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# ON THE RULED SURFACES WHOSE FRAME IS THE BISHOP FRAME IN THE EUCLIDEAN 3−SPACE

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Abstract. In this study we investigate the differential geometric elements (such as Normal vector field $N$, Weingarten Map $S$, curvatures $K$ and $H$, the fundamental forms I,II,III) of a ruled surface according to its Bishop frame or parallel transport frame in the Euclidean 3−space.

## 1. Introduction

[1]In this paper we will study the shape operator and the fundamental forms of the ruled surfaces on the curves with Bishop frame, or with parallel transport frame in the Euclidean 3−space.

It is well-known that, if an Euclidean space curve is differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curve. The set, whose elements are frame vectors and curvatures of a curve, is called Frenet apparatus of the curve. Let $\alpha(s)$ be a space curve, where $s$ is an arc length parameter. Let $\{T(s), N(s), B(s)\}$ be Frenet frame of this curve. Here $T$, $N$ and $B$ are called, the tangent vector field, the principal normal vector field and the binormal vector field of the curve, respectively. $\kappa(s)$ and $\tau(s)$ are called, curvature and torsion of the curve $\alpha(s)$, respectively. The Frenet formulae are also well known as

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

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where

$$\langle T,T\rangle = \langle N,N\rangle = \langle B,B\rangle = 1,$$
$$\langle T,N\rangle = \langle T,B\rangle = \langle N,B\rangle = 0.$$

The curvature functions are defined by

$$\kappa = \kappa(s) = \left\| \dot{T}(s) \right\| \text{ and } \tau(s) = -\left\langle N, \dot{B} \right\rangle = \frac{[\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha}]}{\kappa^2}.$$

The Frenet frame is constructed for the curve of 3-time continuously differentiable non-degenerate curves. Curvature may vanish at some points on the curve. That is, the second derivative of the curve may be zero. In this situation, we can use an alternative frame. We use the tangent vector $T$ and two relatively parallel vector fields to construct this alternative frame such that the normal vector field $N_1$ along the curve is relatively parallel if its derivative is tangential. This frame is called parallel frame or Bishop frame along $T$ . The reason for the name parallel is because the normal component of the derivatives of the normal vector field is zero. The advantages of the parallel frame with the Frenet frame in Euclidean 3-space was given and studied by Bishop [1] .

We can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. $T, N_1, N_2$ will be a smoothly varying right-handed orthonormal frame as we move along the curve.

(To this point, the Frenet frame would work just fine if the curve with $\kappa(s) \neq 0$) But now we want to impose the extra condition that $\langle \dot{N}_1, N_2\rangle = 0$. We say the unit normal vector field $N_1$ is parallel along $\alpha(s)$; this means that the only change of $N_1$ is in the direction of $T$. In this event, $T, N_1, N_2$ is called a Bishop frame for $\alpha(s)$. A Bishop frame can be defined even when a Frenet frame cannot (e.g., when there are points with $\kappa = 0$ ) [11].The parallel transport frame is based on the observation that, while $T(s)$ for a given curve model is unique, we may choose any convenient arbitrary basis $(N_1(s), N_2(s))$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $T(s)$ at each point.

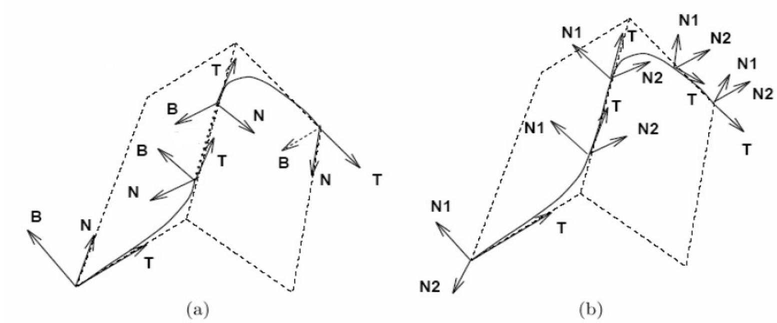


Figure 1, Comparing the Frenet and Bishop frames on a "roof-top." (a) The Frenet frame becomes undefined on the straight line at the peak, then changes abruptly as the curve descends the right side. (b) The Bishop frame is smooth throughout [8].

