International Electronic Journal of Geometry
Volume 6 No.2 pp. 88–99 (2013) ©IEJG

THE NATURAL LIFT CURVES AND GEODESIC CURVATURES OF THE SPHERICAL INDICATRICES OF THE TIMELIKE BERTRAND CURVE COUPLE

SÜLEYMAN ŞENYURT AND ÖMER FARUK ÇALIŞKAN

(Communicated by Murat TOSUN)

Abstract. In this paper, when (α, α^*) timelike Bertrand curve couple is given, the geodesic curves and the arc-lenghts of the curvatures (T^*), (N^*),(B^*) and the fixed pole curve (C^*) which are generated over the S_1^2 Lorentz sphere or the H_0^2 hyperbolic sphere by the Frenet vectors $\{T^*, N^*, B^*\}$ and the unit Darboux vector C^* have been obtained. The condition being the naturel lifts of the spherical indicatrix of the α^* is an integral curve of the geodesic spray has expressed.

1. Preliminaries

Let Minkowski 3-space $\mathbb{R}_1^3$ be the vector space $\mathbb{R}^3$ equipped with the Lorentzian inner product g given by

$$g(X, X) = x_1^2 + x_2^2 - x_3^2,$$

where $X = (x_1, x_2, x_3) \in \mathbb{R}^3$. A vector $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ is said to be timelike if $g(X, X) < 0$, spacelike if $g(X, X) > 0$ and lightlike (or null) if $g(X, X) = 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{R}_1^3$ where s is an arc-length parameter, can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors, $\alpha'(s)$ are respectively timelike, spacelike or null (lightlike) for every $s \in \mathbb{R}$. The norm of a vector $X \in \mathbb{R}_1^3$ is defined by [5]

$$\|X\| = \sqrt{|g(X, X)|}.$$

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve α. Let α be a timelike curve with curvature κ and torsion τ. Let Frenet vector fields of α

Date: Received: March 25, 2013 and Accepted June 24, 2013.
2010 *Mathematics Subject Classification.* 53A04.
Key words and phrases. Lorentz Space, Timelike Bertrand Curve Couple, Natural Lift, Geodesic Spray.

This article is the written version of author's plenary talk delivered on September 03-07, 2012 at 1^{st} International Euroasian Conference on Mathematical Sciences and Applications IECMSA-2012 at Prishtine, Kosovo.

be $\{T, N, B\}$. In this trihedron, T is a timelike vector field, N and B are spacelike vector fields. Then Frenet formulas are given by [8]

$$T' = \kappa N \ \ , N' = \kappa T - \tau B \ \ , B' = \tau N. \tag{1.1}$$

Let α be a timelike vector, the Frenet vectors T be timelike, N and B be spacelike vectors, respectively, such that

$$T \times N = -B, \ \mathrm{N} \times B = T, \ B \times T = -N,$$

and the Frenet instantaneous rotation vector is given by [6]

$$W = \tau T - \kappa B, \ \ \|W\| = \sqrt{|\kappa^2 - \tau^2|}.$$

Let φ be the angle between W and $-B$ vectors and if W is a spacelike vector, then we can write

$$\begin{cases} \kappa = \|W\| \cosh\varphi, \ \ \tau = \|W\| \sinh\varphi, \\ C = \sinh\varphi T - \cosh\varphi B \end{cases} \tag{1.2}$$

and if W is a timelike vector, then we can write

$$\begin{cases} \kappa = \|W\| \sinh\varphi, \ \ \tau = \|W\| \cosh\varphi, \\ C = \cosh\varphi T - \sinh\varphi B \end{cases} \tag{1.3}$$

Let $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ be the vectors in $\mathbb{R}^3_1$. The cross product of X and Y is defined by [1]

$$X \wedge Y = (x_3 y_2 - x_2 y_3, x_1 y_3 - x_3 y_1, x_1 y_2 - x_2 y_1).$$

The Lorentzian sphere and hyperbolic sphere of radius r and center 0 in $\mathbb{R}^3_1$ are given by

$$S^2_1 = \{X = (x_1, x_2, x_3) \in \mathbb{R}^3_1 | \ g(X, X) = r^2, \ r \in \mathbb{R}\}$$

and

$$H^2_0 = \{X = (x_1, x_2, x_3) \in \mathbb{R}^3_1 \ | g(X, X) = -r^2, \ r \in \mathbb{R}\}$$

respectively.

