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ON I-EXTREMALLY DISCONNECTED SPACES

AYNUR KESKIN , SAZIYE YUKSEL AND TAKASHI NOIRI

ABSTRACT. We have introduced and investigated the notion of I-extremal disconnectedness on ideal topological spaces. First, we found that the notions of extremal disconnectedness and I-extremal disconnectedness are independent of each other. About the letter one, we observed that every open subset of an I-extremally disconnected space is also an I-extremally disconnected space. And also, in extremally disconnectedness spaces we have shown that I-open set is equivalent almost I-open and every β-I-open set is preopen. Finally, we have shown that α-I-continuity(resp. pre-I-continuity, I-continuity) is equivalent to semi-I-continuity(resp. strongly β-I-continuity, almost strongly I-continuity) if the domain is I-exteremally disconnected.

1. INTRODUCTION

Throughout the present paper, spaces always mean topological spaces on which no separation property is assumed unless explicitly stated. In a topological space (X, τ), the closure and the interior of any subset A of X will be denoted by $Cl(A)$ and $Int(A)$, respectively. An ideal is defined as a nonempty collection I of subsets of X satisfying the following two conditions: (1) If $A \in I$ and $B \subset A$, then $B \in I$; (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$. Let (X, τ) be a topological space and I an ideal of subsets of X. An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). For a subset $A \subset X$, $A^*(I) = \{x \in X \mid U \cap A \notin I$ for each neighbourhood U of $x\}$ is called the local function of A with respect to I and τ [12]. We simply write A^* instead of $A^*(I)$ in case there is no chance for confusion. X^* is often a proper subset of X. It is well-known that $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$ which is finer than τ. A subset A of (X, τ, I) is called τ **-closed* if $A^* \subset A$ [10].

First we shall recall some lemmas and definitions used in the sequel:

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Lemma 1.1. *Let (X, τ, I) be an ideal topological space and A, B subsets of X. Then the following properties hold:*

a) If $A \subset B$, then $A^* \subset B^*$,
b) $A^* = Cl(A^*) \subset Cl(A)$,
c) $(A^*)^* \subset A^*$,
d) $(A \cup B)^* = A^* \cup B^*$,
e) If $U \in \tau$, then $U \cap A^* \subset (U \cap A)^*$ (Janković and Hamlett [10]) .

Definition 1.2. Let (X, τ, I) be an ideal topological space and S a subset of X. Then (S, $\tau |_S$,I_S) is an ideal topological space with an ideal

$$I_S = \{I \in I \mid I \subset S\} = \{I \cap S \mid I \in I\}$$

on S (Dontchev [3]).

Lemma 1.3. *Let (X, τ, I) be an ideal topological space and $A \subset S \subset X$. Then, $A^*(I_S, \tau |_S) = A^*(I, \tau) \cap S$ holds(Dontchev et al.[6]).*

Definition 1.4. A subset A of an ideal topoogical space (X, τ, I) is said to be

a) *I-open* [1] if $A \subset Int(A^*)$,
b) *pre-I-open* [4] if $A \subset Int(Cl^*(A))$,
c) *α-I-open* [7] if $A \subset Int(Cl^*(Int(A)))$,
d) *semi-I-open* [7] if $A \subset Cl^*(Int(A))$,
e) *β-I-open* [7] if $A \subset Cl(Int(Cl^*(A)))$,
f) *almost I-open* [2] if $A \subset Cl(Int(A^*))$,
g) *strong β-I-open* [8] if $A \subset Cl^*(Int(Cl^*(A)))$,
h) *almost strong I-open* [8] if $A \subset Cl^*(Int(A^*))$.

For the relationship among several sets defined above, Hatir et al. [8] obtained the following dıagram.

DIAGRAM I

open → *α-I-open* → *semi-I-open*
↓ ↓
I-open → *pre-I-open* → *β-I-open*

We recall that a space (X, τ) is said to be extremally disconnected (briefly e.d.) if $Cl(A) \in \tau$ for each $A \in \tau$.

2. *I*-EXTREMALLY DISCONNECTED SPACES

Definition 2.1. A subset A of an ideal topological space (X, τ, I) is said to be weak *regular-I-closed* if $A = Cl^*(Int(A))$.

We denote by $wR_IC(X, \tau)$ (resp. $S_IO(X, \tau)$, $P_IO(X, \tau)$) the family of all *weak regular-I-closed* (resp. semi-*I*-open, pre-*I*-open) subsets of (X, τ, I), when there is no chance for confusion with the ideal.

