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GEODESICS ON THE TANGENT SPHERE BUNDLE OF 3-SPHERE

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Abstract. The Sasaki Riemann metric g^S on the tangent sphere bundle T_1S^3 of the unit 3-sphere S^3 is obtained by using the geodesic polar coordinate of S^3. The connection coefficients of the Levi Civita connection of the Sasaki Riemann manifold $\left(T_1S^3, g^S\right)$ are found. Furthermore, a system of differential equations which gives all geodesics of Sasaki Riemann manifold is obtained.

1. Introduction

The unit 3-sphere and its tangent sphere bundle are important issues of the differential geometry which have attracted the interest of physicists as well as mathematicians.

The unit 3 sphere has been considered as non-relativistic closed universe model by physicists [7]. According to this model, the universe has expanded since Big Bang and this expansion will continue until Big Crunch.

In [5], U. Pincall considered Hopf tori in S^3 which is the inverse image of the closed curves on S^2 by helping the Hopf projection $p : S^3 \to S^2$.

In [6], Sasaki classified geodesics on the tangent sphere bundles of the unit n-sphere S^n and the hyperbolic n-space H^n by using Sasaki metric on T_1S^n and T_1H^n. Moreover, he obtained geodesics of horizontal, vertical and oblique types on the tangent sphere bundles of the unit 3-sphere and the unit hyperbolic 2-space.

In [1], Klingenberg and Sasaki obtained the Sasaki Riemann metric on T_1S^2 by using the geodesic polar coordinate of S^2, and they indicated that the unit vector fields which make a constant angle with the geodesic circles of unit sphere S^2 constitute geodesics of T_1S^2.

In [2] and [3], P. T. Nagy expanded the studies in this field from space forms to Riemann manifolds. He defined a new metric on the tangent sphere bundle of a

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Riemann manifold and examined the geometry of the tangent sphere bundle of the Riemann manifold with respect to this new metric.

In this paper, the Sasaki Riemann metric g^S on the tangent sphere bundle T_1S^3 of the unit 3 sphere S^3 is obtained by using the geodesic polar coordinates of S^3. The connection coefficients of the Levi Civita connection of the Sasaki Riemann manifold $\left(T_1S^3, g^S\right)$ are calculated. Furthermore, a system of differential equations which gives all geodesics on $\left(T_1S^3, g^S\right)$ is obtained.

2. The Riemann Manifold $\left(S^3, g\right)$

This section has been developed by using [2], [4], and [6]. This section consists of some subjects as the representation with respect to the geodesic polar coordinates of the unit 3 sphere, the induced Riemann metric on S^3, the basis vectors of the tangent vector space at any point of S^3, the Christoffel symbols of S^3, and the differential equations system which gives all geodesics of S^3.

Let $<\ ,\ >$ be positive definite, symmetric, bilinear form in 4 dimensional Euclidean space E^4 defined by

$$< u, v >= u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4, \tag{2.1}$$

for any vectors $u, v \in E^4$. S^3 is a surface in E^4 given by

$$S^3 = \left\{u = (x_1, x_2, x_3, x_4) :< u, u >= 1, u \in E^4\right\}. \tag{2.2}$$

S^3 is called as the unit 3 sphere in E^4. The unit 3 sphere is given by the following equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1, \tag{2.3}$$

with respect to Cartesian coordinate system. The unit 3 sphere is also represented by

$$\begin{aligned} x_1 &= \sin\omega \sin a \cos\theta, \\ x_2 &= \sin\omega \sin a \sin\theta, \\ x_3 &= \sin\omega \cos a, \\ x_4 &= \cos\omega, \end{aligned} \tag{2.4}$$

with respect to the geodesic polar coordinate of S^3 if a curve on S^3 is described by giving the following coordinates as a function of a single parameter t.

$$\begin{aligned} a &= a(t), \\ \theta &= \theta(t), \\ \omega &= \omega(t). \end{aligned} \tag{2.5}$$

In order to find the arc length between infinitely close two points on the unit 3-sphere, the covariant derivations of x_1, x_2, x_3, x_4 is used, given by

$$\begin{aligned} dx_1 &= \cos\omega \sin a \cos\theta d\omega + \sin\omega \cos a \cos\theta da - \sin\omega \sin a \sin\theta d\theta, \\ dx_2 &= \cos\omega \sin a \sin\theta d\omega + \sin\omega \cos a \sin\theta da + \sin\omega \sin a \cos\theta d\theta, \\ dx_3 &= \cos\omega \cos a d\omega - \sin\omega \sin a da, \\ dx_4 &= -\sin\omega d\omega. \end{aligned} \tag{2.6}$$

The arc length between infinitely close two points on the surface S^3 (i.e. (x_1, x_2, x_3, x_4) and $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3, x_4 + dx_4)$ is calculated by

$$\begin{aligned} ds^2 &=< (dx_1, dx_2, dx_3, dx_4), (dx_1, dx_2, dx_3, dx_4) > \\ &= (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2 . \end{aligned} \tag{2.7}$$

By using the (2.6), we get for ds^2 the following:

$$ds^2 = d\omega^2 + \sin^2\omega \left(da^2 + \sin^2 a d\theta^2\right), \tag{2.8}$$

and also the matrix representation of the equation in (2.8)

$$(g_{ik}) : \begin{pmatrix} \sin^2\omega & 0 & 0 \\ 0 & \sin^2\omega \sin^2 a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.9}$$

where (g_{ik}), for $i, k \in \{a, \theta, \omega\}$ is called as induced metric on S^3 from E^4. The inverse matrix of (g_{ik}) is given by

$$\left(g^{kj}\right) : \begin{pmatrix} \frac{1}{\sin^2\omega} & 0 & 0 \\ 0 & \frac{1}{\sin^2 a \sin^2\omega} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.10}$$