Let M be a hypersurface in $\mathbb{R}^3_1$. A curve $\alpha : I \to M$ is an integral curve of $X \in \chi(M)$ provided $\alpha' = X_\alpha$; that is

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)) \text{ for all } s \in I \quad [5].$$

For any parametrized curve $\alpha : I \to M$, the parametrized curve, $\overline{\alpha} : I \to TM$ given by

$\overline{\alpha}(s) = (\alpha(s), \alpha'(s)) = \alpha'(s)|_{\alpha(s)}.$ is called the natural lift of α on TM [7]. Thus we can write

$$\frac{d\overline{\alpha}}{ds} = \frac{d}{ds}(\alpha'(s))|_{\alpha(s)}. = D_{\alpha'(s)}\alpha'(s),$$

where D is the standart connection on $\mathbb{R}^3_1$. For $v \in TM$ the smooth vector field $X \in \chi(M)$ defined by

$$X(v) = \varepsilon g(v, S(v))|_{\alpha(s)}, \ \varepsilon = g(\xi, \xi) \quad [3]$$

is called the geodesic spray on the manifold TM, where ξ is the unit normal vector field of M and S is the shape operator of M.

Let $\alpha : I \to \mathbb{R}^3_1$ be a timelike vector. Let us consider the Frenet frame $\{T, N, B\}$ and the vector C. Accorollarydingly, arc-lengths and the geodesic curvatures of the spherical indicatrix curves $(T),(N)$ and (B) with the fixed pole curve (C) with respect to $\mathbb{R}^3_1$, respectively generated by the vectors T,N and B with the unit Darboux vector C are as follows:

$$\left\{\begin{array}{l} s_T=\displaystyle\int_0^s|\kappa|ds \\ s_N=\displaystyle\int_0^s\|W\|ds \end{array}\right. , \left\{\begin{array}{l} s_B=\displaystyle\int_0^s|\tau|ds \\ s_C=\displaystyle\int_0^s|\varphi'|ds \end{array}\right. \tag{1.4}$$

if W is a spacelike vector, then we can write

$$\left\{\begin{array}{l} k_T=\dfrac{1}{\cosh\varphi} \\ k_N=\sqrt{\left|1+\left(\dfrac{\varphi'}{\|W\|}\right)^2\right|} \end{array}\right. , \left\{\begin{array}{l} k_B=\dfrac{1}{\sinh\varphi} \\ k_C=\sqrt{\left|1+\left(\dfrac{\|W\|}{\varphi'}\right)^2\right|} \end{array}\right. \tag{1.5}$$

if W is a timelike vector, then we have

$$\left\{\begin{array}{l} k_T=\dfrac{1}{\sinh\varphi} \\ k_N=\sqrt{\left|1-\left(\dfrac{\varphi'}{\|W\|}\right)^2\right|} \end{array}\right. , \left\{\begin{array}{l} k_B=\dfrac{1}{\cosh\varphi} \\ k_C=\sqrt{\left|-1+\left(\dfrac{\|W\|}{\varphi'}\right)^2\right|} \end{array}\right. \quad [2] \tag{1.6}$$

Definition 1.1. Let α and α^* be two timelike curves in $\mathbb{R}^3_1$. $\{T,N,B\}$ and $\{T^*,N^*,B^*\}$ are Frenet frames, respectively, on these curves. $\alpha(s)$ and $\alpha^*(s)$ are called Bertrand curves if the principal normal vectors N and N^* a re linearly dependent, and the pair (α,α^*) is said to be timelike Bertrand curve couple [4].

Theorem 1.1. *Let (α,α^*) be timelike Bertrand curve couple. For corollaryresponding $\alpha(s)$ and $\alpha^*(s)$ points*

$$d(\alpha(s),\alpha^*(s))=constant,\ \forall s\in I \quad [4].$$

Theorem 1.2. *Let $(\alpha.\alpha^*)$ be timelike Bertrand curve couple. The measure of the angle between the vector fields of Bertrand curve couple is costant* [4].

2. THE NATURAL LIFT CURVES AND GEODISIC CURVATURES OF THE SPHERICAL INDICATRICES OF THE TIMELIKE BERTRAND CURVE COUPLE

Theorem 2.1. *Let (α,α^*) be timelike Bertrand curve couple. The relations between the Frenet vectors of the curve couple are as follows*

$$\left\{\begin{array}{l} T^*=-\cosh\theta T+\sinh\theta B \\ N^*=N \\ B^*=-\sinh\theta T+\cosh\theta B \end{array}\right. .$$

Here, the angle θ is the angle between T and T^*.

Proof. By taking the derivative of $\alpha^*(s)=\alpha(s)+\lambda N(s)$ with respect to arc-lenght s and using the equation (1.1), we get

$$T^*\frac{ds^*}{ds}=T(1+\lambda\kappa)-\lambda\tau B. \tag{2.1}$$

The inner products of the above equation with respect to T and B are respectively defined as

$$\begin{cases} -\cosh\theta\dfrac{ds^*}{ds} = 1+\lambda\kappa, \\ -\sinh\theta\dfrac{ds^*}{ds} = \lambda\tau \end{cases} \tag{2.2}$$

and by substituting these present equations in (2.1), we obtain

$$T^* = -\cosh\theta T + \sinh\theta B. \tag{2.3}$$

Here, by taking derivative and using the equation (1.1), we get

$$N^* = N. \tag{2.4}$$

We can write

$$B^* = -\sinh\theta T + \cosh\theta B \tag{2.5}$$

by availing the equation $B^* = -(T^* \times N^*)$.

