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CONNECTIONS AND SECOND ORDER DIFFERENTIAL EQUATIONS ON INFINITE DIMENSIONAL MANIFOLDS

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Abstract. For a given manifold M, modelled on a Banach space $\mathbb{B}$, second order differential equation provides an alternative way to study geometric structures on M. Firstly for every connection ∇ on M we associate a second order differential equation S in a way that the ∇-geodesics are geodesics with respect to S. In a further step despite of natural difficulties with non-Banach modelled manifolds, and even spaces, we generalize these results to a wide class of Fréchet manifolds. More precisely we show that for a Fréchet manifold M, which can be considered as projective limit of Banach manifolds, for a given initial value there exists a unique geodesic. As an interesting result we propose two criterions to generalize the concept of completeness for a wide class of Fréchet manifolds. The last part of the paper suggests applications of our technique to some well known Fréchet manifolds i.e. manifold of infinite jets and manifold of smooth mappings.

1. Introduction

The theory of connections forms an interesting chapter of differential geometry which has been widely explored by many authors (see e.g. [4], [7], [14] and [18]). Beside the mathematical nature of connection theory it becomes an essential tool due to its important role in mathematical physics [13], quantum field theory [14], control theory [6] etc.

In our work, first we try to present a unified definition of connection and associate to every connection a second order differential equation (for abbreviation $2ODE$) on a Banach or Fréchet projective limit manifold. Then using $2ODE$'s we derive several important geometric properties of the Banach and non-Banach discussing manifolds.

Section 2 is devoted to introduce the basic notations about connections and bundles and makes an integrated theory for different types of connection in the general case. Most of the results of this section are known but we could not find such a unified theory in any reference. First the notion of connection on a Banach

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fibre bundles $\pi : F \longrightarrow M$ as an smooth complementary to the vertical subbundle $V\pi$ is introduced. Then we procure the local forms which fully determine a connection locally and is necessary for the next step i.e. connections on vector bundles and studies on ordinary differential equations.
Recall that this definition of connection on Banach vector bundles yields a covariant derivative but, as it is known, the converse is true just for the finite dimensional case (propositions 2.1 and 2.2). Another, known, type of connection is a metric one which is introduced in the last part of section 1 and local components are derived.

In section 3 homogeneous second order differential equations (for abbreviation 2ODE's) are introduced. These special class os second order vector fields carry significant geometric properties. For example one can associate a 2ODE S to a (possibly nonlinear) connection ∇ in a way that the S-geodesics are geodesic curves of ∇ (theorem 3.2). For the case that the 2ODE is homogeneous of degree two the converse of the theorem 3.2 also is true [12].
Lemmas 3.1 and 3.2 provide two known criterions (and also motivation for further steps) to determine some of infinite dimensional geodesically complete manifolds which are susceptible to be extended for the Fréchet modelled manifolds.

In a further step (section 4) we consider a wide class of Fréchet manifolds i.e. those which can be considered as projective limits on Banach manifolds. In different literatures of mathematical physics like loop quantization of Gauge theories, Quantum Gravity and the 2D Yang-Mills Theory and string theory one often encounters with projective limits manifolds (see references of [1]). Another example in physics is the space of connections via graphs [5] which one has a projective family of compact Hausdorff spaces labelled by a special partially ordered directed set called graphs. Some of the well known projective families of manifolds which arise in differential geometry are space of infinite jets of a given fibre bundle [17], manifold of maps [8] and group of diffeomorphisms[16].

Despite of the natural difficulties with these manifolds [10] we prove an existence and uniqueness theorem for ordinary differential equations on these manifolds (theorem 4.1) which is followed by two completeness criterion for this category of manifolds. They are good motivations for further studies on these manifolds for example the challenging problem of a generalized length structure arising from the components [15].

Finally (section 4) we give some applications of our technique to two well known Fréchet manifolds i.e. manifold of infinite jets and manifold of maps. It would be nice if one engage a family of natural metric to his framework and looks for minimizers of the induced metric on the limit manifold and in this case our theorems will play a key role. It seems to us that this is the missing part of the geometry of these manifolds and there are ongoing research to enrich this field of differential geometry (mathematical physics) as much as possible.

Through this paper all the maps and manifolds, for the sake of simplicity, are assumed to be smooth but less degrees of differentiability may be assumed.

2. Connections

Let $\pi : F \longrightarrow M$ be a fibre bundle with fibres B where B and F are Banach manifolds modelled on the Banach spaces $\mathbb{B}$ and $\mathbb{F}$ respectively. At any point $p \in F$ let $V\pi_p \subseteq T_pF$ be the vertical subspace, i.e. $V\pi_p = kerT_p\pi$, and define the vertical subbundle as $V\pi = \bigcup_{p\in F} V\pi_p$. A **connection** on (F, π, M), for abbreviation on

π, is a smooth choice of $H\pi_p \subseteq T_pF$ (at any point $p \in E$) complementary to $V\pi_p$ such that $TF = H\pi \oplus V\pi$. If π^h and π^v are the horizontal and vertical projections, then by a smooth choice of $H\pi$ we mean that $\pi^h(X)$ is smooth for every vector field $X \in C^\infty(F)$. Let (U, ϕ) be a chart for M such that $(\pi^{-1}(U), \Phi)$ is a local trivialization of π. In fact we consider a family of these trivializations for π which the domains cover M. Then

$$\Phi : \pi^{-1}(U) \longrightarrow \phi(U) \times F$$

and $T\Phi : {\pi_F}^{-1}(\pi^{-1}(U)) \longrightarrow \phi(U) \times F \times \mathbb{B} \times \mathbb{F}$ is the induced local trivialization for $TF|_U$. Let (V, Ψ) be another trivialization chart with $U \cap V \neq \emptyset$, then

$$\begin{aligned} T\Psi \circ T\Phi^{-1}(x, \xi, y, \eta) &= \big(\sigma_{\psi\phi}(x), G_{\psi\phi}(x, \xi), T\sigma_{\psi\phi}(x)y, \\ &\quad T_1G_{\psi\phi}(x, \xi)y + T_2G_{\psi\phi}(x, \xi)\eta\big) \end{aligned} \tag{2.1}$$

where $\Psi \circ \Phi^{-1}(x, \xi) := (\sigma_{\psi\phi}(x), G_{\psi\phi}(x, \xi))$ and for $i = 1, 2$, T_i is the partial derivative with respect to the i-th variable. Clearly

$$\pi^v(x, \xi; y, \eta) := (x, \xi; 0, \eta + \Gamma_\phi(x, \xi)y)$$

where $\Gamma_\phi : \phi(U) \times F \longrightarrow L(\mathbb{B}, \mathbb{F})$ are smooth functions. In fact the differentiability of Γ_ϕ yields from the differentiability of the connection. It perhaps worth remarking that elements of $V\pi$ locally have the form $(x, \xi, 0, \eta)$ and π^v at every point is a linear projection. This last means that

$$\pi^h(x, \xi; y, \eta) = (x, \xi; y, \eta) - \pi^v(x, \xi; y, \eta) = (x, \xi; y, -\Gamma_\phi(x, \xi)y).$$