Let $e_1(a, \theta, \omega)$ be any point on the surface S^3 given by

$$e_1(a, \theta, \omega) = (\sin\omega \sin a \cos\theta,\ \sin\omega \sin a \sin\theta,\ \sin\omega \cos a,\ \cos\omega), \tag{2.11}$$

with respect to standard orthonormal basis of E^4. Since the orthogonal curves on the surface S^3 is described by $a = a(t)$, $\theta = \theta(t)$ and $\omega = \omega(t)$, the unit tangent vectors of orthogonal curves passing through the point $e_1(a, \theta, \omega)$ on the surface S^3 can be defined by

$$f_2 = \frac{\partial}{\partial\omega}, \quad f_3 = \frac{1}{\sin\omega}\frac{\partial}{\partial a}, \quad f_4 = \frac{1}{\sin\omega \sin a}\frac{\partial}{\partial\theta}. \tag{2.12}$$

Moreover, the local expressions of the unit tangent vectors f_2, f_3 and f_4 at the point $e_1(a, \theta, \omega)$ on the surface S^3 are also given by

$$\begin{aligned} f_2(a, \theta, \omega) &= (\cos\omega \sin a \cos\theta, \cos\omega \sin a \sin\theta, \cos\omega \cos a, -\sin\omega), \\ f_3(a, \theta, \omega) &= (\cos a \cos\theta, \cos a \sin\theta, -\cos a, 0), \\ f_4(a, \theta, \omega) &= (-\sin\theta, \cos\theta, 0, 0), \end{aligned} \tag{2.13}$$

with respect to standard orthonormal basis of E^4. Thus f_2, f_3, f_4 are the basis vectors of tangent vector space at any point $e_1(a, \theta, \omega)$ on S^3.

Definition 2.1. Let S^3 be the unit 3 sphere in 4-dimensional Euclidean space and let $T_{e_1}S^3$ be the tangent vector space consisting of the unit tangent vectors at a point $e_1(a, \theta, \omega)$ on S^3. g is a real valuable function on $T_{e_1}S^3$ defined by

$$\begin{aligned} g: \quad T_{e_1}S^3 \times T_{e_1}S^3 &\to IR \\ (X, Y) &\to g(X, Y) = X^T (g_{ik}) Y, \end{aligned} \tag{2.14}$$

where (g_{ik}), $i, k \in \{a, \theta, \omega\}$ is the matrix which corresponds to the metric g given by (2.9). Since g is positive definite, symmetric and bilinear, g must be called as induced Riemann metric on S^3 from E^4.

Theorem 2.1. *Let S^3 be the unit 3 sphere in 4-dimensional Euclidean space and let $\{e_1, f_2, f_3, f_4\}$ be another orthonormal basis in Euclidean space E^4. The covariant derivations of e_1, f_2, f_3, f_4 are given by*

$$\begin{aligned} de_1 &= d\omega f_2 + \sin\omega da f_3 + \sin\omega \sin a d\theta f_4, \\ df_2 &= -d\omega e_1 + \cos\omega da f_3 + \cos\omega \sin a d\theta f_4, \\ df_3 &= -\sin\omega da e_1 - \cos\omega da f_2 + \cos a d\theta f_4, \\ df_4 &= -\sin\omega \sin a d\theta e_1 - \cos\omega \sin a d\theta f_2 - \cos a d\theta f_3. \end{aligned}$$

Proof. We can use the covariant derivations of orthonormal vectors e_1, f_2, f_3, f_4 in order to examine the change of the basis vectors on a point in the other infinite closer of each point $e_1(a, \theta, \omega)$ on S^3. The covariant derivatives of these vectors are calculated by using the partial derivation operation as follows:

$$\begin{aligned} de_1 &= \frac{\partial e_1}{\partial a}da + \frac{\partial e_1}{\partial \theta}d\theta + \frac{\partial e_1}{\partial \omega}d\omega = d\omega f_2 + \sin\omega da f_3 + \sin\omega \sin a d\theta f_4, \\ df_2 &= \frac{\partial f_2}{\partial a}da + \frac{\partial f_2}{\partial \theta}d\theta + \frac{\partial f_2}{\partial \omega}d\omega = -d\omega e_1 + \cos\omega da f_3 + \cos\omega \sin a d\theta f_4, \\ df_3 &= \frac{\partial f_3}{\partial a}da + \frac{\partial f_3}{\partial \theta}d\theta + \frac{\partial f_3}{\partial \omega}d\omega = -\sin\omega da e_1 - \cos\omega da f_2 + \cos a d\theta f_4, \\ df_4 &= \frac{\partial f_4}{\partial a}da + \frac{\partial f_4}{\partial \theta}d\theta + \frac{\partial f_4}{\partial \omega}d\omega = -\sin\omega \sin a d\theta e_1 - \cos\omega \sin a d\theta f_2 - \cos a d\theta f_3. \end{aligned}$$

□

Theorem 2.2. *Let (S^3, g) be Riemann manifold. Let D be Levi Civita connection of (S^3, g) and let $\phi^k_{ij}; i, j, k \in \{a, \theta, \omega\}$ be Christoffel symbols related to the Riemann metric g. Then the non-zero the Christoffel symbols of (S^3, g) are given by*

$$\begin{array}{lll} \phi^\omega_{aa} = -\sin\omega\cos\omega, & \phi^\theta_{a\theta} = \cot a, & \phi^a_{a\omega} = \cot\omega, \\ \phi^a_{\theta\theta} = -\sin a \cos a, & \phi^\omega_{\theta\theta} = -\sin\omega\cos\omega\sin^2 a, & \phi^\theta_{\theta\omega} = \cot\omega, \end{array}$$

where $\phi^k_{ij} = \phi^k_{ji}$ for all $i, j, k \in \{a, \theta, \omega\}$.