Here, we shall call the set $(T, N_1, N_2)$ as Bishop trihedron and $k_1$ and $k_2$ as Bishop curvatures. The relation matrix may be expressed as

$$\begin{aligned} T &= T \\ N &= \cos\theta(s)N_1 + \sin\theta(s)N_2 \\ B &= -\sin\theta(s)N_1 + \cos\theta(s)N_2, \end{aligned}$$

where

$$\begin{aligned} \theta(s) &= arctan\frac{k_2}{k_1} \\ \tau(s) &= \dot{\theta}(s) \\ \kappa(s) &= \sqrt{k_1^2 + k_2^2}. \end{aligned}$$

One can show [1] that

$$\begin{aligned} \theta(s) &= \arctan\left(\frac{k_2}{k_1}\right), \\ \tau(s) &= -\frac{d\theta(s)}{ds} \end{aligned}$$

so that $k_1$ and $k_2$ effectively correspond to a Cartesian coordinate system for the polar coordinates $\kappa$, $\theta$ with

$$\theta = -\int \tau(s)\, ds.$$

The orientation of the parallel transport frame includes the arbitrary choice of integration constant $\theta_0$ , which disappears from $\tau$ (and hence from the Frenet frame) due to the differentiation [2],[8],[9].

Here, Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s) cos\theta(s), \\ k_2 &= \kappa(s) sin\theta(s). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} T &= T, \\ N_1 &= \cos\theta(s)N - \sin\theta(s)B, \\ N_2 &= \sin\theta(s)N + \cos\theta(s)B, \end{aligned}$$

( see [10]).

Bishop trihedron $(T, N_1, N_2)$, have the properties

$$\langle T, T \rangle = \langle N_1, N_1 \rangle = \langle N_2, N_2 \rangle = 1.$$

Suppose we have unit vector fields $N_1(s)$ and $N_2(s) = T(s) \Lambda\, N_1(s)$ along the curve $\alpha$ so that

$$\langle T, N_1 \rangle = \langle T, N_2 \rangle = \langle N_1, N_2 \rangle = 0,$$

i.e. $T, N_1, N_2$ will be a smoothly varying right-handed orthonormal frame as we move along the curve. If the derivatives of $N_1(s)$ and $N_2(s)$ depend only on $T(s)$ and not each other, we can make $N_1(s)$ and $N_2(s)$ vary smoothly throughout the path regardless of the curvature. Therefore, we have the alternative frame equations $\dot{\alpha}(s) = T$ and

$$\dot{T} = k_1 N_1 + k_2 N_2,$$
$$\dot{N}_1 = -k_1 T,$$
$$\dot{N}_2 = -k_2 T$$

or in matrix notation

$$\begin{bmatrix} \dot{T} \\ \dot{N}_1 \\ \dot{N}_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix},$$

where the functions $k_1(s)$ and $k_2(s)$ are the natural curvatures of Bishop frame of the curve $\alpha(s)$, respectively ( see [8], [1].)

## 2. The normal vector field $\boldsymbol{N}$ and the Weingarten map $S$ of the ruled surfaces with the Bishop frame, in the Euclidean $3-$space

A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. A ruled surface is one which can be generated by the motion of a straight line in $IE^3$ [6],[4]. Choosing a directrix on the surface, i.e. a smooth unit speed curve $\alpha(s)$ orthogonal to the straight lines, and then choosing $v(s)$ to be unit vectors along the curve in the direction of the lines, the velocity vector $\alpha_s$ and $v$ satisfy $\langle \dot{\alpha}, v \rangle = 0$. The surface consists of points $\alpha(s) + uv(s)$ as $s$ and $u$ vary. For example, a cone is formed by keeping one point of a line fixed whilst moving another point along a circle. To illustrate the current situation, we bring here the famous example of L. K. Graves (see [7]), the so called B-scroll. This is a surface which can be parametrized as a "ruled surface" in $L^3$ with null directrix curve and null rulings, i.e.,

$$X(s,t) = x(s) + tB(s),$$

$x(s)$ being a null curve and $B(s)$ a null vector field along $x(s)$ satisfying $\langle \dot{x}, B \rangle = -1$.

**Definition 2.1.** In the Euclidean $3-space$, let $\alpha(s)$ be a curve with the Bishop frame.

$$\varphi(s,u) = \alpha(s) + uN_2(s)$$

is the parametrization of the ruled surface, which will call $N_2 - scroll$ with the Bishop frame. The directrix of this $N_2 - scroll$ is the curve $\alpha(s)$ with the Bishop frame. The vector fields $T$, $N_1, and$ $N_2$ are called Bishop trihedron. The generating space of $N_2 - scroll$ is spanned by binormal vector $N_2$. Here $Sp\{T, N_1\}$ is the rectifying plane of $\alpha(s)$.