With a customary abuse of notation let $\{(\pi^{-1}(U), \Phi)\}$ stands the family of local trivialization for $V\pi$ too. The compatibility condition for the local components yields from the fact that $T\Psi \circ T\Phi^{-1} \circ \pi^v|_U = \pi^v|_V \circ T\Psi \circ \Phi^{-1}$ and this holds if and only for every $(x, \xi, y, \eta) \in \phi(U \cap V) \times F \times \mathbb{B} \times \mathbb{F}$

$$\begin{aligned} &T\Psi \circ T\Phi^{-1}(x, \xi, 0, \eta + \Gamma_U(x, \xi)y) = \pi^v|_V\big((\sigma_{\psi\phi})(x), G_{\psi\phi}(x, \xi), \\ &\qquad\qquad T\sigma_{\psi\phi}(x)y, T_1G_{\psi\phi}(x, \xi)y + T_2G_{\psi\phi}(x, \xi)\eta\big) \\ &\iff \big(\sigma_{\psi\phi}(x), G_{\psi\phi}(x, \xi), 0, 0 + T_2G_{\psi\phi}(x, \xi)[\eta + \Gamma_U(x, \xi)y]\big) = \Big(\sigma_{\psi\phi}(x), \\ &G_{\psi\phi}(x, \xi), 0, T_1G_{\psi\phi}(x, \xi)y + T_2G_{\psi\phi}(x, \xi)\eta + \Gamma_V\big(\sigma_{\psi\phi}(x), G_{\psi\phi}(x, \xi)\big)[T\sigma_{\psi\phi}(x)y]\Big) \end{aligned}$$

if and only the last components of both sides are equal i.e.

$$T_1G_{\psi\phi}(x, \xi)y + \Gamma_V\big(\sigma_{\psi\phi}(x), G_{\psi\phi}(x, \xi)\big)[T\sigma_{\psi\phi}(x)y] = T_2G_{\psi\phi}(x, \xi)[\Gamma_U(x, \xi)y] \tag{2.2}$$

2.1. **connections on vector bundles.** Let $\pi : E \longrightarrow M$ be a vector bundle with fibres isomorphic to the Banach space $\mathbb{E}$. Following the formalism of the previous part, for local trivializations (U, Φ) and (V, Ψ) with $U \cap V \neq \emptyset$, $\Psi \circ \Phi^{-1}(x, \xi) = (\sigma_{\psi\phi}(x), G_{\psi\phi}(x)\xi)$ where $G_{\psi\phi} : U \cap V \longrightarrow GL(\mathbb{E})$ are smooth. Here $GL(\mathbb{E})$ is the space of linear and continuous isomorphisms from $\mathbb{E}$ to $\mathbb{E}$. In this situation equation (2.1) takes the form

$$\begin{aligned} T(\Psi \circ \Phi^{-1})(x, \xi, y, \eta) &= \big(\sigma_{\psi\phi}(x), G_{\Psi\Phi}(x)\xi, T\sigma_{\psi\phi}(x)y \\ &\quad , G_{\Psi\Phi}(x)\eta + TG_{\Psi\Phi}(x)(y, \xi)\big). \end{aligned} \tag{2.3}$$

and consequently the compatibility condition for the connection, say ∇, is

$$G_{\Psi\Phi}(\Gamma_U(x)[y, \xi]) = TG_{\Psi\Phi}(x)(y, \xi) + \Gamma_V(\sigma_{\psi\phi}(x))[T\sigma_{\psi\phi}(x)y, G_{\Psi\Phi}(x)\xi]. \tag{2.4}$$

The connection ∇ is linear if the local components are linear with respect to the second variable i.e. for every chart (U, ϕ),

$$\Gamma_U : \phi(U) \longrightarrow \mathcal{L}(\mathbb{E}, \mathcal{L}(\mathbb{B}; \mathbb{E})).$$

If M is finite dimensional and $E = TM$ we have the familiar notation

$$[\Gamma(x)(\xi, \eta)]^i = \sum \Gamma^i_{jk}(x)\xi^i y^i.$$

Moreover we can define the connection map $\nabla : TE \longrightarrow E$ which is locally given by

$$\nabla_\phi(x, \xi, y, \eta) := \Phi \circ \nabla \circ T\Phi^{-1}(x, \xi, y, \eta) = (x, \eta + \Gamma_\phi(x, \xi)y).$$

For every section ζ of π and any vector field X on M, the covariant derivative of ζ along X is defined by $\nabla_X \zeta := \nabla \circ T\zeta \circ X$. However another familiar concept of connections is defined according to the covariant derivative properties. More precisely a connection is considered to be a map $\nabla : \Gamma(\pi) \times \Gamma(\tau_M) \longrightarrow \Gamma(\pi)$ with the following properties;
$\nabla_{X+Y}\zeta = \nabla_X\zeta + \nabla_Y\zeta$, $\nabla_X(\zeta + \zeta') = \nabla_X\zeta + \nabla_X\zeta'$ and $\nabla_{fX}\zeta = f\nabla_X\zeta$ where $\zeta, \zeta' \in \Gamma(\pi)$, $X, Y \in \Gamma(\tau_M)$ and $f \in C^\infty(M)$.