Proof. On the Riemann manifold (S^3, g), there is a unique connection D such that D is torsion free and compatible with the Riemann metric g. This connection is called as Levi Civita connection and characterized by the Kozsul formula:

$$\begin{aligned} 2g\left(D_{\partial_a}\partial_\theta, \partial_\omega\right) &= \partial_a g\left(\partial_\theta, \partial_\omega\right) + \partial_\theta g\left(\partial_\omega, \partial_a\right) - \partial_\omega g\left(\partial_a, \partial_\theta\right) \\ &\quad - g\left(\left[\partial_a, \partial_\theta\right], \partial_\omega\right) + g\left(\left[\partial_\theta, \partial_\omega\right], \partial_a\right) + g\left(\left[\partial_\omega, \partial_a\right], \partial_\theta\right), \end{aligned}$$

where $\partial_a = \frac{\partial}{\partial a}$, $\partial_\theta = \frac{\partial}{\partial \theta}$ and $\partial_\omega = \frac{\partial}{\partial \omega}$. Since D is symmetric, $[\partial_a, \partial_\theta]$, $[\partial_\theta, \partial_\omega]$ and $[\partial_\omega, \partial_a]$ must be zero. If we get $D_{\partial_a}\partial_\theta = \phi^a_{a\theta}\partial_a + \phi^\theta_{a\theta}\partial_\theta + \phi^\omega_{a\theta}\partial_\omega$, Christoffel symbols are obtained by

$$\begin{aligned} \phi^a_{a\theta} &= \frac{1}{2}g^{am}\left(\partial_a g_{m\theta} + \partial_\theta g_{am} - \partial_m g_{a\theta}\right) = 0, \\ \phi^\theta_{a\theta} &= \frac{1}{2}g^{\theta m}\left(\partial_a g_{m\theta} + \partial_\theta g_{am} - \partial_m g_{a\theta}\right) = \cot a, \\ \phi^\omega_{a\theta} &= \frac{1}{2}g^{\omega m}\left(\partial_a g_{m\theta} + \partial_\theta g_{am} - \partial_m g_{a\theta}\right) = 0, \text{ for } m \in \{a, \theta, \omega\}. \end{aligned}$$

The other Christoffel symbols can be obtained by using the similar method. □

Theorem 2.3. *Let $\left(S^3, g\right)$ be Riemann manifold and let $c : t \in R \to c(t) = (a(t), \theta(t), \omega(t))$ be a curve on S^3. c is geodesic if and only if the following second order differential equations are provided:*

$$\begin{aligned} \ddot{a} - \sinh a \cosh a\dot{\theta}^2 + 2\cot\omega\dot{a}\dot{\omega} &= 0, \\ \ddot{\theta} + 2\cot a\dot{a}\dot{\theta} + 2\cot\omega\dot{\theta}\dot{\omega} &= 0, \\ \ddot{\omega} - \sin\omega\cos\omega\dot{a}^2 - \sin\omega\cos\omega\sin^2 a\dot{\theta}^2 &= 0. \end{aligned}$$

Proof. $c(t) = (a(t), \theta(t), \omega(t))$ is geodesic if and only if $D_{\dot{c}}\dot{c}$ is zero. Since $\dot{c}$ is equal to $\dot{a}\partial_a + \dot{\theta}\partial_\theta + \dot{\omega}\partial_\omega$, $D_{\dot{c}}\dot{c}$ must be equal to:

$$D_{\dot{a}\partial_a}\left(\dot{a}\partial_a + \dot{\theta}\partial_\theta + \dot{\omega}\partial_\omega\right) + D_{\dot{\theta}\partial_\theta}\left(\dot{a}\partial_a + \dot{\theta}\partial_\theta + \dot{\omega}\partial_\omega\right) + D_{\dot{\omega}\partial_\omega}\left(\dot{a}\partial_a + \dot{\theta}\partial_\theta + \dot{\omega}\partial_\omega\right).$$

If we calculate $D_{\dot{c}}\dot{c}$ in the following way:

$$\begin{aligned} D_{\dot{c}}\dot{c} &= \left(\ddot{a} - \sinh a\cosh a\dot{\theta}^2 + 2\cot\omega\dot{a}\dot{\omega}\right)\partial_a \\ &+ \left(\ddot{\theta} + 2\cot a\dot{a}\dot{\theta} + 2\cot\omega\dot{\theta}\dot{\omega}\right)\partial_\theta \\ &+ \left(\ddot{\omega} - \sin\omega\cos\omega\dot{a}^2 - \sin\omega\cos\omega\sin^2 a\dot{\theta}^2\right)\partial_\omega, \end{aligned}$$

it can be seen easily that the claim of the theorem is correct. □

3. The Sasaki Riemann Manifold $\left(T_1S^3, g^S\right)$

This section consists of some subjects as the expression with the local coordinate function of any point on T_1S^3, the orthonormal basis at any point on T_1S^3, the covariant derivations of this orthonormal basis elements, the Sasaki Riemann metric g^S on T_1S^3, the adapted basis and dual basis vectors on T_1S^3 with respect to g^S, the coefficients of the Levi Civita connection of the Sasaki Riemann manifold $\left(T_1S^3, g^S\right)$, and a system of the differential equations which gives all geodesics of the Sasaki Riemann manifold..

