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THE DISCRIMINANT OF SECOND FUNDAMENTAL FORM

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Abstract. In this study we consider the discriminant of the second fundamental form. As application we also give necessary condition for Vranceanu surface in $\mathbb{E}^4$ to have vanishing discriminant.

1. Introduction

Let M be an n-dimensional Riemannian manifolds. For the vector fields X, Y, Z on M the curvature tensor R of M is defined by

$$R(X,Y)Z = \bigtriangledown_X(\bigtriangledown_Y Z) - \bigtriangledown_Y(\bigtriangledown_X Z) - \bigtriangledown_{[X,Y]}Z \tag{1.1}$$

where $\bigtriangledown$ is the Levi-Civita connection of M, and [,] is Lie parantheses operator.

Given a point $p \in M$ and a two-dimensional subspace $\sigma \subset T_pM$, the real number

$$K(\sigma) = \frac{g(\ R(X,Y)X,Y\ )}{g(X,X)g(Y,Y) - g(X,Y)^2} \tag{1.2}$$

is called *the Sectional Curvature* of σ at point p, where X, Y is any basis of σ [1].

Let $f : M \to \widetilde{M}$ be an isometric immersion of an n-dimensional connected Riemannian manifold M into an m-dimensional Riemannian manifold $\widetilde{M}$. For all local formulas and computations, we may assume f as an imbedding and thus we shall often identify $p \in M$ with $f(p) \in \widetilde{M}$. The tangent space T_pM is identified with a subspace $f_*(T_pM)$ of $T_p\widetilde{M}$ where f_* is the differential map of f . Letters X, Y and Z (resp. ζ, μ and ξ) vector fields tangent (resp. normal) to M. We denote the tangent bundle of M (resp. $\widetilde{M}$) by TM (resp. $T\widetilde{M}$) , unit tangent bundle of M by UM and the normal bundle by $T^{\perp}M$. Let $\widetilde{\bigtriangledown}$ and $\bigtriangledown$ be the Levi-Civita connections of $\widetilde{M}$ and M , resp. Then the Gauss formula is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{1.3}$$

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where h denotes the second fundamental form. If the Weingarten formula is given by

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{1.4}$$

where A denotes the shape operator and D the normal connection. Clearly $h(X,Y) = h(Y,X)$ and A is related to h as $\langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle$, where $\langle\ ,\rangle$ denotes the Riemannian metrics of M and $\widetilde{M}$.

For the second fundamental form, we define their covariant derivatives by

$$(\bar{\nabla}_X h)(Y,Z) = D_X(h(Y,Z)) - h(\triangledown_X Y, Z) - h(Y, \triangledown_X Z) \tag{1.5}$$

where X, Y, Z tangent vector fields over M and $\overline{\triangledown}$ is the van der Waerden Bortolotti connection [1].The equation of Codazzi implie, that $\overline{\triangledown}h$ is symmetric hence

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z) = (\bar{\nabla}_Z h)(X,Y) \tag{1.6}$$

If $\bar{\nabla}h = 0$ then the second fundamental form of M is called *parallel* [7] (i.e. M is *1-parallel*) [4].

2. Discriminant of The Second Fundamental Form

Let $f : M \to \widetilde{M}$ be an isometric immersion of an n-dimensional connected Riemannian manifold M into an m-dimensional Riemannian manifold $\widetilde{M}$. The main invariant of the second fundamental form h is its discriminant Δ, (see [2]) the real valued function on the planes (through 0) in T_xM such that if the linearly independent tangent vectors X, Y span σ, then

$$\Delta_{XY} = \Delta(\sigma) = \frac{\langle h(X,X), h(Y,Y)\rangle - \|h(X,Y)\|^2}{\|X\|^2 \|Y\|^2 - \langle X,Y\rangle^2}. \tag{2.1}$$

For an isometric immersion $f : M \to \widetilde{M}$, the Gauss equation asserts that

$$K(\sigma) = \Delta(\sigma) + \widetilde{K}(df(\sigma)) \tag{2.2}$$

where K and $\widetilde{K}$ are the sectional curvatures of M and $\widetilde{M}$, and σ is any plane tangent to M [6].

If the vectors in T_xM are orthonormal then, the formula (2.1) reduces to

$$\Delta_{XY} = \langle h(X,X), h(Y,Y)\rangle - \|h(X,Y)\|^2 \tag{2.3}$$

Definition 2.1. We say that h is λ-*isotropic* provided that $\|h(X,X)\| = \lambda$ for all unit vectors X in T_xM . Clearly, an isometric immersion is isotropic provided that all its normal curvature vectors have the same length [5].