The next two statements reveal the relations between the concepts of covariant derivative and connection. To see that, let (U, Φ) be a local trivialization for π. For $\zeta \in \Gamma(\pi)$ and $X \in \Gamma(\tau_M)$ suppose that $\bar{\zeta}_\phi := proj_2 \circ \Phi \circ \zeta$ and $\bar{X}_\phi := proj_2 \circ T\phi \circ X$ be the principal parts of ζ and X respectively.

Proposition 2.1. *Let ∇ be a connection on π. Then a unique covariant derivative can be defined which locally on (U, Φ) is given by*

$$(\nabla_X\zeta)|_U(\phi p) := (\nabla \circ T\zeta \circ X)|_U(\phi p) = d\bar{\zeta}_\phi(\phi p)\bar{X}_\phi(\phi p) + \Gamma_U(\phi p)[\bar{X}_\phi(\phi p), \bar{\zeta}_\phi(\phi p)].$$

Proof. Clearly the result is again a section and this is a covariant derivative. Note that in the case that $E = TM$ and ∇ is a linear connection, we can impose further assumptions for the covariant derivative i.e. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$, $\nabla_X fY = f\nabla_X\zeta + X(f)Y$ and $\nabla_X Y - \nabla_Y X = [X, Y]$. □

Now, just for the case that the fibres of $\pi : E \longrightarrow M$ are finite dimensional vector spaces we propose a suitable converse for the above mentioned proposition (see also [12]).

Proposition 2.2. *Let $\pi : E \longrightarrow M$ be a vector bundle with finite dimensional total space. Then for every covariant derivative on this bundle we can associate a linear connection.*

Proof. Suppose that a covariant derivative ∇ is given. We define the Christoffel symbols of the connection D in the following way

$$\Gamma_\phi(\phi(p))[\xi, \eta] := (\overline{\nabla_X\zeta})|_U(\phi(p)) - d\bar{\zeta}_\phi(\phi(p))\bar{X}_\phi(\phi(p)) \tag{2.5}$$

where $\bar{\zeta}_\phi(\phi(p)) = \eta$ and $\bar{X}_\phi(\phi(p)) = \xi$ and (U, Φ) is a local trivialization. We claim that this definition is independent of the choice of sections. More precisely the right hand side is $C^\infty(M)$ linear in both components. In the other words it is $C^\infty(M)$-linear map from $\Gamma(\pi) \times_M \Gamma(\tau_M)$ to $\Gamma(\pi)$. According to the following Lemma from [12], (5) is independent from the choice of sections.
"*Lemma: Let E and F be vector bundles over M with E finite dimensional and M admitting cut off functions. Let*

$$H : \Gamma E \longrightarrow \Gamma F$$

be a $C^\infty(M)$-linear map, that is $H(f\zeta) = fH(\zeta)$ for $f \in C^\infty(M)$. Given a point $p \in M$, the value $H(\zeta)p$ depends only on the value $\zeta(p)$."

We can see that the local components (2.5) are linear in η and consequently the connections will be linear. Let (U, Φ) and (V, Ψ) be two local trivializations of π with $U \cap V \neq \emptyset$ and p a point in this intersection . By setting $\phi(p) := \phi p$ and $\psi(p) := \psi p$ we observe that

$$\begin{aligned}
&\Gamma_\psi\big((\psi \circ \phi^{-1})(\phi p)\big)[d(\psi \circ \phi^{-1})(\phi p)\bar{X}_\phi(\phi p), G_{\psi\phi}(\phi p)\bar{\zeta}_\phi(\phi p)] \\
&= \Gamma_\psi(\psi p)[\bar{X}_\psi(\psi p), \bar{\zeta}_\psi(\psi p)] = (\nabla_X \zeta)|_V(\psi p) - d\bar{\zeta}_\psi(\psi p)\bar{X}_\psi(\psi p) \\
&= G_{\psi\phi}(\phi p)(\nabla_X \zeta)|_U(\phi p) - d\big(\bar{\zeta}_\psi((\psi \circ \phi^{-1})\phi p)\big)\big(d(\psi \circ \phi^{-1})(\phi p)\bar{X}_\phi(\phi p)\big) \\
&= G_{\psi\phi}(\phi p)(\nabla_X \zeta)|_U(\phi p) - d\big(\bar{\zeta}_\psi \circ (\psi \circ \phi^{-1})\big)(\phi p)\bar{X}_\phi(\phi p) \\
&= G_{\psi\phi}(\phi p)(\nabla_X \zeta)|_U(\phi p) - d\big(G_{\psi\phi}(\phi p)\bar{\zeta}_\phi\big)(\phi p)\bar{X}_x(\phi p) \\
&= G_{\psi\phi}(\phi p)(\nabla_X \zeta)|_U(\phi p) - \{dG_{\psi\phi}(\phi p)\big(\bar{X}_\phi(\phi p), \bar{\zeta}_\phi(\phi p)\big) + G_{\psi\phi}(\phi p)d\zeta_\phi(\phi p)\bar{X}_\phi(\phi p)\} \\
&= G_{\psi\phi}(\phi p)[\nabla_X \zeta)|_U(\phi p) - d\zeta_\phi(\phi p)\bar{X}_\phi(\phi p)] - dG_{\psi\phi}(\phi p)\big(\bar{X}_\phi(\phi p), \bar{\zeta}_\phi(\phi p)\big) \\
&= G_{\psi\phi}(\phi p)[\Gamma_\phi(\phi(p))[\bar{X}_\phi(\phi p), \bar{\zeta}_\phi(\phi p)] - dG_{\psi\phi}(\phi p)\big(\bar{X}_\phi(\phi p), \bar{\zeta}_\phi(\phi p)\big).
\end{aligned}$$

□

Remark 2.1. Note that just for the case that we want to derive a connection from a covariant derivative, the dimension should be finite. In fact the concept of connection is more general than covariant derivative.

Here we state the definition of a metric from [11] which also is stated in [12]. For the vector bundle $\pi : E \longrightarrow M$ we have the associated bundle $L^2_s(\pi) : L^2_s(E) \longrightarrow M$ where $L^2_s(E)_p$ consists of the continuous symmetric bilinear maps from $\mathbb{E} \times \mathbb{E}$ to $\mathbb{R}$. Let $L^2_\pi(\mathbb{E})$ be the model of the fibres. It contains as an open subset $Ri(\mathbb{E})$ the positive definite forms, i.e. those forms which are $\geq \epsilon$ (Hilbert metric on $\mathbb{E}$), for some $\epsilon \geq 0$ [11].