Let $T_1S^3 = \bigcup_{\forall e_1 \in S^3} T_{e_1}S^3$ be the disjoint union of the tangent vector spaces including all unit tangent vectors at every point of S^3. Then T_1S^3 is called as the tangent sphere bundle of S^3. Since S^3 has 3-dimensional manifold structure, T_1S^3 should be 5 dimensional manifold structure. Let $\pi : T_1S^3 \to S^3$ be a canonical projection map. Assuming that e_2 is an element of T_1S^3 at the point $e_1(a, \theta, \omega)$ of S^3. At the same time, e_2 may be considered as a tangent vector in the tangent vector space spanned by the orthonormal frame $\{f_2, f_3, f_4\}$ at the point $e_1(a, \theta, \omega)$ of S^3. If we denote the angle between f_4 and e_2 by δ and the angle between f_2 and the projected vector of e_2 to the tangent plane spanned by the vectors f_2 and f_3 by φ, then $(a, \theta, \omega, \varphi, \delta)$ can be considered as local coordinates for e_2 in $\pi^{-1}(S^3)$. Therefore, e_2, e_3 and e_4 have the following local expression:

$$\begin{aligned} e_2(a, \theta, \omega, \varphi, \delta) &= \cos\varphi\sin\delta f_2 + \sin\varphi\sin\delta f_3 + \cos\delta f_4, \\ e_3(a, \theta, \omega, \varphi, \delta) &= \cos\varphi\cos\delta f_2 + \sin\varphi\cos\delta f_3 - \sin\delta f_4, \\ e_4(a, \theta, \omega, \varphi, \delta) &= -\sin\varphi f_2 + \cos\varphi f_3, \end{aligned} \tag{3.1}$$

where $e_3 = \frac{\partial}{\partial \delta}$ and $e_4 = \frac{1}{\sin\delta}\frac{\partial}{\partial \varphi}$ are considered as the unit tangent vectors at any point (e_1, e_2) of T_1S^3 or elements of T_1S^3. We assume that e_1, e_2, e_3, e_4 are the unit orthogonal elements of T_1S^3.

Theorem 3.1. *Let T_1S^3 be the tangent sphere bundle of the unit 3 sphere in 4 dimensional Euclidean space and let e_1, e_2, e_3, e_4 be the unit orthogonal elements of T_1S^3. The covariant derivations of these elements are given by*

$$\begin{aligned} de_1 &= w_{12}e_2 + w_{13}e_3 + w_{14}e_4, \\ de_2 &= -w_{12}e_1 + w_{23}e_3 + w_{24}e_4, \\ de_3 &= -w_{13}e_1 - w_{23}e_2 + w_{34}e_4, \\ de_4 &= -w_{14}e_1 - w_{24}e_2 - w_{34}e_3, \end{aligned}$$

where

$$\begin{aligned} w_{12} &= \sinh\omega \sin\varphi \sin\delta da + \sin\omega \sin a \cos\delta d\theta + \cos\varphi \sin\delta d\omega, \\ w_{13} &= \sinh\omega \sin\varphi \cos\delta da + \sin\omega \sin a \sin\delta d\theta + \cos\varphi \cos\delta d\omega, \\ w_{14} &= \sinh\omega \cos\varphi da - \sin\varphi d\omega, \\ w_{23} &= \left(-\sin a \cos\omega \cos\varphi - \cos a \sin\varphi\right) d\theta + d\delta, \\ w_{24} &= \cos\omega \sin\delta da + \left(\sin a \cos\omega \sin\varphi \cos\delta - \cos a \cos\varphi \cos\delta\right) d\theta + \sin\delta d\varphi, \\ w_{34} &= \cos\omega \cos\delta da - \left(\sin a \cos\omega \sin\varphi \sin\delta - \cos a \cos\varphi \sin\delta\right) d\theta + \cos\delta d\varphi. \end{aligned}$$

Proof. We can use the covariant derivations of the unit orthogonal elements e_1, e_2, e_3, e_4 in order to examine the change between infinitely close two points on T_1S^3. The covariant derivatives of these elements can be obtained by using the partial derivation easily. □

Definition 3.1. The 1-forms providing the equation $w_{ij} =< de_i, e_j >$ for $i, j \in \{1, 2, 3, 4\}$ are called as the connection 1-forms on the cotangent space $T^*_{(e_1,e_2)}T_1S^3$.

