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ALMOST CONFORMAL TRANSFORMATION IN A FOUR-DIMENSIONAL RIEMANNIAN MANIFOLD WITH AN ADDITIONAL STRUCTURE

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Abstract. We are interested in four-dimensional Riemannian manifolds which admit an affinor structure. The local coordinates of the metric and the structure of these manifolds are special matrices. We define an almost conformal transformation using the metric and the structure. In this case we find some geometrical properties.

1. Introduction

It is well known that many works are devoted to manifolds with additional structure, whose square is the identity. Our purpose is to investigate cases where the degree of the affinor structure is more then two. On the other hand we think it is interesting to show how the circulation of the local coordinates of the structure reflects on the geometrical properties.

The papers [1], [2] contain the results in a class of three-dimensional Riemannian manifolds with an additional structure, whose third degree is the identity and the local coordinates of the structure are circulant.

In the present paper we consider a Riemannian manifold M admitting affinor structure q, whose forth degree is identity. We note, the local coordinates of the structures are circulant matrices. We study an almost conformal transformation in such manifolds.

Almost conformal transformation in B-manifolds was introduced in [5]. We use that idea and construct another metric $\tilde{g}$ on M with the help of the metric g and the affinor structure q. We find the conditions for $\tilde{g}$ to be a positively defined metric, and for q to be a parallel structure with respect to the Riemannian connection of $\tilde{g}$. We consider an arbitrary no-eigenvector w in χM and its images qw, q^2w. We

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express the angles between an w and qw, w and q^2w with respect to $\tilde{g}$, with the help of the angles between w and qw, w and q^2w with respect to g.

Also, we construct two series $\{\varphi_n\}$ and $\{\phi_n\}$ of the angles between w and qw, w and q^2w. We prove that they are convergent and tend to zero.

2. Preliminaries

We consider a 4-dimensional Riemannian manifold M with a metric g and an affinor structure q. The local coordinates of g and q are circulant matrices. The next conditions and results have been discussed in [3].

The metric g has coordinates:

$$(g_{ij}) = \begin{pmatrix} A & B & C & B \\ B & A & B & C \\ C & B & A & B \\ B & C & B & A \end{pmatrix}, \qquad A > C > B > 0 \tag{2.1}$$

in the local coordinate system (x_1, x_2, x_3, x_4), and $A = A(p), B = B(p), C = C(p)$, where $p(x_1, x_2, x_3, x_4) \in F \subset R^4$. Naturally, A, B, C are smooth functions of a point p. We calculate that $det(g_{ij}) = (A-C)^2((A+C)^2 - 4B^2) \neq 0$.

Further, let the local coordinates of q be

$$(q_i^{.j}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

We will use the notation $\Phi_i = \dfrac{\partial \Phi}{\partial x^i}$ for every smooth function Φ defined in F.

We know from [3] that the following identities are true

$$q^4 = E; \quad q^2 \neq \pm E; \tag{2.3}$$

$$g(qw, qv) = g(w, v), \quad w, v \in \chi M, \tag{2.4}$$

where E is the unit matrix;

$$0 < B < C < A \quad \Rightarrow \quad g \textit{ is possitively defined.} \tag{2.5}$$

Now, let $w = (x, y, z, u)$ be a vector in χM. Using (2.1) and (2.2) we calculate that

$$g(w, w) = A(x^2 + y^2 + z^2 + u^2) + 2B(xy + xu + yz + zu) + 2C(xz + yu) \tag{2.6}$$

$$g(w, qw) = (A + C)(xu + xy + yz + zu) + B(x^2 + y^2 + z^2 + u^2 + 2xz + 2yu) \tag{2.7}$$

$$g(w, q^2w) = 2A(xz + yu) + 2B(xu + xy + zy + zu) + C(x^2 + y^2 + z^2 + u^2). \tag{2.8}$$

Let M be the Riemannian manifold with a metric g and an affinor structure q, defined by (2.1) and (2.2), respectively. Let $w(x, y, z, u)$ be non-eigenvector on χM (i.e. $w(x, y, z, u) \neq (x, x, x, x)$, $w(x, y, z, u) \neq (x, -x, x, -x)$). If φ is the angle between w and qw, and ϕ is the angle between w and q^2w, then we have $\cos\varphi = \dfrac{g(w, qw)}{g(w, w)}$, $\cos\phi = \dfrac{g(w, q^2w)}{g(w, w)}$, $\varphi \in (0, \pi)$, $\phi \in (0, \pi)$.