**Theorem 2.1.** *In the Euclidean $3-space$, the Normal vector field $\boldsymbol{N}$ to the ruled surface with the Bishop frame is*

$$\boldsymbol{N} = -N_1.$$

*Proof.* It is well known that the Normal vector field $\boldsymbol{N}$ to the surface $\varphi(s,u)$ is

$$\boldsymbol{N} = \frac{\bar{\varphi}_s \wedge \bar{\varphi}_u}{\|\bar{\varphi}_s \wedge \bar{\varphi}_u\|},$$

where

$$\bar{\varphi}_s = \frac{\varphi_s}{\|\varphi_s\|} = \frac{(1-uk_2)\,T}{\|(1-uk_2)\|} = \pm T,$$

and

$$\bar{\varphi}_u = \varphi_u = N_2$$

are orthonormal base vectors of tangent space $T_{\alpha(s)}(M)$. the Let we take $(1-uk_2) > 0$, then $\bar{\varphi}_s = T$ and

$$\begin{aligned} \boldsymbol{N} &= \frac{\bar{\varphi}_s \Lambda \bar{\varphi}_u}{\|\bar{\varphi}_s \Lambda \bar{\varphi}_u\|}, \\ \boldsymbol{N} &= \frac{T \Lambda N_2}{\|T \Lambda N_2\|}, \\ \boldsymbol{N} &= -N_1. \end{aligned}$$

□

**Theorem 2.2.** *In the Euclidean* $3-$*space, the matrix corresponding to the Weingarten map ( Shape Operator )* $S$ *of the ruled surface with the Bishop frame, is*

$$S = \begin{bmatrix} \frac{k_1}{(1-uk_2)} & 0 \\ 0 & 0 \end{bmatrix}.$$

*Proof.* The Weingarten map $S$ is an extrinsic curvature and can be calculated as follows; Since

$$S(\overline{\varphi_s}) = \frac{1}{(1-uk_2)} (-N_1)_s = \frac{k_1}{(1-uk_2)} T,$$

and

$$S(\overline{\varphi_u}) = (N_1)_u = 0,$$

we have

$$\begin{aligned} S(d\varphi) &= S(\overline{\varphi_s})\, ds + S(\overline{\varphi_u})\, du \ ; \ \ (1-uk_2) > 0 \\ &= \frac{k_1}{(1-uk_2)} T ds, \end{aligned}$$

or in the matrix notation

$$S = \begin{bmatrix} \frac{k_1}{(1-uk_2)} & 0 \\ 0 & 0 \end{bmatrix}.$$

□

**Corollary 2.1.** ***(Gaussian curvature)*** *It is well known that the Gaussian curvature of a ruled surface is given by determinant of* $S$*. So the Gaussian curvature of the ruled surface with the Bishop frame is*

$$K = \det S = 0.$$

**Corollary 2.2.** ***(Mean curvature)*** *It is well known that the mean curvature of a ruled surface is given by the trace of* $S$*. So the mean curvature of the ruled surface with the Bishop frame is*

$$H = traceS = \frac{k_1}{1-uk_2}.$$

Minimal surfaces are classically defined as surfaces of zero mean curvature in the Euclidean $3-space$. Physically this means that for a given boundary a minimal surface can not be changed without increasing the area of the surface. The only ruled minimal surfaces are the plane and the helicoid [3]

**Theorem 2.3.** *In the Euclidean* $3-space$, *for a minimal ruled surface with the Bishop frame*

$$\begin{aligned} T &= T, \\ N_1 &= \pm B, \\ N_2 &= \mp N. \end{aligned}$$

directrix curve must be a planar curve.