Theorem 3.2. *The square of line element between infinitely close two points on T_1S^3 is given by*

$$\begin{aligned} d\sigma^2 = da^2 + d\theta^2 + d\omega^2 + d\varphi^2 + d\delta^2 + 2\cos\omega da d\varphi \\ - 2\left(\cos a \sin\varphi + \sin a \cos\omega \cos\varphi\right) d\theta d\delta. \end{aligned} \tag{3.2}$$

Proof. From the study in [1] with analogy, we obtained the square of the line element between infinitely close two points on T_1S^3 as follow:

$$\begin{aligned} d\sigma^2 &= < de_1, de_1 > + < de_2, e_3 >^2 + < de_2, e_4 >^2 + < de_3, e_4 >^2 \\ &= w_{12} \wedge w_{12} + w_{13} \wedge w_{13} + w_{14} \wedge w_{14} + w_{23} \wedge w_{23} + w_{24} \wedge w_{24} + w_{34} \wedge w_{34} \\ &= -da^2 + d\theta^2 + d\omega^2 + d\varphi^2 + d\delta^2 + 2\cos\omega da d\varphi \\ &\quad -2\left(\cos a \sin\varphi + \sin a \cos\omega \cos\varphi\right) d\theta d\delta. \end{aligned}$$

□

The square of the line element between infinitely close two points on T_1S^3 has the matrix representation as follows:

$$g_{\alpha\beta}:\begin{pmatrix} 1 & 0 & 0 & \cos\omega & 0\\ 0 & 1 & 0 & 0 & -A\\ 0 & 0 & 1 & 0 & 0\\ \cos\omega & 0 & 0 & 1 & 0\\ 0 & -A & 0 & 0 & 1\end{pmatrix} \text{ for } \alpha,\beta\in\{a,\theta,\omega,\varphi,\delta\} \tag{3.3}$$

where $A=\cos a\sin\varphi+\sin a\cos\omega\cos\varphi$. The inverse matrix of $g_{\alpha\beta}$ is given by

$$g^{\beta\alpha}:\begin{pmatrix} \csc^2\omega & 0 & 0 & -\cos\omega\csc^2\omega & 0\\ 0 & \frac{1}{1-A^2} & 0 & 0 & \frac{A}{1-A^2}\\ 0 & 0 & 1 & 0 & 0\\ -\cos\omega\csc^2\omega & 0 & 0 & \csc^2\omega & 0\\ 0 & \frac{A}{1-A^2} & 0 & 0 & \frac{1}{1-A^2}\end{pmatrix}. \tag{3.4}$$

Definition 3.2. g^S, which has the components $g_{\alpha\beta}$ for $\alpha,\beta\in\{a,\theta,\omega,\varphi,\delta\}$, is called as induced metric on the manifold T_1S^3. The characteristic vectors of matrix $(g_{\alpha\beta})$ which has type 5x5 are base vectors of the tangent vector space at point (e_1,e_2) of T_1S^3 defined by

$$\begin{aligned}
\xi_1 &= \frac{1}{\sqrt{2}}\frac{1}{1-\cos\omega}\left(\partial_a+\partial_\varphi\right),\\
\xi_2 &= \frac{1}{\sqrt{2}}\frac{1}{1+\cos\omega}\left(-\partial_a+\partial_\varphi\right),\\
\xi_3 &= \partial_\omega,\\
\xi_4 &= \frac{1}{\sqrt{2}}\frac{1}{1-\cos\omega\sin a\cos\varphi-\cos a\sin\varphi}\left(\partial_\theta+\partial_\delta\right),\\
\xi_5 &= \frac{1}{\sqrt{2}}\frac{1}{1+\cos\omega\sin a\cos\varphi-\cos a\sin\varphi}\left(-\partial_\theta+\partial_\delta\right),
\end{aligned}$$

where $\partial_k=\frac{\partial}{\partial k}$ for $k\in\{a,\theta,\omega,\varphi,\delta\}$. $\xi_i; i\in\{1,2,3,4,5\}$ is called as adapted basis vector of the tangent space $T_{(e_1,e_2)}T_1S^3$ with respect to the induced metric g^S. If the 1-form $\eta^i; i\in\{1,2,3,4,5\}$ provides the following equation:

$$\eta^i(\xi_j)=g^S\left(\xi_i,\xi_j\right)=\delta^i_j \tag{3.5}$$

1-form η^i is called as adapted dual basis vector of the cotangent space $T^*_{(e_1,e_2)}T_1S^3$ with respect to the induced metric g^S.The local expressions of 1-form η^i are given by

$$\begin{aligned}
\eta^1 &= \frac{1}{\sqrt{2}}\left(1-\cos\omega\right)\left(da+d\varphi\right),\\
\eta^2 &= \frac{1}{\sqrt{2}}\left(1+\cos\omega\right)\left(-da+d\varphi\right),\\
\eta^3 &= d\omega,\\
\eta^4 &= \frac{1}{\sqrt{2}}\left(1-\cos\omega\sin a\cos\varphi-\cos a\sin\varphi\right)\left(d\theta+d\delta\right),\\
\eta^5 &= \frac{1}{\sqrt{2}}\left(1+\cos\omega\sin a\cos\varphi-\cos a\sin\varphi\right)\left(-d\theta+d\delta\right).
\end{aligned} \tag{3.6}$$

Theorem 3.3. *Let T_1S^3 be the tangent sphere bundle of the unit 3 sphere and let $T_{(e_1,e_2)}T_1S^3$ be a tangent vector space at any point (e_1, e_2) on T_1S^3. g^S, a real valuable function on $T_{(e_1,e_2)}T_1S^3$, is a Riemann metric on the manifold T_1S^3 defined by*

$$g^S : \begin{array}{ccc} T_{(e_1,e_2)}T_1S^3 \times T_{(e_1,e_2)}T_1S^3 & \to & IR \\ (X,Y) & \to & g^S(X,Y). \end{array} \tag{3.7}$$