We apply (2.6), (2.7) and (2.8) in the above equations and we get

$$\cos\varphi = \frac{(A + C)(xu + xy + yz + zu) + B(x^2 + y^2 + z^2 + u^2 + 2xz + 2yu)}{A(x^2 + y^2 + z^2 + u^2) + 2B(xy + xu + yz + zu) + 2C(xz + yu))},$$

$$\cos\phi = \frac{C(x^2+y^2+z^2+u^2)+2B(xy+xu+yz+zu)+2A(xz+yu)}{A(x^2+y^2+z^2+u^2)+2B(xy+xu+yz+zu)+2C(xz+yu))}.$$

3. Almost Conformal Transformation in M

Let M satisfies (2.1)– (2.5). We note $f_{ij} = g_{ik}q_t^k q_j^t$, i.e.

$$(f_{ij}) = \begin{pmatrix} C & B & A & B \\ B & C & B & A \\ A & B & C & B \\ B & A & B & C \end{pmatrix}. \tag{3.1}$$

We calculate $det(f_{ij}) = (C-A)^2((A+C)^2 - 4B^2) \neq 0$, so we accept f_{ij} for local coordinates of another metric f. The metric f is necessary undefined. Further, we suppose α and β are two smooth functions in $F \subset R^4$ and we construct the metric $\tilde{g}$, as follows:

$$\tilde{g} = \alpha.g + \beta.f. \tag{3.2}$$

We say that equation (3.2) define an almost conformal transformation, noting that if $\beta = 0$ then (3.2) implies the case of the classical conformal transformation in M [4].

From (2.1), (2.2), (3.1) and (3.2) we get the local coordinates of $\tilde{g}$:

$$(\tilde{g}_{ij}) = \begin{pmatrix} \alpha A+\beta C & (\alpha+\beta)B & \alpha C+\beta A & (\alpha+\beta)B \\ (\alpha+\beta)B & \alpha A+\beta C & (\alpha+\beta)B & \alpha C+\beta A \\ \alpha C+\beta A & (\alpha+\beta)B & \alpha A+\beta C & (\alpha+\beta)B \\ (\alpha+\beta)B & \alpha C+\beta A & (\alpha+\beta)B & \alpha A+\beta C \end{pmatrix}. \tag{3.3}$$

We see that (f_{ij}) and $(\tilde{g}_{ij})$ are both circulant matrices.

Theorem 3.1. [3] *Let M be a Riemannian manifold with a metric g from (2.1) and an affinor structure q from (2.2). Let ∇ be the Riemannian connection of g. Then $\nabla q = 0$ if and only if*

$$gradA = (gradC)q^2; \quad 2gradB = (gradC)(q+q^3). \tag{3.4}$$

Theorem 3.2. *Let M be a Riemannian manifold with a metric g from (2.1) and an affinor structure q from (2.2). Also, let $\tilde{g}$ be a metric of M, defined by (3.2). Let ∇ and $\tilde{\nabla}$ be the corresponding connections of g and $\tilde{g}$, and $\nabla q = 0$. Then $\tilde{\nabla} q = 0$ if and only if*

$$grad\alpha = grad\beta.q^2; \quad grad\beta = -grad\beta.q^2. \tag{3.5}$$

Proof. At first we suppose (3.5) is valid. Using (3.5) and (3.4) we can verify that the following identity is true:

$$\text{grad}(\alpha A+\beta C) = \text{grad}(\alpha C+\beta A).q^2,\ 2\text{grad}(\alpha+\beta)B = \text{grad}(\alpha C+\beta A).(q+q^3) \tag{3.6}$$

The identity (3.6) is analogue to (3.4), and consequently we conclude $\tilde{\nabla} q = 0$.

Inversely, if $\tilde{\nabla} q = 0$ then analogously to (3.4) we have (3.6). Now, (3.4) and (3.6) imply the system

$$A\text{grad}\alpha + C\text{grad}\beta = (C\text{grad}\alpha + A\text{grad}\beta)q^2 \tag{3.7}$$

$$2B(\text{grad}\alpha + \text{grad}\beta) = (C\text{grad}\alpha + A\text{grad}\beta)(q+q^3). \tag{3.8}$$

From (3.7) we find the only solution $\text{grad}\alpha = \text{grad}\beta.q^2$, and from (3.8) we get the only solution $\text{grad}\beta = -\text{grad}\beta.q^2$. So the theorem is proved. □

Lemma 3.1. *Let $\tilde{g}$ be the metric given by (3.2). If $0 < \beta < \alpha$ and g is positively defined, then $\tilde{g}$ is also positively defined.*

Proof. From the condition $(\alpha - \beta)(A - C) > 0$ we get $\alpha A + \beta C > \beta A + \alpha C > 0$. Also, we see that $\beta A + \alpha C > (\alpha + \beta)B > 0$ and finely $(\alpha A + \beta C) > \beta A + \alpha C > (\alpha + \beta)B > 0$. Analogously to (2.5) we state that $\tilde{g}$ is positively defined. □

Lemma 3.2. *Let $w = w(x(p), y(p), z(p), u(p))$ be in χM, $qw \neq w$, $q^2 w \neq w$ and g and $\tilde{g}$ be the metrics of M, related by (3.2). Then we have:*

$$\begin{aligned}
\tilde{g}(w,w) &= (\alpha A + \beta C)(x^2 + y^2 + z^2 + u^2) \\
&\quad + 2(\alpha+\beta)B(xy + xu + yz + zu) + 2(\alpha C + \beta A)(yu + xz) \\
\tilde{g}(w,qw) &= (\alpha+\beta)(A + C)(xu + xy + yz + zu) \\
&\quad + (\alpha+\beta)B(x^2 + y^2 + z^2 + u^2 + 2xz + 2yu) \\
\tilde{g}(w,q^2w) &= 2(\alpha A + \beta C)(xz + yu) + 2(\alpha+\beta)B(xu + xy + zy + zu) \\
&\quad + (\alpha C + \beta A)(x^2 + y^2 + z^2 + u^2).
\end{aligned} \tag{3.9}$$