Lemma 2.2. [5]*Suppose that h is λ- isotropic on T_xM and let X, Y be orthonormal vectors in T_xM. Then*

$$\langle h(X,X), h(Y,Y)\rangle + 2\|h(X,Y)\|^2 = \lambda^2. \tag{2.4}$$

The assertation (2.1) in the preceding lemma yields the following result.

Lemma 2.3. [5] *If h is λ- isotropic then for orthonormal vectors X, Y in T_xM*
i) $\Delta_{XY} + 3\|h(X,Y)\|^2 = \lambda^2$.
ii) $2\Delta_{XY} + \lambda^2 = 3\langle h(X,X), h(Y,Y)\rangle$.

In the case of $\widetilde{M} = \mathbb{E}^{n+d}$ the sectional curvature $K(\sigma)$ of M reduces to

$$K(\sigma) = \Delta_{XY}. \tag{2.5}$$

Remark 2.4. Let K be a Gaussian curvature of the surface $M \subseteq \mathbb{E}^m$. Then $K = \Delta_{XY}$. If $\Delta_{XY} = 0$ then M is said to be *flat.*

Proposition 1. [7] *Let $f : M^2 \to \mathbb{E}^{2+d}$ be isometric immersion. If the second fundamental form of M^2 is 1-parallel (i.e. $\overline{\nabla} h = 0$) then $f(M^2)$ is one of the following surfaces*

i)$\mathbb{E}^2$
ii) $S^2 \subset \mathbb{E}^3$
iii) $IR^1 \times S^1 \subset \mathbb{E}^3$
iv) $S^1(a) \times S^1(b) \subset \mathbb{E}^4$
v) $V^2 \subset \mathbb{E}^5$.

Proposition 2. *Let M be a ruled surface of the form*

$$x(u,v) = \beta(u) + v\delta(u).$$

If $\Delta xy = 0$ (i.e M is flat) then M is one of the following surfaces;

i) a cone of the form $x(u,v) = p + v\delta(u)$ or,
ii) a cylinder of the form $x(u,v) = \beta(u) + vq$ or,
iii) a tangent developable surface of the form $x(u,v) = \beta(u) + v\beta'(u)$, $(v > 0)$.

Proof. (see [6]).

For more details for the following Examples see [3].

Example 2.5. For the following surfaces $K = \Delta_{XY} = 0$;
1) The torus T^2 embedded in $\mathbb{E}^4$ by

$$T^2 = \{(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi) : \theta, \varphi \in IR\}$$

2) The helicel cylinder H^2 embedded in $\mathbb{E}^4$ by

$$H^2 = \{(u, c\cos v, c\sin v, dv) : u, v \in IR\}$$

3)The cylinder C embedded in $\mathbb{E}^3$ by

$$C = \{(a\cos s, a\sin s, t) : s, t \in IR\}.$$

Example 2.6. For the following surfaces $K = \Delta_{XY} \neq 0$;

1) The sphere S^2 embedded in $\mathbb{E}^3$ by

$$\begin{aligned} S^2 &= \{(a\cos s\cos t, a\cos s\sin t, a\sin s) : s, t \in IR\}, \\ \Delta_{XY} &= \frac{1}{a^2}. \end{aligned}$$

2)The helicoid H embedded in $\mathbb{E}^3$ by

$$\begin{aligned} H &= \{(s\cos t, s\sin t, at) : s, t \in IR\} \\ \Delta_{XY} &= -\frac{a^2}{(s^2+a^2)^2}. \end{aligned}$$

Proposition 3. *The Veronese surface parametrized by*

$$V^2 = \{\frac{1}{\sqrt{3}}yz, \frac{1}{\sqrt{3}}zx, \frac{1}{\sqrt{3}}xy, \frac{1}{2\sqrt{3}}(x^2-y^2), \frac{1}{6}(x^2+y^2-2z^2)\}$$

is spherical.

Proof. The parametric representation of V^2 defines an isometric immersion of $S^2(\sqrt{3})$ into $S^4(1)$. Two points (x, y, z) and $(-x, -y, -z)$ of $S^2(\sqrt{3})$ are mapped into the same point of $S^4(1)$, and this mapping defines an imbedding of the real projective plane into $S^4(1)$. This real projective plane imbedded in $S^4(1)$ is called *the Veronese surface* [1] which is minimal in $S^4(1) \subset \mathbb{E}^5$.