Definition 2.1. A Riemannian metric on $\pi : E \longrightarrow M$ is a differentiable section $g : M \longrightarrow L^2_s(E)$ such that for every $p \in M$, $g(p)$ is positive definite. If we have a Riemannian metric g on $\tau_M : TM \longrightarrow M$ then we call M a Riemannian manifold and we also call g a Riemannian metric on M.

Proposition 2.3. *Let M be a manifold modelled on a self dual Banach space and ∇ a covariant derivative on M such that*

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for any $X, Y, Z \in \Gamma(\tau_M)$. Then a unique torsion free connection can be defined on M, known as the Levi-Civita connection.

Proof. Let (U, ϕ) be a local chart on M. For any $(x, \xi, y, \eta) \in U \times \mathbb{B} \times \mathbb{B} \times \mathbb{B}$ the relation

$$g_U(\Gamma_U(x)[\xi, y], \eta) = \frac{1}{2}\big(dg_U(x).\xi(y, \eta) - dg_U(x).\eta(\xi, y) + dg_U(x).y(\xi, \eta)\big)$$

defines the continuous and also smooth map $\Gamma_\phi : \phi(U) \longrightarrow L^2(\mathbb{B}, \mathbb{B})$ as the Christoffel symbols (for more details see [11] and [12]). □

3. Connections and second order differential equations

Let M be a smooth manifold modelled on the Banach space $\mathbb{E}$ with the atlas $\mathcal{A} = \{(U_\alpha, \psi_\alpha)\}_{\alpha\in I}$. $\mathcal{A}$ induces the canonical atlases $\mathcal{B} = \{(\pi_M^{-1}(U_\alpha), \Psi_\alpha)\}_{\alpha\in I}$ and $\mathcal{C} = \{(\pi_{TM}^{-1}(\pi_M^{-1}(U_\alpha)), \widetilde{\Psi}_\alpha)\}_{\alpha\in I}$ for TM and $T(TM)$ respectively. (Here $\pi_M : TM \longrightarrow M$ and $\pi_{TM} : T(TM) \longrightarrow TM$ are the canonical projections.) Let $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ be a smooth curve. A lifting of γ to TM is a curve β such that $\pi \circ \beta = \gamma$. Such lifting always exists, for example the canonical lifting γ'.

A connection on a manifold M is a connection on its tangent bundle i.e. a vector bundle morphism $\nabla : T(TM) \longrightarrow TM$ with the local representation;

$$\begin{aligned}\nabla_\alpha : \psi_\alpha(U_\alpha) \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} &\longrightarrow \psi_\alpha(U_\alpha) \times \mathbb{E}\\ (x, \xi, y, \eta) &\longmapsto (x, \eta + \Gamma_\alpha(x, \xi)y)\end{aligned}$$

where $\nabla_\alpha = \Psi_\alpha \circ \nabla \circ \tilde{\Psi}_\alpha^{-1}$ and $\Gamma_\alpha : \psi_\alpha(U_\alpha) \times \mathbb{E} \longrightarrow \mathcal{L}(\mathbb{E}, \mathbb{E})$, $\alpha \in I$, are the local forms of ∇. The connection ∇ is linear if $\{\Gamma_\alpha\}_{\alpha\in I}$ are linear with respect to the second variable i.e.

$$\Gamma_\alpha : \psi_\alpha(U_\alpha) \longrightarrow \mathcal{L}(\mathbb{E}, \mathcal{L}(\mathbb{E}, \mathbb{E}));\ \alpha \in I.$$

(For a detailed study see [4] or [18].)

Let $\pi_* : TTM \longrightarrow TM$ be differential of the projection $\pi : TM \longrightarrow M$ which locally sends (x, ξ, y, η) to (x, y). It is known that there are two vector bundle structures for TTM on TM i.e. (TTM, π_*, TM) and (TTM, π_{TM}, TM).

Definition 3.1. A vector field $S : TM \longrightarrow T(TM)$ is called a second order differential equation, for abbreviation $2ODE$, if each integral curve β of S is equal to the canonical lifting of $\pi \circ \beta$.

Consider the involution map $\iota : TTM \longrightarrow TTM$; locally given by $(x, \xi, y, \eta) \longmapsto (x, y, \xi, \eta)$.

Theorem 3.1. *The following statements are equivalent.*
1) S is a $2ODE$.
2) S is a section of (TTM, π_{TM}, TM) with $\pi_* \circ S = id_{TM}$.
3) S is a section of (TTM, π_{TM}, TM) with $\iota \circ S = S$.

Proof. Suppose that S is a $2ODE$. For any $v \in TM$ there exists a unique integral curve $\beta_v : I = (\epsilon, \epsilon) \longrightarrow TM$ of S with $\beta_v(0) = v$. Since β_v is an integral curve for S then,

$$\pi_* \circ S(v) = \pi_* \circ S \circ \beta_v(0) = \pi_* \circ \beta_v'(0) = (\pi \circ \beta_v)'(0) = \beta_v(0) = v$$

which proves 2. Conversely, let 2 holds true. Then for any $t \in I$,

$$(\pi \circ \beta)'(t) = \pi_* \circ \beta'(t) = \pi_* \circ S \circ \beta(t) = \beta(t).$$

The equivalence of the conditions 2 and 3 is an immediate result of their local representation. □

The parts 1 and 2 are used by Lang [12] and 3 is used by Del Riego and Parker [7] to define a $2ODE$. Let (U_α, ψ_α) be a chart of M. Then the local expression of S on this chart is

$$\begin{aligned}S_\alpha := \tilde{\Psi}_\alpha \circ S \circ \Psi_\alpha^{-1} : U_\alpha \times \mathbb{E} &\longrightarrow U_\alpha \times \mathbb{E} \times \mathbb{E} \times \mathbb{E}\\ (x, \xi) &\longmapsto (x, \xi, \xi, \mathcal{S}_\alpha(x, \xi))\end{aligned}$$

where $\{\mathcal{S}_\alpha\}_{\alpha\in I}$ are smooth $\mathbb{E}$-valued functions. Here we state a definition from [7].