Corollary 2.1. *Let (α, α^*) be a timelike Bertrand curve couple. Between the curvature κ and the torsion τ of the α, there is a relationship*

$$\mu\tau + (-\lambda)\kappa = 1 \;\; \textit{and} \;\; \mu = \lambda\coth\theta, \tag{2.6}$$

where λ and μ are nonzero real numbers.

Proof. From equation (2.2), we obtain

$$\frac{\cosh\theta}{1+\lambda\kappa} = \frac{\sinh\theta}{\lambda\tau},$$

and by arranging this equation, we get

$$\coth\theta = \frac{1+\lambda\kappa}{\lambda\tau}$$

and if we choose $\mu = \lambda\coth\theta$ for brevity, then we obtain

$$\mu\tau + (-\lambda)\kappa = 1.$$

Theorem 2.2. *There are connections between the curvatures κ and κ^* and the torsions τ and τ^* of the timelike Bertrand curve couple (α, α^*), which are shown as follows*

$$\begin{cases} \kappa^* = \dfrac{-\sinh^2\theta + \lambda\kappa}{\lambda(1+\lambda\kappa)}, \\ \tau^* = -\dfrac{\sinh^2\theta}{\lambda^2\tau} \end{cases} . \tag{2.7}$$

Proof. If α and α^* are Bertrand curve couple, we can write $\alpha(s) = \alpha^*(s) - \lambda N^*(s)$. By taking the derivative of this equation with respect to s^* and using equation (1.1) we obtain

$$T = T^*\frac{ds^*}{ds}(1-\lambda\kappa^*) + \lambda\tau^* B^*\frac{ds^*}{ds}.$$

The inner products of the above equation with respect to T^* and B^* are as followings

$$(2.8)\qquad \begin{cases} \cosh\theta = -(1-\lambda\kappa^*)\dfrac{ds^*}{ds} \\ \sinh\theta = \lambda\tau^*\dfrac{ds^*}{ds} \end{cases}$$

respectively. The proof can easily be completed by using and rearranging the equations (2.2) and (2.8).

Corollary 2.2. *Let (α, α^*) be a timelike Bertrand curve couple. Then*

$$(2.9)\qquad \kappa^* = \frac{\lambda\kappa - \sinh^2\theta}{\lambda^2\tau\coth\theta}$$

Proof. By using the equations (2.6) and with substition of them in (2.7), we get the desired result.

Theorem 2.3. *Let (α, α^*) be a timelike Bertrand curve couple. There are following relations between Darboux vector W of curve α and Darboux vector W^* of curve α^**

$$(2.10)\qquad W^* = \frac{\sinh\theta}{\lambda\tau}W.$$

Proof. For the Darboux vector W^* of timelike curve α^*, we can write

$$W^* = \tau^*T^* - \kappa^*B^*.$$

By substituting (2.3), (2.5), (2.7) and (2.9) into the last equation, we obtain

$$W^* = \frac{\sinh\theta}{\lambda\tau}[\frac{1}{\lambda}\tanh\theta(1+\lambda\kappa)T - \kappa B].$$

By substituting (2.6) into the above equation, we get

$$W^* = \frac{\sinh\theta}{\lambda\tau}W.$$

This completes the proof.

Now, let compute the arc-lengths of the spherical indicatrix curves, (T^*), (N^*), (B^*) and of the fixed pole curve, (C^*), and then calculate the geodesic curvatures of these in IR^3_1 and H^2_0 or S^2_1. of with the

Firstly, for the arc-length s_{T^*} of tangents indicatrix (T^*) of the curve α^*, we can write

$$s_{T^*} = \int_0^s \left\|\frac{dT^*}{ds}\right\| ds.$$

By taking the derivative of equation (2.3), we have

$$s_{T^*} \le |\cosh\theta| \int_0^s |\kappa| ds + |\sinh\theta| \int_0^s |\tau| ds.$$

By using equation (1.4) we obtain

$$s_{T^*} \le |\cosh\theta| s_T + |\sinh\theta| s_B.$$

For the arc-length s_{N^*} of principal normals indicatrix (N^*) of the curve α^*, we can write

$$s_{N^*} = \int_0^s \left\|\frac{dN^*}{ds}\right\| ds.$$

By substituting (2.4) into the above equation, we get

$$s_{N^*} = s_N.$$

Similarly, for the arc-length s_{B^*} of binormals indicatrix (B^*) of the curve α^*, we can write

$$s_{B^*} = \int_0^s \left\| \frac{dB^*}{ds} \right\| ds.$$

By taking the derivative of equation (2.5), we have

$$s_{B^*} \leq |\sinh\theta| \int_0^s |\kappa| ds + |\cosh\theta| \int_0^s |\tau| ds.$$