A submanifolds (or immersion) is called *non-spherical* in the fact that it does not lie in a sphere.

Theorem 2.7. *Let $f : M^n \to \mathbb{E}^{n+d}$ be non-spherical isometric immersion. If M is 1-parallel then $\Delta_{XY} = 0$.*

Proof. Since $f(M)$ is not spherical therefore by Proposition 3 the possible non-spherical 1-parallel surfaces are cylinder $IR^1 \times S^1 \subset \mathbb{E}^3$and torus $S^1(a) \times S^1(b) \subset \mathbb{E}^4$. On the other hand , both of them have vanishing sectional curvature.

Definition 2.8. The Vranceanu surface is defined by the parametrized

$$x(s,t) = \{u(s)\cos s\cos t, u(s)\cos s\sin t, u(s)\sin s\cos t, u(s)\sin s\sin t\}. \tag{2.6}$$

Theorem 2.9. *Let the Vranceanu surface is given by the parametrized (2.6) . The Vranceanu surface has vanishing Gaussian curvature ($K = \Delta_{XY} = 0$) if and only if $(u')^2 - uu'' = 0$ (i.e. $u = Ce^{ks}$ for the real constant $0 \neq C$ and k).*

Proof. We choose a moving frame e_1, e_2, e_3, e_4 such that e_1, e_2 are tangent to M and e_3, e_4 are normal to M as given by the following

$$\begin{aligned} e_1 &= (-\cos s \sin t, \cos s \cos t, -\sin s \sin t, \sin s \cos t) \\ e_2 &= \frac{1}{A}(B \cos t, B \sin t, C \cos t, C \sin t) \\ e_3 &= \frac{1}{A}(-C \cos t, -C \sin t, B \cos t, B \sin t) \\ e_4 &= (-\sin s \sin t, \sin s \cos t, \cos s \sin t, -\cos s \cos t) \end{aligned} \tag{2.7}$$

where we put $A = \sqrt{u^2 + (u')^2}, B = u' \cos s - u \sin s, C = u' \sin s + u \cos s$. Then we have

$$e_1 = \frac{1}{u}\frac{\partial}{\partial t},\ e_2 = \frac{1}{A}\frac{\partial}{\partial s}. \tag{2.8}$$

Then the structure equations of $\mathbb{E}^m$ are obtained as follows:

$$\tilde{\nabla}_{e_i} e_j = w_j^k(e_i)e_k + h_{ij}^\alpha e_\alpha,\ 1 \leq i, j, k \leq 2 \tag{2.9}$$

$$\begin{aligned} \tilde{\nabla}_{e_i} e_\alpha &= -h_{ij}^\alpha e_j + w_\alpha^\beta(e_i)e_\beta,\ 3 \leq \alpha, \beta \leq 4 \\ D_{e_\alpha} e_\beta &= w_\alpha^\beta(e_i)e_\beta \end{aligned} \tag{2.10}$$

where D is the normal connection and h_{ij}^α the coefficients of the second fundamental form h. Using (2.7), (2.8), (2.9) and (2.10) we can get that the coefficients of the second fundamental form h and the connection form w_B^A are as following:

$$\begin{aligned} h_{11}^3 &= \frac{1}{\sqrt{u^2 + (u')^2}} = \alpha,\ h_{12}^3 = h_{21}^3 = 0 \\ h_{22}^3 &= \frac{2(u')^2 - uu'' + u^2}{(u^2 + (u')^2)^{3/2}} = \beta \\ h_{11}^4 &= h_{22}^4 = 0,\ h_{12}^4 = h_{21}^4 = -\frac{1}{\sqrt{u^2 + (u')^2}}. \end{aligned}$$

The Gauss curvature is given by

$$\begin{aligned} K &= \det(h_{ij}^3) + \det(h_{ij}^4),\ 1 \leq i, j \leq 2 \\ &= \frac{(u')^2 - uu''}{(u^2 + (u')^2)^2}. \end{aligned} \tag{2.11}$$

Thus this completes the proof of the theorem.

ÖZET: Bu çalışmada, ikinci temel formun diskriminantı gözönünde bulunduruldu. $\mathbb{E}^4$ de Vranceanu yüzeyinin sıfır diskriminantlı olması için gerekli koşul verildi.

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