Definition 2.2. An ideal topological space (X, τ, I) is said to be I-extremally disconnected (briefly I.e.d.) if $\mathrm{Cl}^*(A) \in \tau$ for each $A \in \tau$.

Proposition 1. *For an ideal topological space (X, τ, I), the following properties are equivalent:*

a) (X, τ, I) is I.e.d.,
b) $S_I O(X, \tau) \subset P_I O(X, \tau)$,
c) $wR_I C(X, \tau) \subset \tau$.

Proof. a)$\Longrightarrow$b): Let A$\in$ $S_I O(X, \tau)$.Then $A \subset Cl^*(Int(A))$ and by a) $Cl^*(Int(A)) \in \tau$. Therefore, we have $A \subset Cl^*(Int(A)){=}Int(Cl^*(Int(A))) \subset Int(Cl^*((A))$. This shows that A$\in P_I O(X, \tau)$.

b)$\Longrightarrow$c): Let A$\in wR_I C(X, \tau)$. Then A$=Cl^*(Int(A))$ and hence A$\in S_I O(X, \tau)$. By b), A$\in$ $P_I O(X, \tau)$ and $A \subset Int(Cl^*((A))$. Morever, A is τ^*-closed and $A \subset Int(Cl^*((A)) = Int(A)$. Therefore, we obtain A$\in \tau$.

c)$\Longrightarrow$a): For A$\in\tau$, we show that Cl*(A)$\in$ $wR_I C(X, \tau)$. Since Int(Cl*(A)) $\subset$Cl*(A), we have (Int(Cl*(A)))* $\subset$(Cl*(A))*=(A$\cup$A*)*=A$^*\cup$(A*)* $\subset$A$^*\cup$A*=A* $\subset$Cl*(A) by using Lemma 1d), c) respectively and hence (Int(Cl*(A)))* $\subset$Cl*(A). So, we have Cl*(Int(Cl*(A)))=Int(Cl*(A))$\cup$(Int(Cl*(A)))* $\subset$Cl*(A) and hence

$$\mathrm{Cl}^*(\mathrm{Int}(\mathrm{Cl}^*(\mathrm{A}))) \subset \mathrm{Cl}^*(\mathrm{A}). \tag{2.1}$$

On the other hand, since A is open , according to Diagram I, it is a pre-I-open set and hence we have A$\subset$Int(Cl*(A)). Then, we have

$$\mathrm{Cl}^*(\mathrm{A}) \subset \mathrm{Cl}^*(\mathrm{Int}(\mathrm{Cl}^*(\mathrm{A}))). \tag{2.2}$$

By using (2.1) and (2.2), we have Cl*(A)=Cl*(Int(Cl*(A))). This shows that Cl*(A) is weak regular -I-closed by using Definition 3. Furthermore, since $wR_I C(X, \tau) \subset \tau$, we have Cl*(A)$\in \tau$. This shows that (X, τ, I) is I.e.d. by Definition 4. □

Example 2.3. Let (X, τ, I) is an ideal topological space. If $I = P(X)$, then (X, τ, I) is I.e.d. .

Remark 2.4. I-extremally disconnectedness and extremally disconnectedness are independent of each other as following examples show.

Example 2.5. Let X={a,b,c}, τ={X,∅,{a},{b},{a,b}} and I={∅,{a},{b},{a,b}}. Then (X, τ, I) is an I.e.d. space which is not e.d. For A$\in \tau$, since $\mathrm{A}^* = \varnothing$, we have $Cl^*(A) = A \cup A^* = A$. This shows that (X, τ, I) is an I.e.d. space. On the other hand, for A={a}$\in \tau$, since Cl(A)=Cl({a})={a,c}$\notin \tau$, (X, τ, I) is not e.d..

Example 2.6. Let X={a,b,c,d,e}, τ={∅,X,{a},{a,c},{a,b,d},{a,b,c,d}} and I={∅,{a},{d},{a,d}}. Then, (X,τ,I) is e.d. which is not I.e.d. It is obvious that for every A $\in \tau$, since Cl(A)=X, (X,τ,I) is e.d. On the other hand, for A={a,b,d}$\in \tau$, since A*={b,d,e}, we have Cl*(A)=A$\cup$A*={a,b,d}$\cup${b,d,e}={a,b,d,e} is not open set in (X,τ,I). This shows that (X,τ,I) is not I.e.d. by using Definition 4.