By using equation (1.4), we obtain

$$s_{B^*} \leq |\sinh\theta| s_T + |\cosh\theta| s_B.$$

Finally, for the arc-length s_{C^*} of the fixed pole curve (C^*), we can write

$$s_{C^*} = \int_0^s \| \frac{dC^*}{ds} \| ds.$$

If W^* is a spacelike vector, we can write $C^* = \sinh\varphi^* T^* - \cosh\varphi^* B^*$ from the equation (1.2). By taking the derivative of this equation, we obtain

$$s_{C^*} = \int_0^s |(\varphi^*)'| ds. \tag{2.11}$$

On the other hand, from equation (1.3) and by using

$$\cosh\varphi^* = \frac{\kappa^*}{\|W^*\|} \text{ ve } \sinh\varphi^* = \frac{\tau^*}{\|W^*\|}$$

we can set

$$\tanh\varphi^* = \frac{\tau^*}{\kappa^*}.$$

By substituting (2.7) and (2.9) into the last equation and after differentiation, we obtain

$$(\varphi^*)' = \frac{\lambda\kappa' \sinh\theta\cosh\theta}{\lambda^2\kappa^2 - (1+2\lambda\kappa)\sinh^2\theta}. \tag{2.12}$$

By substituting (2.12) into (2.11), we have

$$s_{C^*} = \int_0^s \left| \frac{\lambda\kappa' \sinh\theta\cosh\theta}{\lambda^2\kappa^2 - (1+2\lambda\kappa)\sinh^2\theta} \right| ds.$$

If W^* is a timelike vector, we have the same result. Thus the following corollary-ollary can be drawn.

Corollary 2.3. *Let (α, α^*) be a timelike Bertrand curve couple and $\{T^*, N^*, B^*\}$ be the Frenet frame of the curve α^*. For the arc-lengths of the spherical indicatrix curves (T^*), (N^*) and (B^*) with the fixed pole curve (C^*) with respect to $\mathbb{R}^3_1$, we have*

i. $s_{T^*} \leq |\cosh\theta| s_T + |\sinh\theta| s_B$,
ii. $s_{N^*} = s_N$,
iii. $s_{B^*} \leq |\sinh\theta| s_T + |\cosh\theta| s_B$,
iv. $s_{C^*} = \int_0^s \left| \frac{\lambda\kappa' \sinh\theta\cosh\theta}{\lambda^2\kappa^2 - (1+2\lambda\kappa)\sinh^2\theta} \right| ds.$

Now, let us compute the geodesic curvatures of the spherical indicatrix curves (T^*), (N^*) and (B^*) with the fixed pole curve (C^*) with respect to $\mathbb{R}^3_1$.

For the geodesic curvature k_{T^*} of the tangents indicatrix (T^*) of the curve α^*, we can write

$$k_{T^*} = \|D_{T_{T^*}} T_{T^*}\|. \tag{2.13}$$

By differentiating the curve $\alpha_{T^*}(s_{T^*}) = T^*(s)$ with the respect to s_{T^*} and normalizing, we obtain

$$T_{T^*} = N.$$

By taking derivative of the last equation we get

$$D_{T_{T^*}} T_{T^*} = \frac{\kappa T - \tau B}{|-\kappa \cosh\theta + \tau \sinh\theta|}. \tag{2.14}$$

By substituting (2.14) into (2.13) we have

$$k_{T^*} = \frac{\|W\|}{|-\kappa \cosh\theta + \tau \sinh\theta|}.$$

Here, if W is a spacelike vector, by substituting (1.2) and (1.5) into the last equation we have

$$k_{T^*} = \left|\frac{k_T \cdot k_B}{k_T \cdot \sinh\theta - k_B \cdot \cosh\theta}\right|,$$

if W is a timelike vector, then by substituting (1.3) and (1.6) we have the same result.