Theorem 3.3. *Let $w = w(x(p), y(p), z(p), u(p))$ be a vector in χM, $qw \neq w$, $q^2 w \neq w$. Let g and $\tilde{g}$ be two positively defined metrics of M, related by (3.2). If φ and φ_1 are the angles between w and qw, with respect to g and $\tilde{g}$, ϕ and ϕ_1 are the angles between w and $q^2 w$, with respect to g and $\tilde{g}$, then the following equations are true:*

$$\cos\varphi_1 = \frac{(\alpha+\beta)\cos\varphi}{\alpha + \beta\cos\phi}, \tag{3.10}$$

$$\cos\phi_1 = \frac{\alpha\cos\phi + \beta}{\alpha + \beta\cos\phi}. \tag{3.11}$$

Proof. Since g and $\tilde{g}$ are both positively defined metrics we can calculate $\cos\varphi$ and $\cos\varphi_1$, respectively. Then by using (3.3) and (3.9) we get (3.10). Also, we calculate $\cos\phi$ and $\cos\phi_1$, respectively. Then by using (3.3) and (3.9) we get (3.11). □

Theorem 3.3 implies immediately the assertions:

Corollary 3.1. *Let φ and φ_1 be the angles between w and qw with respect to g and $\tilde{g}$. Let ϕ and ϕ_1 be the angles between w and $q^2 w$ with respect to g and $\tilde{g}$. Then*

1) $\varphi = \frac{\pi}{2}$ if and only if $\varphi_1 = \frac{\pi}{2}$;

2) if $\phi = \frac{\pi}{2}$ then $\phi_1 = \arccos\frac{\beta}{\alpha}$;

3) if $\phi_1 = \frac{\pi}{2}$ then $\phi = \arccos(-\frac{\beta}{\alpha})$.

Further, we consider an infinite series of the metrics of M as follows:

$$g_0,\ g_1,\ g_2, \dots,\ g_n, \dots$$

where

$$\begin{gathered}
g_0 = g, \quad g_1 = \tilde{g}, \quad g_n = \alpha g_{n-1} + \beta f_{n-1}, \\
f_{n-1,is} = g_{n-1,ka} q_s^a q_i^k, \quad 0 < \beta < \alpha.
\end{gathered} \tag{3.12}$$

By the method of the mathematical induction we can see that the matrix of every g_n is circulant one and every g_n is positively defined.

Theorem 3.4. *Let M be a Riemannian manifold with metrics g_n from (3.12) and an affinor structure q from (2.2). Let $w = w(x(p), y(p), z(p))$ be in χM, $qw \neq w$, $q^2 w \neq w$. Let φ_n be the angle between w and qw, with respect to g_n, and ϕ_n be the angle between w and $q^2 w$ with respect to g_n. Then two infinite series*

$$\varphi_0,\ \varphi_1,\ \varphi_2, \dots,\ \varphi_n, \dots \ \text{and} \quad \phi_0,\ \phi_1,\ \phi_2, \dots,\ \phi_n, \dots$$

are convergent and tend to zero.

Proof. Using the method of the mathematical induction and Theorem 2.3 we obtain:

$$\cos\varphi_n = \frac{(\alpha+\beta)\cos\varphi_{n-1}}{\alpha+\beta\cos\phi_{n-1}} \tag{3.13}$$

as well as $\varphi_n \in (0, \pi)$. From (3.13) we get:

$$\frac{\cos\varphi_n}{\cos\varphi_{n-1}} = \frac{\alpha+\beta}{\alpha+\beta\cos\varphi_{n-1}} \geq 1. \tag{3.14}$$

The equation (3.14) implies $\cos\varphi_n \geq \cos\varphi_{n-1}$, so the series $\{\cos\varphi_n\}$ is increasing one and since $\cos\varphi_n < 1$ then it converges. From (3.13) we have $\lim\cos\varphi_n = 1$, so $\lim\varphi_n = 0$.

Now, we find

$$\cos\phi_n = \frac{\alpha\cos\phi_{n-1}+\beta}{\alpha+\beta\cos\phi_{n-1}} \tag{3.15}$$

as well as $\phi_n \in (0, \pi)$. From (3.15) we get:

$$\cos\phi_n - \cos\phi_{n-1} = \frac{\beta\sin^2\phi_{n-1}}{\alpha+\beta\cos\phi_{n-1}} \geq 0. \tag{3.16}$$

The equation (3.16) implies $\cos\phi_n > \cos\phi_{n-1}$, so the series $\{\cos\phi_n\}$ is increasing one and since $\cos\phi_n < 1$ then it converges. From (3.15) we have $\lim\cos\phi_n = 1$, so $\lim\phi_n = 0$. □

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