Proposition 2. *Let (X, τ, I) be an ideal topological space and I=$\{\varnothing\}$. Then (X, τ, I) is an I.e.d. space if and only if (X, τ, I) is an e.d. space.*

Proof. If I=$\{\varnothing\}$, then it is well-known that A* = $Cl(A)$ and $Cl^*(A) = A \cup A^* = A \cup Cl(A)$ =$Cl(A)$. Consequently, we obtain Cl(A)=$Cl^*(A) \in \tau$ for every A$\in \tau$. This shows that (X, τ, I) is an I.e.d. space if and only if it is e.d.. □

Lemma 2.7. *Let (X, τ, I) be an ideal topological space. If A∩B=∅ for every A, B$\in \tau$, then A∩$Cl^*(B)$=∅.*

Proof. Since A∩B=∅, we have A∩$Cl^*(B)$ ⊂A∩$(B \cup B^*)$=(A∩B) ∪ $(A \cap B^*)$ ⊂ $(A \cap B) \cup (A \cap B)^* = Cl^*(A \cap B)$ by using Lemma 1.e). On the other hand, since $\varnothing^* = \varnothing$ and $Cl^*(\varnothing) = \varnothing$, we have A∩$Cl^*(B) \subset Cl^*(A \cap B)$ =∅. Thus, we obtain that A∩$Cl^*(B)$=∅. □

Lemma 2.8. *Let (X, τ, I) be an I.e.d. space. If A∩B=∅ for every A, B$\in \tau$, then $Cl^*(A) \cap Cl^*(B)$=∅.*

Proof. The proof is obvious from Lemma 3 and Definition 4. □

Lemma 4 is important because it is given that in any I.e.d. space every two disjoint τ-open sets have disjoint τ^*-closures.

Lemma 2.9. *Let (X, τ, I) be an ideal topological space. If $Cl^*(A)$∩$Cl^*(B)$=∅ for any subsets A and B, then A∩B=∅.*

Proof. Since A⊂Cl*(A) and B⊂ $Cl^*(B)$, we have $A \cap B \subset Cl^*(A) \cap Cl^*(B)$=∅ . Then, we have A∩B=∅. □

Theorem 2.10. *Let (X, τ, I) be an I.e.d. space. For open subsets A, B of X, the following property hold: A∩B =∅ if and only if $Cl^*(A) \cap Cl^*(B)$=∅ .*

Proof. This is an immediate consequence of Lemmas 4 and 5. □

3. I-EXTREMALLY DISCONNECTEDNESS ON SUBSPACES

Theorem 3.1. *Let (X, τ, I) be an I.e.d. space and S an open set in X. Then (S, $\tau |_S, I_S$) is an I.e.d. space.*

Proof. Let A be any open set in S. Since S is open in X and A⊂S⊂X, A is an open set in X. Since (X, τ, I) is an I.e.d. space, $Cl^*(A)$ is open in X by using Definition 4. Furthermore, we can say that $Cl^*_S(A)$ is open in S using Lemma 2. Therefore (S, $\tau |_S, I_S$) is an I.e.d. space. □

4. Functions on I.e.d. spaces

By $\alpha_I O(X,\tau)$ (resp. $\beta_I O(X,\tau)$)we denote the family of all α-I-open (resp. β-I-open) sets of (X,τ,I), when there is no chance for confusion with the ideal. Furthermore, for almost I-open (resp. I-open, strong β-I-open, almost strong I-open) sets of (X,τ,I) we will use $AIO(X,\tau)$ (resp. $IO(X,\tau)$, s$\beta I(X,\tau)$, as$I(X,\tau)$) follow to [1], [2] and [8].

Hatir et al.[8] introduced notions of almost strong I-open sets and strong β-I-open sets and obtained the following diagram.

DIAGRAM II

open $\longrightarrow$ *α-I-open* $\longrightarrow$ *semi-I-open*
$\downarrow$ $\downarrow$
pre-I-open $\longrightarrow$ *strong β-I-open* $\longrightarrow$ *β-I-open*
$\uparrow$ $\uparrow$ $\uparrow$
I-open $\longrightarrow$ *almost strong I-open* $\longrightarrow$ *almost I-open*

Proposition 3. *Let (X,τ,I) be an I.e.d. space and A a subset of X. Then, the following properties hold:*

a) A$\in S_I O(X,\tau)$ if and only if A$\in \alpha_I O(X,\tau)$,
b) A$\in P_I O(X,\tau)$ if and only if A$\in$s$\beta I(X,\tau)$,
c) A$\in IO(X,\tau)$ if and only if A$\in$as$I(X,\tau)$.