Similarly, by differentiating the curve $\alpha_{N^*}(s_{N^*}) = N^*(s)$ with the respect to $s_{N^*}$and by normalizing we obtain

$$T_{N^*} = \frac{\kappa}{\|W\|} T - \frac{\tau}{\|W\|} B.$$

If W is a spacelike vector, then by using equation (1.2) we have

$$T_{N^*} = \cosh\varphi T - \sinh\varphi B,$$

$$D_{T_{N^*}} T_{N^*} = \frac{\varphi'}{\|W\|}(\sinh\varphi T - \cos\varphi B) + N, \;\; k_{N^*} = k_N = \sqrt{\left(\frac{\varphi'}{\|W\|}\right)^2 + 1}. \tag{2.15}$$

If W is a timelike vector, then by using of the equations (1.3) and (1.5) we have

$$D_{T_{N^*}} T_{N^*} = \frac{\varphi'}{\|W\|}(\cosh\varphi T - \sinh\varphi B) - N, \tag{2.16}$$

$$k_{N^*} = k_N = \sqrt{\left|1 - \left(\frac{\varphi'}{\|W\|}\right)^2\right|}.$$

By differentiating the curve $\alpha_{B^*}(s_{B^*}) = B^*(s)$ with the respect to s_{B^*} and by normalizing, we obtain

$$T_{B^*} = N.$$

By taking the derivative of the last equation we get

$$D_{T_{B^*}} T_{B^*} = \frac{\kappa T - \tau B}{|-\kappa \sinh\theta + \tau \cosh\theta|} \tag{2.17}$$

or by taking the norm of equation (2.17), we obtain

$$k_{B^*} = \frac{\|W\|}{|-\kappa \sinh\theta + \tau \cosh\theta|}.$$

If W is a spacelike vector, then by substituting (1.2) and (1.5) we have

$$k_{B^*} = \Big|\frac{k_T \cdot k_B}{k_T \cdot \cosh\theta - k_B \cdot \sinh\theta}\Big|,$$

if W is a timelike vector, then by substituting (1.3) and (1.6) we have the same result.

By differentianting the curve $\alpha_{C^*}(s_{C^*}) = C^*(s)$ with the respect to s_{C^*} and normalizing, if W^* is a spacelike vector, then by substituting (1.2) we obtain

$$T_{C^*} = \cosh\varphi^* T^* - \sinh\varphi^* B^*,$$

$$D_{T_{C^*}} T_{C^*} = (\sinh\varphi^* T^* - \cosh\varphi^* B^*) + \frac{\|W^*\|}{(\varphi^*)'} N^* \tag{2.18}$$

$$k_{C^*} = \sqrt{1 + \Big(\frac{\|W^*\|}{(\varphi^*)'}\Big)^2}. \tag{2.19}$$

By substituting (2.10) and (2.12) into (2.19) and rearranging we have

$$k_{C^*} = \sqrt{\Big|\frac{(\kappa^2 - \tau^2)[\lambda^2\kappa^2 - (1 + 2\lambda\kappa)\sinh^2\theta]^2}{(\lambda^2\tau\kappa')^2\cosh^2\theta} + 1\Big|}.$$

If W^* is a timelike vector, then by substituting (1.1) and (1.3) we get

$$T_{C^*} = \sinh\varphi^* T^* - \cosh\varphi^* B^*,$$

$$D_{T_{C^*}} T_{C^*} = (\cosh\varphi^* T^* - \sinh\varphi^* B^*) + \frac{\|W^*\|}{(\varphi^*)'} N^*, \tag{2.20}$$

$$k_{C^*} = \sqrt{\Big| - 1 + (\frac{\|W^*\|}{(\varphi^*)'})^2\Big|}. \tag{2.21}$$

By substituting (2.10) and (2.12) into (2.21) we have

$$k_{C^*} = \sqrt{\Big|\frac{(\tau^2 - \kappa^2)[\lambda^2\kappa^2 - (1 + 2\lambda\kappa)\sinh^2\theta]^2}{(\lambda^2\tau\kappa')^2\cosh^2\theta} - 1\Big|}.$$

Then the following corollaryollary can be given.

Corollary 2.4. *Let (α, α^*) be a timelike Bertrand curve cuople and $\{T^*, N^*, B^*\}$ be Frenet frame of the curve α^*. For the geodesic curvatures of the spherical indicatrix curves $(T^*), (N^*)$ and (B^*) with the fixed pole curve (C^*) with the respect to $\mathbb{R}^3_1$ we have*

i. $k_{T^*} = \Big|\dfrac{k_T \cdot k_B}{k_T \cdot \sinh\theta - k_B \cdot \cosh\theta}\Big|,$

ii. $$\begin{cases} k_{N^*} = k_N = \sqrt{\Big|\Big(\dfrac{\varphi'}{\|W\|}\Big)^2 + 1\Big|}, & W \ \textit{spacelike} \\ k_{N^*} = k_N = \sqrt{\Big|1 - \Big(\dfrac{\varphi'}{\|W\|}\Big)^2\Big|}, & W \ \textit{timelike}, \end{cases}$$

iii. $k_{B^*} = \Big|\dfrac{k_T \cdot k_B}{k_T \cdot \cosh\theta - k_B \cdot \sinh\theta}\Big|,$

$$iv.\ \begin{cases} k_{C^*} = \sqrt{\Big|\dfrac{(\kappa^2-\tau^2)[\lambda^2\kappa^2-(1+2\lambda\kappa)\sinh^2\theta]^2}{(\lambda^2\tau\kappa')^2\cosh^2\theta}+1\Big|}, & W^* \ \textit{spacelike} \\ k_{C^*} = \sqrt{\Big|\dfrac{(\tau^2-\kappa^2)[\lambda^2\kappa^2-(1+2\lambda\kappa)\sinh^2\theta]^2}{(\lambda^2\tau\kappa')^2\cosh^2\theta}-1\Big|}, & W^* \ \textit{timelike.} \end{cases}$$