Definition 3.2. A $2ODE$ S is called **homogeneous** of order $m \in \mathbb{R}$ if for every $\alpha \in I$ and every $a \in \mathbb{R}$, $\mathcal{S}_\alpha(x, a\xi) = a^m \mathcal{S}_\alpha(x, \xi)$.

Theorem 3.2. *Let ∇ be a connection on M. Then there exists an induced $2ODE$ S_∇ on M given by*

$$S_\nabla(x,\xi) := \pi_*|^{-1}_{H\pi_{(x,\xi)}}(x,\xi).$$

Proof. Remind that $\pi_* : TTM \longrightarrow TM$ locally sends (x,ξ,y,η) to (x,y) and elements of $H\pi_{(x,\xi)}$ have the form $(x,\xi,y,-\Gamma(x,\xi)y)$ for some $y \in \mathbb{E}$. Let $S_\nabla(x,\xi) := \pi_*|^{-1}_{H\pi_{(x,\xi)}}(x,\xi) = (\bar{x},\bar{\xi},\bar{y},\bar{\eta})$. Since $(\bar{x},\bar{\xi},\bar{y},\bar{\eta})$ belongs to $\pi_*^{-1}(x,\xi)$ then $\bar{x} = x$ and $\bar{y} = \xi$. On the other hand $(x,\bar{\xi},\xi,\bar{\eta}) \in H\pi_{(x,\xi)}$ i.e. $\bar{\xi} = \xi$ and $\bar{\eta} = -\Gamma(x,\xi)\xi := \mathcal{S}(x,\xi)$ which shows that S_∇ is a $2ODE$. □

Definition 3.3. For the smooth curve $\gamma : (-\epsilon,\epsilon) \longrightarrow M$ and a $2ODE$ S we define γ to be a geodesic with respect to S (or S-geodesic) if γ' is an integral curve for S i.e.

$$\gamma''(t) = S(\gamma'(t)).$$

Remark 3.1. For $m = 2$ the $2ODE$ S is called a **spray** [12]. Let $\{\mathcal{S}_\alpha\}_{\alpha\in I}$ be the family of local components for S. Since for every $\alpha \in I$, $\mathcal{S}_\alpha$ is homogeneous of degree 2 in second variable, then

$$\mathcal{S}_\alpha(x,\xi) = \frac{1}{2} d_2^2 \mathcal{S}_\alpha(x,0)(\xi,\xi)$$

where d_2^2 means the second partial derivative with respect to the second variable (see also [12] pp. 9 and 101). Define the bilinear symmetric map $B_\alpha(x) := \frac{1}{2} d_2^2 \mathcal{S}_\alpha(x,0)$ and consequently for $\xi,\eta \in \mathbb{E}$ we have

$$B_\alpha(x)(\xi,\eta) = \frac{1}{2}\{B_\alpha(x)(\xi+\eta,\xi+\eta) - B_\alpha(x)(\xi,\xi) - B_\alpha(x)(\eta,\eta)\}$$

Hence every spray S locally can be uniquely determined by symmetric bilinear maps

$$B_\alpha : U_\alpha \longrightarrow L^2_{sym}(\mathbb{E},\mathbb{E})$$

where $B_\alpha(x)(\xi,\xi) = \mathcal{S}_\alpha(x,\xi)$.

Del Riego and Parker [7] discussed second order vector fields and their homogeneity properties. Several important results relate to the case of sprays and second order vector fields. For example in the case of finite dimensional manifolds one can associate a second order vector field S_∇ to a (possibly nonlinear) connection ∇ such that the geodesic curves with respect to S_∇ are the geodesic curves of ∇ [7]. Moreover every spray S determines a unique torsion-free linear connection, and conversely every spray S arises from a linear connection with arbitrary torsion. If there is a Riemannian metric on the base manifold then quasi-geodesics and geodesics, coincide.

The latter holds also in our frame work since for every linear connection ∇ on M by theorem 3.2 there exists a unique spray (2ODE in general) S_∇ such that geodesic curves of the connection ∇ are geodesics with respect to S_∇. To see that, it is enough to set $B_\alpha = B|_{\psi_\alpha(U_\alpha)} = -\Gamma_\alpha$.

The converse of the theorem 3.2 is also true in Banach case and the proof can be derived from [7] theorem 4.4.

The local expression of the equation $\gamma' = S(\gamma'')$ is

$$\widetilde{\Psi}_\alpha \circ \gamma''(t) = ((\psi_\alpha\circ\gamma)(t), (\psi_\alpha\circ\gamma)'(t), (\psi_\alpha\circ\gamma)'(t), (\psi_\alpha\circ\gamma)''(t)).$$

and the following system of ordinary differential equations

$$(\psi_\alpha \circ \gamma)''(t) = \mathcal{S}_\alpha\big((\psi_\alpha \circ \gamma)(t), (\psi_\alpha \circ \gamma)'(t)\big);\ \alpha \in I. \tag{3.1}$$

determine the S-geodesics for given initial values. On the other hand let $\nabla : T(TM) \to TM$ be a connection. It is known (see e.g. [2]) that the local differential equations satisfied by a ∇-geodesic curve $\gamma : I \subseteq \mathbb{R} \to M$ are:

$$\begin{aligned} \nabla_{\alpha \gamma'_\alpha(t)} \gamma'_\alpha(t) &= \nabla_\alpha \circ \gamma''_\alpha(t) = (\psi_\alpha \circ \gamma)''(t) \\ &+ \Gamma_\alpha((\psi_\alpha \circ \gamma)(t))[(\psi_\alpha \circ \gamma)'(t), (\psi_\alpha \circ \gamma)'(t)] = 0; \quad \alpha \in I. \end{aligned}$$

which coincide with the equations (3.1) if we notice that $\Gamma_\alpha(x)(y,y) = -B_\alpha(x)(y,y) = \mathcal{S}_\alpha(x)$ for all $\alpha \in I$. Since for every connection we can consider its associated $2ODE$ hereinafter we state our results just for $2ODE$'s which automatically will be true for connections also.

Proposition 3.1. *For any $2ODE$ S there exists a unique geodesic γ satisfying the initial conditions of the form $\gamma(0) = x$ and $T_t\gamma(\partial_t) = y$, for any choice of $x \in M$ and $y \in T_xM$.*

The other interesting part of studying $2ODE$'s and their geodesics is the completeness of their geodesics. On the other hand we need to resolve that under which conditions the geodesic γ is defined on the whole of real line $\mathbb{R}$.