Proof. a) Sufficient condition is given in Proposition 2.2.b) of [7]. On the other hand, let A$\in S_I O(X,\tau)$. Then, we have A$\subset$Cl*(Int(A)). Since (X,τ,I) is an I.e.d. space, for Int(A)$\in \tau$, we have Cl*(Int(A)$\in \tau$. Therefore, we have

$$A \subset Cl^*(Int(A)) \subset Int(Cl^*(Int(A)))$$

and hence A is α-I-*open*.

b) Necessary condition is obvious from Diagram II. On the other hand, let A$\in$s$\beta I(X,\tau)$ and hence A$\subset$Cl*(Int(Cl*(A))). Since (X,τ,I) is an I.e.d. space, for Int(Cl*(A))$\in \tau$, we have Cl*(Int(Cl*(A)))$\in \tau$. So, we have

$$A \subset Cl^*(Int(Cl^*(A))) \subset Int(Cl^*(Int(Cl^*(A)))),$$

that is

$$A \subset Int(Cl^*(Int(Cl^*(A)))). \tag{4.1}$$

Besides, since Int(Cl*(A))$\subset$Cl*(A) and Cl**is* Krotowski closure operator, we have Cl*(Int(Cl*(A)))$\subset$Cl*(Cl*(A))=Cl*(A) and hence

$$Int(Cl^*(Int(Cl^*(A)))) \subset Int(Cl^*(A)). \tag{4.2}$$

Consequently, by using (4.1) and (4.2) we have A$\subset$Int(Cl*(A))and hence A is *pre-I-open*.

c) Necessity condition is obvious from Diagram II. On the other hand, let A$\in$as$I(X,\tau)$, then we have A$\subset$Cl*(Int(A*)). Since (X,τ,I) is an I.e.d. space, for

Int(A*)$\in$ τ, we have Cl*(Int(A*)$\in$ τ. Then, we have A$\subset$Cl*(Int(A*)) $\subset$Int(Cl*(Int(A*)))$\subset$Int(Cl*(A*))=Int(A$^*\cup$(A*)*)$\subset$Int(A$^*\cup$A*)=Int(A*) and hence A$\subset$Int(A*). This shows that A is I-open. □

We recall that a subset A of a topological space (X,τ) is said to be preopen if A$\subset$ $Int(Cl(A))$ ([13]).The family of all preopen sets of (X,τ) is denoted by $PO(X,\tau)$.

Proposition 4. *Let (X,τ,I) be an e.d. space and A a subset of X. Then, the following properties hold:*

a) A$\in$ $IO(X,\tau)$ if and only if A$\in$ $AIO(X,\tau)$,
b) If A$\in$ $\beta_I O(X,\tau)$, then A$\in$ $PO(X,\tau)$.

Proof. a) Necessary condition is obvious from Diagram II. On the other hand, let A$\in$ $AIO(X,\tau)$. Since (X,τ,I) is an e.d. space, for Int(A*) $\in$ τ, we have Cl(Int(A*))$\in$ τ. Since A$\in$ $AIO(X,\tau)$, we obtain

A $\subset$Cl(Int(A*))=Int(Cl(Int(A*)))$\subset$Int(Cl(A*))$\subset$Int(A*)

by using Lemma 1.b). This shows that A is I-open.

b) Let A$\in$ $\beta_I O(X,\tau)$, then we have A$\subset$Cl(Int(Cl*(A))). Since (X,τ,I) is an e.d. space, for Int(Cl*(A))$\in$ τ, we have Cl(Int(Cl*(A))) $\in$ τ. So, we have

A$\subset$Cl(Int(Cl*(A)))
$\subset$Int(Cl(Int(Cl*(A))))
$\subset$Int(Cl(Cl*(A)))
$\subset$Int(Cl(A$\cup$A*))
=Int(Cl(A)$\cup$Cl(A*))
$\subset$Int(Cl(A))

by using Lemma 1.b). Therefore, A$\subset$Int(Cl(A)) and hence A is preopen. □

Corollary 1. *Let (X,τ,I) be an ideal topological space such that I={∅} and A a subset of X. Then, the following properties hold:*

a) A$\in$ $IO(X,\tau)$ if and only if A$\in$ $AIO(X,\tau)$,
b) If A$\in$ $\beta_I O(X,\tau)$, then A$\in$ $PO(X,\tau)$.