Now let us compute the geodesic curvatures $(T^*), (N^*)$ and (B^*) with the fixed pole curve (C^*) with respect to H_0^2 or S_1^2.

For the geodesic curvature γ_{T^*} of the tangents indicatrix curve (T^*) of the curve α^* with respect to H_0^2, we can write

$$\gamma_{T^*} = \|\overline{\overline{D}}_{T_{T^*}}T_{T^*}\|. \tag{2.22}$$

Here, $\overline{\overline{D}}$ becomes a covariant derivative operator. By (2.3) and (2.14) we obtain

$$D_{T_{T^*}}T_{T^*} = \overline{\overline{D}}_{T_{T^*}}T_{T^*} + \varepsilon g(S(T_{T^*}), T_{T^*})T^*,$$

$$\begin{aligned} \overline{\overline{D}}_{T_{T^*}}T_{T^*} &= \Big(\frac{\kappa}{|-\kappa\cosh\theta+\tau\sinh\theta|}+\cosh\theta\Big)T \\ &+\Big(\frac{-\tau}{|-\kappa\cosh\theta+\tau\sinh\theta|}-\sinh\theta\Big)B. \end{aligned} \tag{2.23}$$

By substituting (2.23) into (2.22) we get

$$\gamma_{T^*} = \sqrt{\Big|\frac{\tau^2-\kappa^2}{(-\kappa\cosh\theta+\tau\sinh\theta)^2}+1\Big|}.$$

If W is a spacelike vector, then by using of the equations (1.2) and (1.5) we have

$$\gamma_{T^*} = \sqrt{\Big|-\Big(\frac{k_T k_B}{-k_B\cosh\theta+k_T\sinh\theta}\Big)^2+1\Big|},$$

if W is a timelike vector, then by using of the equations (1.3) and (1.6) we have

$$\gamma_{T^*} = \sqrt{\Big(\frac{k_T k_B}{-k_B\cosh\theta+k_T\sinh\theta}\Big)^2+1}.$$

If the curve $(\overline{T^*})$ is an integral curve of the geodesic spray, then $\overline{\overline{D}}_{T_{T^*}}T_{T^*} = 0$. Thus, by (2.23) we can write

$$\begin{cases} \dfrac{\kappa}{|-\kappa\cosh\theta+\tau\sinh\theta|}+\cosh\theta = 0 \\ \dfrac{-\tau}{|-\kappa\cosh\theta+\tau\sinh\theta|}-\sinh\theta = 0 \end{cases}$$

and here, we obtain $\kappa > 0$, $\tau = 0$ and $\theta = 0$. So, we can give following corollaryollary.

Corollary 2.5. *Let (α, α^*) be a timelike Bertrand curve couple. If the curve α is a plenary curve and frames are equivalent, the natural lift $(\overline{T^*})$ of the tangent indicatrix (T^*) is an integral curve of the geodesic spray.*

For the geodesic curvature γ_{N^*} of the principal normals indicatrix curve (N^*) of the curve α^* with respect to S_1^2 we can write

$$\gamma_{N^*} = \|\overline{D}_{T_{N^*}}T_{N^*}\|. \tag{2.24}$$

Here, $\overline{D}$ becomes a covariant derivative operator. If W is a spacelike vector, by using of the equation (2.15) we obtain

$$\overline{D}_{T_{N^*}} T_{N^*} = \frac{\varphi'}{\|W\|}(\sinh\varphi T - \cosh\varphi B). \tag{2.25}$$