*Proof.* It can be calculated as

$$\begin{aligned} &H = 0, \\ &k_1 = 0 = \kappa(s)\ cos\theta(s), \end{aligned}$$

since $\kappa(s) \neq 0$; $\cos\theta(s) = 0$

$$if \qquad \cos\theta(s) = 0 \quad \text{then} \qquad \theta(s) = \frac{\pi}{2} + k\pi,\ k \in Z$$

and

$$\dot{\theta}(s) = 0 = \tau(s),$$

with the condition

$$\theta(s) = \frac{\pi}{2} + k\pi,\ k \in Z$$

Bishop frame can be written as

$$\begin{aligned} &T = T, \\ &N_1 = -\sin\left(\frac{\pi}{2} + k\pi\right) B,\ k \in Z, \\ &N_2 = \sin\left(\frac{\pi}{2} + k\pi\right) N,\ k \in Z. \end{aligned}$$

Hence we have the following equations for a minimal ruled surface with Bishop frame.

$$\begin{aligned} T &= T, \\ N_1 &= \pm B, \\ N_2 &= \mp N. \end{aligned}$$

□

## 3. The Fundamental forms of the ruled surface with the Bishop frame in the Euclidean 3−space

It is well known that the fundamental forms of a surface characterize the basic intrinsic properties of the surface and the way it is located in space in a neighborhood of a given point; one usually singles out the so-called first, second and third fundamental forms. The first and the second fundamental forms define two important common scalar quantities which are invariant under a transformation of the coordinates on the surface.

3.1. **The first fundamental forms of the ruled surface with the Bishop frame in the Euclidean 3−space.** In this subsection we will examine the first fundamental form of the ruled surface $\varphi(s,u) = \alpha(s) + uN_2(s)$; with Bishop frame. The first fundamental form characterizes the interior geometry of the surface in a neighborhood of a given point $M$; and

$$d\varphi = \varphi_s ds + \varphi_u du$$

is the differential of the radius vector of $\varphi$ along a chosen direction from a point $M$ to an infinitesimally close point $M'$. The principal linear part of growth of the arc length $MM'$ is expressed by the square of $d\varphi$ [12]. The form $I$ is the first fundamental form of the surface and

$$I = \langle d\varphi, d\varphi \rangle.$$

**Theorem 3.1.** *In the Euclidean* $3-$ *space, the first fundamental form of the* $N_2-$ *scroll with the Bishop frame is*

$$I = dsds + dudu.$$

*Proof.* For the ruled surface $\varphi(s,u) = \alpha(s) + uN_2(s)$,we have

$$\begin{aligned} \bar{\varphi}_s &= \frac{\varphi_s}{\|\varphi_s\|} \\ &= T(s); \qquad if \;\; (1-uk_2) > 0 \\ \bar{\varphi}_u &= \varphi_u = N_2(s), and\ \bar{\varphi}_u = \varphi_u = N_2(s), \end{aligned}$$

hence

$$d\varphi = Tds + N_2 du$$

and

$$I = dsds + dudu.$$

We can write this quadratic form in the matrix form as

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad and \quad \det \bar{I} = 1.$$

□

3.2. **The second fundamental form of the ruled surface with the Bishop frame in the Euclidean 3−space.** In this subsection we will examine the second fundamental form of the $N_2 - scroll$ with the Bishop frame , already defined. The second fundamental form characterizes the local structure of the surface in a neighborhood of a regular point.

Let $\boldsymbol{N}$ be the unit normal vector of the surface at the point $M$. The doubled principal linear part $2\rho$, of the deviation of the point $M'$ on the surface from the tangent plane at the point $M$ is the second fundamental form of the surface [12]. It is also given by

$$II = \langle d\varphi, d\boldsymbol{N} \rangle = 2\rho.$$

**Theorem 3.2.** *In the Euclidean* $3-$ *space, the second fundamental form of the ruled surface with the Bishop frame is*

$$II = -k_1 dsds.$$

*Proof.* It is obvious. □

3.3. **The third fundamental form of the ruled surface with the Bishop frame in the Euclidean $3-$ space.** The third fundamental form of a surface is equal to the principal linear part of the growth of the angle between the vectors $\boldsymbol{N}$ and $\boldsymbol{N}'$ under displacement along the surface from the point $M$ to $M'$.The third fundamental form of the surface is the square of the differential of the unit normal vector $\boldsymbol{N}$ of the surface at the point $M$ which is denoted by $III$ [12] and given by

$$III = \langle d\boldsymbol{N}, d\boldsymbol{N} \rangle.$$

**Theorem 3.3.** *In the Euclidean $3-space$, the third fundamental form of the ruled surface with the Bishop frame is*

$$III = k_1^2 ds ds.$$

*Proof.* It is obvious. □

**Corollary 3.1.** *For the second and the third fundamental forms of the ruled surface with the Bishop frame in the Euclidean $3-space$ we have the relation*

$$III = -k_1 II.$$

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