Proof. Let $\widetilde{X} = x^i\xi_i$, $\widetilde{Y} = y^j\xi_j$ and $\widetilde{Z} = z^k\xi_k$ for $i, j, k \in \{1, 2, 3, 4, 5\}$ be the unit tangent vectors at any point on T_1S^3 where $\{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5\}$ is adapted basis of $T_{(e_1,e_2)}T_1S^3$. For all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in T_{(e_1,e_2)}T_1S^3$ and $\alpha, \beta \in IR$, we get

$$\begin{aligned} g^S(\alpha\widetilde{X} + \beta\widetilde{Y}, \widetilde{Z}) &= g^S(\{\alpha\left[x^i\xi_i\right] + \beta\left[y^i\xi_i\right]\}, z^j\xi_j) \\ &= \alpha x^i z^i \varepsilon_i + \beta y^i z^i \varepsilon_i \\ &= \alpha g^S(\widetilde{X}, \widetilde{Z}) + \beta g^S(\widetilde{Y}, \widetilde{Z}). \end{aligned}$$

Similarly, we get $g^S(\widetilde{X}, \alpha\widetilde{Y} + \beta\widetilde{Z}) = \alpha g^S(\widetilde{X}, \widetilde{Y}) + \beta g^S(\widetilde{X}, \widetilde{Z})$. Thus, g^S is bilinear transformation. Since the following equality is held

$$g^S(\widetilde{X}, \widetilde{Y}) = g^S(x^i\xi_i, y^j\xi_j) = y^i x^i \varepsilon_i = g^S(\widetilde{Y}, \widetilde{X}).$$

g^S must be symmetric map. Finally, g^S is a positive definite map because g^S provides the following identities:

$$g^S(\widetilde{X}, \widetilde{X}) = 0 \quad \text{if and only if } \widetilde{X} = 0 \quad \vee \quad g^S(\widetilde{X}, \widetilde{X}) > 0 \quad \text{for every } \widetilde{X} \neq 0.$$

Since g^S is positive definite, symmetric and bilinear form, g^S must a Riemann metric on the tangent sphere bundle T_1S^3. Thus, g^S is called as the Sasaki Riemann metric. Moreover, $\left(T_1S^3, g^S\right)$ is also called as Sasaki Riemann manifold. □

Theorem 3.4. *Let $\left(T_1S^3, g^S\right)$ be the Sasaki Riemann manifold. Let be Levi Civita connection of $\left(T_1S^3, g^S\right)$ and let $\Gamma^{\gamma}_{\alpha\beta}; \alpha, \beta, \gamma \in \{a, \theta, \omega, \varphi, \delta\}$ be the connection coefficients of the Levi Civita connection (i.e. Christoffel symbols) related to the matrix $(g_{\alpha\beta})$, α, $\beta \in \{a, \theta, \omega, \varphi, \delta\}$which corresponds to the Sasaki Riemann metric g^S. Then the non-zero the Christoffel symbols of $\left(T_1S^3, g^S\right)$ are given by*

$$\begin{aligned} \Gamma^a_{a\omega} &= \frac{1}{2}\cot\omega, \Gamma^a_{\theta\delta} = -\frac{1}{2}\csc^2\omega\left(A_a + \cos\omega A_\varphi\right), \Gamma^a_{\omega\varphi} = -\frac{1}{2}\csc\omega, \\ \Gamma^\theta_{a\theta} &= -\frac{A}{2\left(1-A^2\right)}A_a, \Gamma^\theta_{a\delta} = \frac{1}{2\left(1-A^2\right)}A_a, \Gamma^\theta_{\theta\omega} = -\frac{A}{2\left(1-A^2\right)}A_\omega, \\ \Gamma^\theta_{\theta\varphi} &= -\frac{A}{2\left(1-A^2\right)}A_\varphi, \Gamma^\theta_{\omega\delta} = \frac{1}{2\left(1-A^2\right)}A_\omega, \Gamma^\theta_{\varphi\delta} = \frac{1}{2\left(1-A^2\right)}A_\varphi, \\ \Gamma^\omega_{a\varphi} &= -\frac{1}{2}\sin\omega, \Gamma^\omega_{\theta\delta} = -\frac{1}{2}A_\omega, \Gamma^\varphi_{a\omega} = -\frac{1}{2}\csc\omega, \\ \Gamma^\varphi_{\theta\delta} &= \frac{1}{2}\csc^2\omega\left(\cos\omega A_a - A_\varphi\right), \Gamma^\varphi_{\omega\varphi} = \frac{1}{2}\cot\omega, \Gamma^\delta_{a\theta} = \frac{1}{2\left(1-A^2\right)}A_a, \\ \Gamma^\delta_{a\delta} &= -\frac{A}{2\left(1-A^2\right)}A_a, \Gamma^\delta_{\theta\omega} = \frac{1}{2\left(1-A^2\right)}A_\omega, \Gamma^\delta_{\theta\varphi} = \frac{1}{2\left(1-A^2\right)}A_\varphi, \\ \Gamma^\delta_{\omega\delta} &= -\frac{A}{2\left(1-A^2\right)}A_\omega, \Gamma^\delta_{\varphi\delta} = -\frac{A}{2\left(1-A^2\right)}A_\varphi. \end{aligned}$$

where $\Gamma^{\gamma}_{\alpha\beta}=\Gamma^{\gamma}_{\beta\alpha}$ *for all* $\alpha,\beta,\gamma\in\{a,\theta,\omega,\varphi,\delta\}$ *and* $A_k=\frac{\partial A}{\partial k}$ *for* $k\in\{a,\theta,\omega,\varphi,\delta\}$*.*