By substituting (2.25) into (2.24) we get

$$\gamma_{N^*} = \frac{\varphi'}{\|W\|}. \tag{2.26}$$

On the other hand, from the equation (1.2), by using

$$\sinh\varphi = \frac{\tau}{\|W\|} \text{ and } \cosh\varphi = \frac{\kappa}{\|W\|},$$

we can set

$$\tanh\varphi = \frac{\tau}{\kappa}.$$

By taking the derivative of the last equation we get

$$\varphi' = \frac{\tau'\kappa - \kappa'\tau}{\|W\|^2}.$$

By substituting the e above quation into (2.26) we have

$$\gamma_{N^*} = \gamma_N = \frac{\tau'\kappa - \kappa'\tau}{\|W\|^3}.$$

If W is a timelike vector, by using of the equation (2.16) we obtain

$$\overline{D}_{T_{N^*}} T_{N^*} = \frac{\varphi'}{\|W\|}(\cosh\varphi T - \sinh\varphi B), \tag{2.27}$$

$$\gamma_{N^*} = \frac{\varphi'}{\|W\|}.$$

On the other hand, from equation (1.3) by using

$$\sinh\varphi = \frac{\kappa}{\|W\|} \text{ and } \cosh\varphi = \frac{\tau}{\|W\|},$$

we can set

$$\tanh\varphi = \frac{\kappa}{\tau}.$$

By taking the derivative of the last equation we get

$$\varphi' = \frac{\kappa'\tau - \tau'\kappa}{\|W\|^2}$$

or

$$\gamma_{N^*} = \gamma_N = \frac{\kappa'\tau - \tau'\kappa}{\|W\|^3}.$$

If the curve $(\overline{N^*})$ is an integral curve of the geodesic spray, then $\overline{D}_{T_{N^*}} T_{N^*} = 0$. Thus, by (2.25) and (2.27) we can write $\varphi' = 0$ and here, we obtain $\frac{\kappa}{\tau} = \text{constant}$. So, we can give following corollaryollary.

Corollary 2.6. *Let (α, α^*) be a timelike Bertrand curve couple. If the curve α is a helix curve, the natural lift $(\overline{N^*})$ of the pirincipal normal indicatrix (N^*) is an integral curve of the geodesic spray.*

For the geodesic curvature γ_{B^*} of the binormal indicatrix curve (B^*) of the curve $\alpha^*$with respect to S_1^2 and with substitution of (2.5) and (2.17) we obtain

$$D_{T_{B^*}}T_{B^*} = \overline{D}_{T_{B^*}}T_{B^*} + \varepsilon g(S(T_{B^*}), T_{B^*})B^*,$$

$$\begin{aligned}\overline{D}_{T_{B^*}}T_{B^*} \quad = \quad & \Big(\frac{\kappa}{|-\kappa\sinh\theta + \tau\cosh\theta|} - \sinh\theta\Big)T \\ & + \Big(-\frac{\tau}{|-\kappa\sinh\theta + \tau\sinh\theta|} + \cosh\theta\Big)B,\end{aligned} \tag{2.28}$$

$$\gamma_{B^*} = \sqrt{\Big| -1 + \frac{\tau^2 - \kappa^2}{(-\kappa\sinh\theta + \tau\cosh\theta)^2}\Big|}.$$

If W is a spacelike vector, then by using of the equations (1.2) and (1.5) we have

$$\gamma_{B^*} = \sqrt{\Big| -1 - \Big(\frac{k_T k_B}{-k_B\sinh\theta + k_T\cosh\theta}\Big)^2\Big|},$$

if W is a timelike vector, then by using of the equations (1.3) and (1.6) we get

$$\gamma_{B^*} = \sqrt{\Big| -1 + \Big(\frac{k_T k_B}{-k_B\sinh\theta + k_T\cosh\theta}\Big)^2\Big|}.$$

If the curve $(\overline{B^*})$ is an integral curve of the geodesic spray, then $\overline{D}_{T_{B^*}}T_{B^*} = 0$. Thus, by (2.28) we can write

$$\begin{cases} \dfrac{\kappa}{|-\kappa\sinh\theta + \tau\cosh\theta|} - \sinh\theta = 0, \\ \dfrac{-\tau}{|-\kappa\sinh\theta + \tau\cosh\theta|} + \cosh\theta = 0 \end{cases}$$

and here, we obtain $\kappa = 0$, $\tau \neq 0$ and $\theta = 0$. So, we can give following corollaryollary.

Corollary 2.7. *Let (α, α^*) be a timelike Bertrand curve couple. The natural lift $(\overline{B^*})$ of the binormal indicatrix (B^*) is never an integral curve of the geodesic spray.*

If W^* is a spacelike vector, for the geodesic curvature γ_{C^*} of the fixed pole curve (C^*) of the curve α^* with respect to S_1^2 and by using of the equations (1.2) and (2.18) we obtain

$$D_{T_{C^*}}T_{C^*} = \overline{D}_{T_{C^*}}T_{C^*} + \varepsilon g(S(T_{C^*}), T_{C^*})C^*,$$

$$\overline{D}_{T_{C^*}}T_{C^*} = \frac{\|W^*\|}{(\varphi^*)'}N^*, \tag{2.29}$$

$$\gamma_{C^*} = \Big\|\frac{\|W^*\|}{(\varphi^*)'}\Big\|.$$

By substituting (2.10) and (2.12) into the last equation we have

$$\gamma_{c^*} = \frac{\|W\| \cdot [\lambda^2\kappa^2 - (1 + 2\lambda\kappa)\sinh^2\theta]}{\lambda^2\tau\kappa'\cosh\theta}.$$