Proof. This is an immediate consequence of Propositions 2 and 4. □

Definition 4.1. A function f:(X,τ,I)$\longrightarrow$(Y,φ) is said to be almost strongly I-continuous (resp. weakly regular-I-continuous) if for every V$\in$ φ, f^{-1}(V) is almost strong I-open (resp. weak regular-I-closed) in (X,τ,I).

Definition 4.2. A function f:(X,τ,I)$\longrightarrow$(Y,φ) is said to be I-continuous [1] (resp. almost I-continuous [2], pre-I-continuous [4] , semi-I-continuous [7], α-I-continuous [7] , strongly β-I-continuous [8]) if for every V$\in$ φ, f^{-1}(V) is I-open, almost I-open, pre-I-open, *semi*-I-open, α-I-open, strong β-I-open) in (X,τ,I).

Theorem 4.3. *Let (X, τ, I) be an I.e.d. space. For a function f:(X,τ,I)$\longrightarrow$(Y,φ), then the following properties hold:*

a) If f is semi-I-continuous, then it is pre-I-continuous,
b) If f is weakly regular-I-continuous, then it is continuous.

Proof. The proof is obvious from Proposition 1. □

Theorem 4.4. *Let (X, τ, I) be an I.e.d. space. For a function f:(X,τ,I)$\longrightarrow$(Y,φ), then the following properties hold:*

a) f is semi-I-continuous if and only if it is α-I-continuous,
b) f is pre-I-continuous if and only if it is strongly β-I-continuous,
c) f is I-continuous if and only if it is almost strongly I-continuous.

Proof. The proof is obvious from Proposition 3. □

We recall the following definition:A function f:(X,τ)$\longrightarrow$(Y,φ) is said to be precontinuous ([13]) if for every V$\in \varphi$, f^{-1}(V) is preopen in (X,τ)..

Theorem 4.5. *Let (X, τ, I) be an e.d. and I.e.d. such that I=$\{\varnothing\}$, respectively. For a function f:(X,τ,I)$\longrightarrow$(Y,φ), the following properties hold:*

a) f is I-continuous if and only if it is almost I-continuous,
b) If f is β-I-continuous, then it is precontinuous.

Proof. The proof is obvious from Proposition 4 and Corollary 1. □

ÖZET:İdeal topolojik uzaylarda; I-extremal (sonderece) disconnectedness (bağlantısızlık) kavramını tanımladık ve inceledik. İlk olarak; extremal (sonderece) disconnectedness (bağlantısızlık) ve I-extremally (sonderece) disconnectedness (bağlantısızlık) kavramlarının birbirinden bağımsız olduklarını elde ettik. Sonra; I-extremally (son dereceli) disconnected (bağlantısız) bir uzayın her açık alt kümesinin de I-extremally (son dereceli) disconnected (bağlantısız) uzay olduğunu gözledik. Aynı zamanda; extremally(son dereceli) disconnected (bağlantısız) bir uzayda I-açık kümenin almost I-açık kümeye denk olduğunu ve her β-I-açık kümenin pre(ön) açık küme olduğunu da gösterdik. Son olarak; eğer tanım uzayı, I-extremally(son dereceli) disconnected(bağlantısız) bir uzay ise; sırasıyla α-I-süreklilik ile semi(yarı)-I-sürekliliğin, pre(ön)-I-süreklilik ile strongly(kuvvetli) β-I-sürekliliğin, I-süreklilik ile almost(hemen hemen) strongly(kuvvetli) I-sürekliliğin denk olduklarını gösterdik.

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Current address: Aynur KESKIN,Selcuk University, Department of Mathematics, 42075 Campus-Konya TURKEY, Sazıye YUKSEL, Selcuk University, Department of Mathematics, 42031 Campus-Konya TURKEY, Takashi NOIRI,2949-1 Shiokita-cho, Hinagu Yatsushiro-shi, Kumamoto-ken 869-5142 JAPAN

E-mail address: akeskin@email.selcuk.edu.tr, syuksel@selcuk.edu.tr, t.noiri@nifty.com