]+\alpha_i]<+\infty$ implies that

$$\prod_{i=0}^{\infty}(1-[(1-\alpha_i)[(1-\beta_i)\gamma_i+\beta_i]+\alpha_i])=\prod_{i=0}^{\infty}(1-\alpha_i)(1-\beta_i)(1-\gamma_i)\in(0,1).$$

Let $1-\prod_{i=0}^{\infty}(1-\alpha_i)(1-\beta_i)(1-\gamma_i)=\ell\in(0,1)$. Thus, from (9), we obtain

$$\left\|E_n^{(1)}\right\|\leq B\left[1-\prod_{i=0}^{\infty}(1-\alpha_i)(1-\beta_i)(1-\gamma_i)\right]\leq \ell B \text{ for all } n\in\mathbb{N}. \qquad (12)$$

On the other hand, the conditions $\lim_{n\to\infty}\beta_n=0$ and $\lim_{n\to\infty}\gamma_n=0$ imply that $\lim_{n\to\infty}(\gamma_n+\beta_n-\beta_n\gamma_n)=0$ which implies the existence of an $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ we have $\gamma_n+\beta_n-\beta_n\gamma_n\leq \ell/(1-\ell)$. Using this fact together with (10) and (12), we get

$$\begin{aligned}
\left\|E_n^{(2)}\right\| &\leq B[(1-\beta_n)\gamma_n+\beta_n+(1-\beta_n)(1-\gamma_n)\ell]\\
&= B[\ell+(\gamma_n+\beta_n-\beta_n\gamma_n)(1-\ell)]\\
&\leq B\left[\ell+\frac{\ell}{1-\ell}(1-\ell)\right]=2B\ell, \qquad (13)
\end{aligned}$$

Similarly, the condition $\lim_{n\to\infty}\gamma_n = 0$ implies the existence of an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\gamma_n \leq \ell/(1-\ell)$. Hence, from (11) and (12), we have

$$\begin{aligned} \left\|E_n^{(3)}\right\| &\leq \gamma_n \|w_n\| + (1-\gamma_n)\left\|E_{n-1}^{(1)}\right\| \\ &\leq \gamma_n B(1-\ell) + B\ell \\ &\leq \frac{\ell}{1-\ell} B(1-\ell) \ + B\ell = 2B\ell \text{ for all } n \geq n_0. \end{aligned} \tag{14}$$

Thus, we conclude that $\left\|E_n^{(1)}\right\|$, $\left\|E_n^{(2)}\right\|$ and $\left\|E_n^{(3)}\right\|$ can be controlled for suitable choice of the parameter sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ for all $n \geq n_0$. □

Example 0.4. Let $\alpha_n = \frac{1}{n^5+100}$, $\beta_n = \frac{1}{2n^4+500}$, $\gamma_n = \frac{1}{(n^2+300)^8}$ for all $n \in \mathbb{N}$. Then, by the comparison test, the series $\sum_{i=0}^{\infty}[(1-\alpha_i)[(1-\beta_i)\gamma_i+\beta_i]+\alpha_i]$ converges and the Wolfram Mathematica 9 software package implies that

$$\sum_{i=0}^{\infty}[(1-\alpha_i)[(1-\beta_i)\gamma_i+\beta_i]+\alpha_i] \approx 0.0416295 < +\infty$$

and

$$\ell = 1 - \prod_{i=0}^{\infty}(1-\alpha_i)(1-\beta_i)(1-\gamma_i) \approx 0.040967 \in (0,1).$$

Hence, it follows from (12)-(14) that $\left\|E_n^{(1)}\right\| \leq 0.040967 \times B$, $\left\|E_n^{(2)}\right\| \leq 0.081934 \times B$ and $\left\|E_n^{(3)}\right\| \leq 0.081934 \times B$ for all $n \in \mathbb{N}$.

Especially, for any $\varepsilon \in (0,1)$, if $(1-\alpha_n)[(1-\beta_n)\gamma_n+\beta_n]+\alpha_n = \frac{1}{2(2017^n+2017^{-n})}\varepsilon$ for all $n \in \mathbb{N}$, then

$$\begin{aligned} &\prod_{i=0}^{\infty}(1-\alpha_i)(1-\beta_i)(1-\gamma_i) \\ =\ &\prod_{i=0}^{\infty}(1-[(1-\alpha_i)[(1-\beta_i)\gamma_i+\beta_i]+\alpha_i]) \\ \geq\ &1-\sum_{i=0}^{\infty}[(1-\alpha_i)[(1-\beta_i)\gamma_i+\beta_i]+\alpha_i] \approx 1-0.250248\times\varepsilon, \end{aligned}$$

which yields $\ell < 0.250248 \times \varepsilon$, so that

$$\left\|E_n^{(1)}\right\| \leq 0.250248 \times \varepsilon \times B \text{ for all } n \in \mathbb{N},$$

and

$$\left\|E_n^{(2)}\right\| \leq 0.500496 \times \varepsilon \times B \text{ for all } n \geq n_0,$$

$$\left\|E_n^{(3)}\right\| \leq 0.500496 \times \varepsilon \times B \text{ for all } n \geq n_0,$$

where n_0 belong to $\mathbb{N}$ and the inequalities $\beta_n \leq \frac{\varepsilon}{3.99604-\varepsilon}$ and $\gamma_n \leq \frac{\varepsilon}{3.99604-\varepsilon}$ hold. Hence, the random errors is controllable in a permissible range for suitable choice of the parameter sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ for all $n \geq n_0$.

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