If W^* is a timelike vector, for the geodesic curvature γ_{C^*} of the fixed pole curve (C^*) with respect to H_0^2 and by using of the equations (1.3) and (2.20) we have the same result. If the curve $(\overline{C^*})$ is an integral curve of the geodesic spray, then

$\overline{\overline{D}}_{T_{C^*}} T_{C^*} = 0$. Thus by (2.29) we can write $\|W^*\| = 0$ and here, we get $\kappa^* = \tau^* = 0$ or $\kappa^* = \tau^*$. Thus, by using of the equation (2.7) and (2.9) we obtain

$$\kappa = \frac{\sinh^2\theta - \sinh\theta\cosh\theta}{\lambda}.$$

So, we can give following corollaryollary.

Corollary 2.8. *Let (α, α^*) be a timelike Bertrand curve couple. If the curve α is a curve that provides the requirement $\kappa = \frac{\sinh^2\theta - \sinh\theta\cosh\theta}{\lambda}$, the natural lift $(\overline{C^*})$ of the fixed pole curve (C^*) is an integral curve of the geodesic spray.*

Corollary 2.9. *Let (α, α^*) be a timelike Bertrand curve couple and $\{T^*, N^*, B^*\}$ be Frenet frame of the curve α^*. For the geodesic curvatures of the spherical indicatrix curves $(T^*), (N^*)$ and (B^*) with the fixed pole curve (C^*) with respect to H_0^2 or S_1^2, we have*

$$i.\ \begin{cases} \gamma_{T^*} = \sqrt{\left| -\left(\dfrac{k_T k_B}{-k_B\cosh\theta + k_T\sinh\theta}\right)^2 + 1\right|}, & W \text{ spacelike} \\ \gamma_{T^*} = \sqrt{\left| \left(\dfrac{k_T k_B}{-k_B\cosh\theta + k_T\sinh\theta}\right)^2 + 1\right|}, & W \text{ timelike,} \end{cases}$$

$$ii.\ \begin{cases} \gamma_{N^*} = \gamma_N = \dfrac{\tau'\kappa - \kappa'\tau}{\|W\|^3}, & W \text{ spacelike} \\ \gamma_{N^*} = \gamma_N = \dfrac{\kappa'\tau - \tau'\kappa}{\|W\|^3}, & W \text{ timelike,} \end{cases}$$

$$iii.\ \begin{cases} \gamma_{B^*} = \sqrt{\left| -1 - \left(\dfrac{k_T k_B}{-k_B\sinh\theta + k_T\cosh\theta}\right)^2\right|}, & W \text{ spacelike} \\ \gamma_{B^*} = \sqrt{\left| -1 + \left(\dfrac{k_T k_B}{-k_B\sinh\theta + k_T\cosh\theta}\right)^2\right|}, & W \text{ timelike,} \end{cases}$$

$$iv.\ \gamma_{c^*} = \frac{\|W\| \cdot [\lambda^2\kappa^2 - (1 + 2\lambda\kappa)\sinh^2\theta]}{\lambda^2\tau\kappa'\cosh\theta}.$$

References

[1] Akutagawa, K. and Nishikawa S., The Gauss Map and Spacelike Surfaces with Prescribed Mean Curvature in Minkowski 3-space, Tohoku Math., J. **42**(1990), 67-82.

[2] Bilici, M., Ph.d. Dissertation, Ondokuzmayıs University Institute of Science and Technology, Samsun, 2009.

[3] Çalışkan M., Sivridağ A.İ., Hacısalihoğlu H. H., Some Characterizations For The Natural Lift Curves And The Geodesic Sprays, Commun. Fac. Sci. Ank. Series A1 **33**(1984), 235-242.

[4] Ekmekçi, N. and Ilarslan K., On Bertrand Curves and Their Characterization, Differential Geometry-Dynamical Systems, **3**(2001), 17-24.

[5] O'Neill, B., Semi Riemann Geometry, Academic Press, New York, London, 1983.

[6] Uğurlu, H.H., On the Geometry of Timelike Surfaces, Commun. Fac. Sci. Ank. Series A1 **46**(1997), 211-223.

[7] Thorpe, J.A., Elemantary Topics in Differential Geometry, Springer. New York. 1979.

[8] Woestijne, V.D.I., Minimal Surfaces of the 3-dimensional Minkowski space. Proc. Congres Geometrie differentielle et aplications, Avignon (30 May 1988), Wold Scientific Publishing. Singapore. 344-369, 1990.

Department of Mathematics, University of Ordu-TURKEY

E-mail address: senyurtsuleyman@hotmail.com