Definition 3.4. The smooth manifold M is called geodesically complete with respect to the connection ∇ (or a 2ODE S), if its geodesics are defined on the whole of real line $\mathbb{R}$.

Based on the above constructions we may state the following nice criterion to characterize some of the geodesically complete Banach manifolds.

Lemma 3.1. *Let M be a smooth manifold modelled on the Banach space $\mathbb{E}$ and ∇ a connection on M. If the associated $2ODE$ to ∇ has compact support, then M is geodesically complete.*

Proof. Let S be a $2ODE$ associated to ∇ and γ be a geodesic for S. Since S has compact support and γ' is an integral curve for S, then by [13], γ' is complete which yields the completeness of γ and proves the lemma. □

Lemma 3.2. *Let (M, g) be a Riemannian manifold modelled on a self dual Banach space. If $(M, dist_g)$ is complete then all geodesics with respect to the Levi-Civita connection ∇_g (and also the associated 2ODE S_g) are complete. (For the proof see [12].)*

4. Fréchet case

In the sequel we introduce our notations about a wide class of Fréchet manifolds i.e. those which can be considered as projective limits of Banach manifolds. Suppose that $\{(M^i, \varphi^{ji})\}_{i,j\in\mathbb{N}}$ be a projective system of Banach manifolds with the limit $M = \varprojlim M^i$ such that M^i is modelled on the Banach space $\mathbb{E}^i$ and $\{\mathbb{E}^i, \rho^{ji}\}_{i\in\mathbb{N}}$ also forms a projective system of Banach spaces. (Here $\rho^{ji} : \mathbb{E}^j \longrightarrow \mathbb{E}^i$ are linear and continuous maps for $j \geq i$.)
Remind from [1] that elements of M are $(x^i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} M^i$ such that for $j \geq i$, $\varphi^{ji}(x^j) = x^i$. Let $\{(M^i, \varphi^{ji})\}_{i,j\in\mathbb{N}}$ and $\{(N^i, \vartheta^{ji})\}_{i,j\in\mathbb{N}}$ be projective family

of manifolds. A family of maps $\{f^i : M^i \longrightarrow N^i\}_{i\in\mathbb{N}}$ is a projective system of mappings if $\vartheta^{ji} \circ f^j = f^i \circ \varphi^{ji}$ for $i, j \in \mathbb{N}$ with $j \geq i$.

Furthermore suppose that for any $x = (x)_{i\in\mathbb{N}} \in M$ there exists a projective system of local charts $\{(U^i, \psi^i)\}_{i\in\mathbb{N}}$ such that $x^i \in U^i$ and $U = \varprojlim U^i$ is open in M. Two example of projective limit manifolds are given at the last section.

The vector bundle structure of TM for a Fréchet manifold M links to pathological structure of general linear group $GL(\mathbb{F})$ and this causes troubles. It is shown in [9] that by replacing the generalized topological Lie group

$$\mathcal{H}_0(\mathbb{F}) = \{(l^i)_{i\in\mathbb{N}} \in \prod^{i\in I} GL(\mathbb{E}^i) \ : \ \varprojlim l^i \ exists\}$$

rather than $GL(\mathbb{F})$ this obstacle also can be solved. Moreover as we will see in the rest of the paper, the problems related to the lack of a general solvability for differential equations on Fréchet manifolds overcome with the introduced technique (see also [2], [3] and [4]).

Proposition 4.1. *Let $\{\nabla^i\}_{i\in\mathbb{N}}$ be a projective system of connections on $\{M^i\}_{i\in\mathbb{N}}$. If $\{S^i\}_{i\in\mathbb{N}}$ is the corresponding family of 2ODE's, then $\{S^i\}_{i\in\mathbb{N}}$ also form a projective systems.*

Proof. Since $\{\nabla^i\}_{i\in\mathbb{N}}$ is a projective system of maps then for $j \geq i$, $T\varphi^{ji} \circ \nabla^j = \nabla^i \circ TT\varphi^{ji}$ or locally on a given limit chart

$$\rho^{ji} \circ \Gamma^j_{U^j}(x^j, \xi^j)(y^j) = \Gamma^i_{U^i}(\varphi^{ji}(x^j), \rho^{ji}\xi^j)(\rho^{ji}y^j).$$

For any $i \in \mathbb{N}$, let S^i be the corresponding $2ODE$ to ∇^i. Then according to the theorem 3.2 $S^i|_{U^i}(x^i, \xi^i) = (x^i, \xi^i, \xi^i, \mathcal{S}^i_{U^i}(, x^i, \xi))$ where $\mathcal{S}^i_{U^i}(x^i, \xi^i) = \Gamma^i_{U^i}(x^i, \xi^i)(\xi^i)$ and as a result

$$\rho^{ji} \circ \mathcal{S}^j_{U^j}(x^j, \xi^j) = \mathcal{S}^i_{U^i}(\varphi^{ji}(x^j), \rho^{ji}\xi^j)$$

which means that $\{S^i\}_{i\in\mathbb{N}}$ is a projective system. □

Here we state the following main theorem which is a generalization of proposition 3.1 for Fréchet manifolds.

Theorem 4.1. *Let S be a 2ODE obtained as projective limit of 2ODE's $\{S^i\}_{i\in\mathbb{N}}$. Then for any choice of $x \in M = \varprojlim M^i$ and $y \in T_xM$ there exists a unique geodesic γ satisfying initial conditions of the form $\gamma(0) = x = (x^i)_{i\in\mathbb{N}}$ and $\gamma'(0) = y = (y^i)_{i\in\mathbb{N}}$.*

Proof. For any $i \in \mathbb{N}$, S^i is a 2ODE on the Banach manifold M^i. Hence by proposition 3.1 for initial conditions $x^i = \varphi^i(x)$ and $y^i = T_x\varphi^i(y)$, there exists a unique S^i-geodesic γ^i such that:

$$(\psi^i_\alpha \circ \gamma^i)''(t) = \mathcal{S}^i_\alpha((\psi^i_\alpha \circ \gamma^i)(t), (\psi^i_\alpha \circ \gamma^i)'(t)) \tag{4.1}$$

for which $\gamma^i(0) = x^i$ and ${\gamma^i}'(0) = y^i$.(Here $\varphi^i : M \longrightarrow M^i$; $x = (x^i)_{i\in\mathbb{N}} \longmapsto x^i$ is the canonical projection.)