Proof. On the Sasaki Riemann manifold $\left(T_1S^3,g^S\right)$, there is a unique connection ∇ such that ∇ is torsion free and compatible with Riemann metric g^S. This connection is called Levi Civita connection and characterized by the Kozsul formula:

$$\begin{aligned}2g^S\left(\nabla_{\partial_a}\partial_\theta,\partial_\omega\right) &= \partial_a g^S\left(\partial_\theta,\partial_\omega\right)+\partial_\theta g^S\left(\partial_\omega,\partial_a\right)-\partial_\omega g^S\left(\partial_a,\partial_\theta\right)+\\ &\quad -g^S\left(\left[\partial_a,\partial_\theta\right],\partial_\omega\right)+g^S\left(\left[\partial_\theta,\partial_\omega\right],\partial_a\right)+g^S\left(\left[\partial_\omega,\partial_a\right],\partial_\theta\right)\end{aligned}$$

where $\partial_a=\frac{\partial}{\partial a}$, $\partial_\theta=\frac{\partial}{\partial\theta}$, $\partial_\omega=\frac{\partial}{\partial\omega}$, $\partial_\varphi=\frac{\partial}{\partial\varphi}$ and $\partial_\delta=\frac{\partial}{\partial\delta}$. Since Levi Civita connection ∇ is symmetric, $\left[\partial_a,\partial_\theta\right],\left[\partial_\theta,\partial_\omega\right],\left[\partial_\omega,\partial_a\right]$ must be zero. By using the following identity:

$$\nabla_{\partial_a}\partial_\theta=\Gamma^a_{a\theta}\partial_a+\Gamma^\theta_{a\theta}\partial_\theta+\Gamma^\omega_{a\theta}\partial_\omega+\Gamma^\varphi_{a\theta}\partial_\varphi+\Gamma^\delta_{a\theta}\partial_\delta,$$

and Kozsul formula, Christoffel symbols are obtained by

$$\begin{aligned}\Gamma^a_{a\theta} &= \frac{1}{2}g^{ak}\left(\partial_a g_{k\theta}+\partial_\theta g_{ak}-\partial_k g_{a\theta}\right)=0,\\ \Gamma^\theta_{a\theta} &= \frac{1}{2}g^{\theta k}\left(\partial_a g_{k\theta}+\partial_\theta g_{ak}-\partial_k g_{a\theta}\right)=-\frac{A}{2\left(1-A^2\right)}A_a,\\ \Gamma^\omega_{a\theta} &= \frac{1}{2}g^{\omega k}\left(\partial_a g_{k\theta}+\partial_\theta g_{ak}-\partial_k g_{a\theta}\right)=0,\\ \Gamma^\varphi_{a\theta} &= \frac{1}{2}g^{\varphi k}\left(\partial_a g_{k\theta}+\partial_\theta g_{ak}-\partial_k g_{a\theta}\right)=0,\\ \Gamma^\delta_{a\theta} &= \frac{1}{2}g^{\delta k}\left(\partial_a g_{k\theta}+\partial_\theta g_{ak}-\partial_k g_{a\theta}\right)=\frac{1}{2\left(1-A^2\right)}A_a,\ \text{for } k\in\{a,\theta,\omega,\varphi,\delta\}.\end{aligned}$$

The other Christoffel symbols can be obtained by using the similar method. □

Theorem 3.5. *Let* $\left(T_1S^3,g^S\right)$ *be the Sasaki Riemann manifold and* $c:t\in R\to c(t)=(a(t),\theta(t),\omega(t),\varphi(t),\delta(t))$ *be a curve on the tangent sphere bundle. c is geodesic if and only if the following second order differential equations are provided:*

$$\begin{aligned}\ddot{a}+\cot\omega\dot{a}\dot{\omega}-\csc^2\omega\left(A_a+\cos\omega A_\varphi\right)\dot{\theta}\dot{\delta}-\csc\omega\dot{\omega}\dot{\varphi} &= 0,\\ \ddot{\theta}-\frac{1}{1-A^2}\left\{AA_a\dot{a}\dot{\theta}+A_a\dot{a}\dot{\delta}-AA_\omega\dot{\theta}\dot{\omega}-AA_\varphi\dot{\theta}\dot{\varphi}+A_\omega\dot{\omega}\dot{\delta}+A_\varphi\dot{\varphi}\dot{\delta}\right\} &= 0,\\ \ddot{\omega}-\sin\omega\dot{a}\dot{\varphi}-A_\omega\dot{\theta}\dot{\delta} &= 0,\\ \ddot{\varphi}-\csc^2\omega\dot{a}\dot{\omega}+\csc^2\omega\left(\cos\omega A_a-A_\varphi\right)\dot{\theta}\dot{\delta}+\cot\omega\dot{\omega}\dot{\varphi} &= 0,\\ \ddot{\delta}+\frac{1}{1-A^2}\left\{A_a\dot{a}\dot{\theta}-AA_a\dot{a}\dot{\delta}+A_\omega\dot{\theta}\dot{\omega}+A_\varphi\dot{\theta}\dot{\varphi}-AA_\omega\dot{\omega}\dot{\delta}-AA_\varphi\dot{\varphi}\dot{\delta}\right\} &= 0.\end{aligned}$$

Proof. $c(t)=(a(t),\theta(t),\omega(t),\varphi(t),\delta(t))$ is geodesic if and only if $\nabla_{\dot{c}}\dot{c}$ is zero. Since $\dot{c}$ is equal to $\dot{a}\partial_a+\dot{\theta}\partial_\theta+\dot{\omega}\partial_\omega+\dot{\varphi}\partial_\varphi+\dot{\delta}\partial_\delta$, $\nabla_{\dot{c}}\dot{c}$ is equal to:

$$\begin{aligned}&\nabla_{\dot{a}\partial_a}\left(\dot{a}\partial_a+\dot{\theta}\partial_\theta+\dot{\omega}\partial_\omega+\dot{\varphi}\partial_\varphi+\dot{\delta}\partial_\delta\right)+\nabla_{\dot{\theta}\partial_\theta}\left(\dot{a}\partial_a+\dot{\theta}\partial_\theta+\dot{\omega}\partial_\omega+\dot{\varphi}\partial_\varphi+\dot{\delta}\partial_\delta\right)\\ &+\nabla_{\dot{\omega}\partial_\omega}\left(\dot{a}\partial_a+\dot{\theta}\partial_\theta+\dot{\omega}\partial_\omega+\dot{\varphi}\partial_\varphi+\dot{\delta}\partial_\delta\right)+\nabla_{\dot{\varphi}\partial_\varphi}\left(\dot{a}\partial_a+\dot{\theta}\partial_\theta+\dot{\omega}\partial_\omega+\dot{\varphi}\partial_\varphi\right)\\ &+\nabla_{\dot{\varphi}\partial_\varphi}\left(\dot{\delta}\partial_\delta\right)+\nabla_{\dot{\delta}\partial_\delta}\left(\dot{a}\partial_a+\dot{\theta}\partial_\theta+\dot{\omega}\partial_\omega+\dot{\varphi}\partial_\varphi+\dot{\delta}\partial_\delta\right)\end{aligned}$$

Therefore we get

$$\begin{aligned}\nabla_{\dot c}\dot c &= \ddot a\partial_a + \dot a\dot\theta\left(-\frac{AA_a}{1-A^2}\partial_\theta + \frac{A_a}{1-A^2}\partial_\delta\right) + \dot a\dot\omega\left(\cot\omega\partial_a - \csc\omega\partial_\varphi\right)\\ &+\dot a\dot\varphi\left(-\sin\omega\partial_\omega\right) + \dot a\dot\delta\left(-\frac{AA_a}{1-A^2}\partial_\delta\right) + \ddot\theta\partial_\theta\\ &+\dot\theta\dot\omega\left(-\frac{AA_\omega}{1-A^2}\partial_\theta + \frac{A_\omega}{1-A^2}\partial_\delta\right) + \dot\theta\dot\varphi\left(-\frac{AA_\varphi}{1-A^2}\partial_\theta + \frac{A_\varphi}{1-A^2}\partial_\delta\right)\\ &+\dot\theta\dot\delta\left\{-A_\omega\partial_\omega + \csc^2\omega\left(\cos\omega A_a - A_\varphi\right)\partial_\varphi\right\} + \ddot\omega\partial_\omega\\ &+\dot\omega\dot\varphi\left(-\csc\omega\partial_a + \cot\omega\partial_\varphi\right) + \dot\omega\dot\delta\left(\frac{A_\omega}{1-A^2}\partial_\theta - \frac{AA_\omega}{1-A^2}\partial_\delta\right)\\ &+\ddot\varphi\partial_\varphi + \dot\varphi\dot\delta\left(\frac{A_\varphi}{1-A^2}\partial_\theta - \frac{AA_\varphi}{1-A^2}\partial_\delta\right) + \ddot\delta\partial_\delta\end{aligned}$$

If we arrange $\nabla_{\dot c}\dot c$ in the following way:

$$\begin{aligned}&\left(\ddot a + \cot\omega\dot a\dot\omega - \csc^2\omega\left(A_a + \cos\omega A_\varphi\right)\dot\theta\dot\delta - \csc\omega\dot\omega\dot\varphi\right)\partial_a\\ &+\left\{\ddot\theta - \frac{1}{1-A^2}\left(AA_a\dot a\dot\theta + A_a\dot a\dot\delta - AA_\omega\dot\theta\dot\omega - AA_\varphi\dot\theta\dot\varphi + A_\omega\dot\omega\dot\delta + A_\varphi\dot\varphi\dot\delta\right)\right\}\partial_\theta\\ &+\left(\ddot\omega - \sin\omega\dot a\dot\varphi - A_\omega\dot\theta\dot\delta\right)\partial_\omega\\ &+\left\{\ddot\varphi - \csc^2\omega\dot a\dot\omega + \csc^2\omega\left(\cos\omega A_a - A_\varphi\right)\dot\theta\dot\delta + \cot\omega\dot\omega\dot\varphi\right\}\partial_\varphi\\ &+\left\{\ddot\delta + \frac{1}{1-A^2}\left(A_a\dot a\dot\theta - AA_a\dot a\dot\delta + A_\omega\dot\theta\dot\omega + A_\varphi\dot\theta\dot\varphi - AA_\omega\dot\omega\dot\delta - AA_\varphi\dot\varphi\dot\delta\right)\right\}\partial_\delta,\end{aligned}$$

it can be seen that the claim of the theorem is true straightforwardly. □

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