We claim that $\gamma = \varprojlim \gamma^i$ exists and fulfils the conditions of the theorem. For this aim we show that for $j \geq i$, $\varphi^{ji} \circ \gamma^i$ is also a solution for (7) since:

$$\begin{aligned}
(\psi^i_\alpha \circ \varphi^{ji} \circ \gamma^j)''(t) &= (\rho^{ji} \circ \psi^j_\alpha \circ \gamma^j)''(t) = \rho^{ji} \circ ((\psi^j_\alpha \circ \gamma^j)''(t) \\
&= \rho^{ji} \circ \mathcal{S}^j_\alpha((\psi^j_\alpha \circ \gamma^j)(t), (\psi^j_\alpha \circ \gamma^j)'(t)) \\
&= \mathcal{S}^i_\alpha(\rho^{ji} \circ (\psi^j_\alpha \circ \gamma^j)(t), \rho^{ji} \circ (\psi^j_\alpha \circ \gamma^j)'(t)) \\
&= \mathcal{S}^i_\alpha((\psi^i_\alpha \circ \gamma^i)(t), (\psi^i_\alpha \circ \gamma^i)'(t)).
\end{aligned}$$

Furthermore $\varphi^{ji} \circ \gamma^j(0) = \varphi^{ji}(x^j) = x^i$ and $T_x\varphi^{ji}\gamma^j(0) = T_x\varphi^{ji}(y^j) = y^i$. Hence again by 3.1 for $j \geq i$ we have $\varphi^{ji} \circ \gamma^j = \gamma^i$ i.e. $\{\gamma^i\}_{i\in\mathbb{N}}$ is a projective system of curves and $\gamma = \varprojlim \gamma^i$ exists. Moreover

$$\begin{aligned}(\psi_\alpha \circ \gamma)''(t) &= \{(\psi^i_\alpha \circ \gamma^i)''(t)\}_{i\in\mathbb{N}}\\ &= \{\mathcal{S}^i_\alpha((\psi^i_\alpha \circ \gamma^i)(t), (\psi^i_\alpha \circ \gamma^i)'(t))\}_{i\in\mathbb{N}}\\ &= \mathcal{S}_\alpha((\psi_\alpha \circ \gamma)(t), (\psi_\alpha \circ \gamma)'(t)).\end{aligned}$$

which means that γ is the desired geodesic.

For the last part of the proof i.e. uniqueness, suppose that θ be another S-geodesic satisfying $\theta(0) = x$ and $\theta'(0) = y$, then for each $i \in \mathbb{N}$, $\varphi^i \circ \theta$ is another geodesic for S^i which satisfying $\varphi^i \circ \theta = x^i$ and $(\varphi^i \circ \theta)'(0) = y^i$. Using proposition 3.1 we observe that $\varphi^i \circ \theta = \gamma^i$, for any $i \in \mathbb{N}$, and consequently $\theta = \gamma$. □

Finally we state two criteria to characterize some of the geodesically complete Fréchet manifolds.

Lemma 4.1. *Suppose that $M = \varprojlim M^i$ be a projective limit manifold modelled on the Fréchet space $\mathbb{F} = \varprojlim \mathbb{E}^i$ with a 2ODE $S = \varprojlim S^i$. If S has compact support, then M is geodesically complete.*

The proof is a direct consequence of lemma 3.1 and theorem 4.1.

Lemma 4.2. *For any $i \in \mathbb{N}$, let (M^i, g^i) be a complete Riemannian manifold. If $M = \varprojlim M^i$ and $\nabla_g = \varprojlim \nabla^i_{g^i}$ (or $S_g = \varprojlim S^i_{g^i}$) then, for every given initial value, ∇_g- geodesics (S_g-geodesic) uniquely exists and it is complete.*

The proof is clear. Just note that here g could be $g = \sum_{i=1}^{\infty} \frac{1}{2^i}\frac{g^i}{1+g^i}$ (see also [15]).

5. Applications and examples

5.1. **Infinite jets.** Let $\pi : F \longrightarrow M$ be a finite dimensional fibre bundle with $dimM = m$ and $dimF = m+n$. Consider an atlas of adopted coordinates for F i.e. if (U, Φ) is a local chart around $a \in F$ then $pr_1 \circ \Phi = x \circ \pi$ where x is a coordinate chart around $\pi(a) \in M$ and $pr_1 : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^m$ is projection to the first m factors.. We denote the space of the local sections of π with $\Gamma_p(\pi)$ for $p \in M$.

For two local sections ξ and η in $\Gamma_p(\pi)$ we say that they are 1-equivalent if $\xi(p) = \eta(p)$ and for some (and consequently for every) adopted coordinate $\Phi = (x^i, \phi^\alpha)$ around $\xi(p)$

$$\frac{\partial \xi^\alpha}{\partial x^i} = \frac{\partial \eta^\alpha}{\partial x^i}$$

for $i = 1, ..., m$ and $\alpha = 1, ..., n$ (Here $\xi^\alpha = \phi^\alpha \circ \xi$ and (x^i) is a local chart for M around p). The equivalence class containing ξ is called the first order jet ξ at p and is denoted by $j^1_p\xi$. The first order jet manifold of the fibre bundle π is $J^1\pi := \{j^1_p\xi;\ p \in M \ and \ \xi \in \Gamma_p(\pi)\}$ is an $m + n + mn$ dimensional manifold. The canonical local chart for $j^1\pi$ with respect to the adopted coordinate (U, Φ) is (U^1, Φ^1) where $U^1 = \{j^1_p\xi;\ \xi(p) \in U\}$ and $\Phi^1 = (x^i, \phi^\alpha, \phi^\alpha_i)$ with

$$x^i(j^1_p\xi) = x^i(p),\ \phi^\alpha(j^1_p\xi) = \phi^\alpha(\xi(p)),\ \phi^\alpha_i(j^1_p\xi) = \frac{\partial \xi^\alpha}{\partial x^i}|_p.$$

In a similar way we can define $J^2\pi := \{j^2_p\xi;\ p \in M\ and\ \xi \in \Gamma_p(\pi)\}$ using the following equivalent relation on the local sections

$$j^2_p\xi = j^2_p\eta \Longleftrightarrow \xi(p) = \eta(p),\ \frac{\partial \xi^\alpha}{\partial x^i} = \frac{\partial \eta^\alpha}{\partial x^i},\ \frac{\partial^2 \xi^\alpha}{\partial x^i \partial x^j} = \frac{\partial^2 \eta^\alpha}{\partial x^i \partial x^j},$$

for all $1 \leq \alpha \leq n$ and $1 \leq i,j \leq m$. In this way $J^2\pi$ becomes a smooth manifold with a canonical atlas of charts (U^2, Φ^2) where $U^2 = \{j^2_p\xi;\ \xi(p) \in U\}$, $\Phi^2 = (x^i, \phi^\alpha, \phi^\alpha_i, \phi^\alpha_{ij})$ and

$$\phi^\alpha_{ij} : j^2_p\xi \longmapsto \frac{\partial^2 \xi^\alpha}{\partial x^i \partial x^j}.$$

Using multi index notation for every natural number $k \geq 2$ we define the k-th order jet manifold, $J^k\pi$ with the equivalence relation

$$j^2_p\xi = j^2_p\eta \Longleftrightarrow \xi(p) = \eta(p),\ \frac{\partial^{|I|}\xi^\alpha}{\partial x^{|I|}} = \frac{\partial^{|I|}\eta^\alpha}{\partial x^{|I|}}$$

where I stands a multi-index with $1 \leq |I| \leq k$. $J^k\pi$ is also an smooth finite dimensional manifold with local charts $U^k = \{j^k_p\xi;\ \xi(p) \in U\}$, $\Phi^k = (x^i, \phi^\alpha, \phi^\alpha_{|I|})$ [17].

For any $k \in \mathbb{N}$, the map $\pi_{k+1,k} : J^{k+1}\pi \longrightarrow J^k\pi;\ j^{k+1}_p\xi \longmapsto j^k_p\xi$ is surjective submersion. For any $k \in \mathbb{N}$ let $dimJ^k\pi = N(k)$ and $\rho_{k+1,k} : \mathbb{R}^{N(K+1)} \longrightarrow \mathbb{R}^{N(k)}$ be the natural projection to the first $N(k)$-component. With these notations the family $\{J^k\pi, \pi_{k+1,k}\}_{k\in\mathbb{N}}$ forms a projective system (inverse system) of finite dimensional manifolds modelled on $\{\mathbb{R}^{N(k)}, \rho_{k+1,k}\}_{k\in\mathbb{N}}$. If $J^\infty\pi := \varprojlim J^k\pi$, then $J^\infty\pi$ is a Fréchet manifold modelled on the Fréchet space $\mathbb{R}^\infty = \varprojlim \mathbb{R}^k$. More precisely $J^\infty\pi$ is a subset of $\Pi_{k\in\mathbb{N}}J^k\pi$ consisting strings of the form $(j^k_p\xi)_{k\in\mathbb{N}}$ for $p \in M$ and $\xi \in \Gamma_p(\pi)$ and $(U^k, \Phi^k)_{k\in\mathbb{N}}$ is a projective system of charts with the limit $(U^\infty = \varprojlim U^k, \Phi^\infty = \varprojlim \Phi^k)$ as a projective limit chart for $J^\infty\pi$. Clearly the collection of these charts forms an atlas modelling $J^\infty\pi$ on $\mathbb{R}^\infty$.

In what follows ∇^k stands for the Levi-Civita connection associated to the Riemannian metric g_k. Now suppose that we have a system of Riemannian metrics $\{g^k\}_{k\in\mathbb{N}}$ on $\{J^k\pi\}_{k\in\mathbb{N}}$ such that one can construct a projective system of connections ($2ODE$'s) $\{\nabla^k\}_{k\in\mathbb{N}}$ with the limit $\nabla^\infty = \varprojlim \nabla^k$ as a generalized connection on $j^\infty\pi$. The existence and uniqueness of geodesics is a direct consequence of theorem 4.1. Moreover one can directly consider a family of (possibly nonlinear) connection which are not necessarily Levi-Civita connections and deduce the same results. (For an example of such connection on $J^\infty\pi$ see [14])

5.2. **Manifold of mappings.** Let N be a compact finite dimensional manifold of class C^r (for $r \geq 1$) and M a C^{r+s} Banach manifold with $s \geq 3$ admitting a connection ∇ of class C^{r+s-2}. Suppose that $C^k(N,M)$, $k \geq 1$, denote the space of C^k functions between manifolds N and M. For a C^k map $h : N \longrightarrow M$ we use the exponential map of ∇ to model $C^k(N,M)$ on the Banach (or Sobolev) space of C^k sections $C^k(h^*TM)$ where h^*TM is the pullback bundle of TM via h.

Eliasson in [8] theorem 5.4 proved that every C^{r+s-2} connection ∇ on M induces a natural connection $C^K(\nabla)$ (and therefore a $2ODE$ S^k_∇) on $C^k(N,M)$. The exponential map for $C^K(\nabla)$ is $C^k(exp_\nabla)$ where $C^k(exp_\nabla)\zeta := exp_\nabla \circ \zeta$. More precisely ζ belongs to $h^*\mathcal{D}$ where $\mathcal{D}$ is an open neighborhood of the set of zero vectors in TM.

By these means $\{C^k(N,M)\}_{k\in\mathbb{N}}$ is a projective system of Banach manifolds and the connecting morphisms are simply inclusions. Moreover we have a projective system of connections $\{C^k(\nabla)\}_{k\in\mathbb{N}}$ with the limit $C^\infty(\nabla)=\varprojlim C^k(\nabla)$ which is a limit connection on the Fréchet manifold $C^\infty(N,M)$. Now with our method we can construct the corresponding system of 2ODE and theorem 4.1 proposes an existence and uniqueness theorem for geodesics of $C^\infty(N,M)$.

Note that one can use the benefits of our method in the Eliasson's framework to study partial differential equations for maps $N\longrightarrow M$. Moreover one can apply our technique to study ordinary differential equations the group of diffeomorphisms [16].

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