

ON CONVERGENCE OF CHLODOWSKY-TYPE MKZD OPERATORS

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ABSTRACT. In the present paper, we define a new kind of MKZD operators for functions defined on $[0, b_n]$, named Chlodowsky type MKZD operators and give an estimate, by means of the techniques of probability theory, on the rate of convergence of operators M_n^* for functions of bounded variation on the interval $[0, b_n]$, ($n \rightarrow \infty$) extending infinity.

1. Introduction

For a function defined on the interval $[0,1]$, the Meyer-König and Zeller operators $M_n(f, x)$ [1] are defined as

$$M_n(f; x) = \sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{n+k}\right) \quad (1.1)$$

where $m_{n,k}(x) = \binom{n+k-1}{k} x^k (1-x)^n$. In 1989 Guo [2] introduced the integrated Meyer-König and Zeller operators \tilde{M}_n by the means of the operators (1.1), to approximate Lebesgue integrable functions on the interval $[0,1]$, are defined as

$$\tilde{M}_n(f; x) = \sum_{k=0}^{\infty} \tilde{m}_{n,k}(x) \int_{I_k} f(t) dt \quad (1.2)$$

where $I_k = \left[\frac{k}{n+k}, \frac{k+1}{n+k+1}\right]$ and $\tilde{m}_{n,k}(x) = (n+1) \binom{n+k+1}{k} x^k (1-x)^n$. The rates of pointwise convergence of operators (1.2) and some other integral modifications of the MKZ operators defined by (1.1) has been studied for functions of bounded variation on $[0,1]$. We refer to the reader [3]-[8]. We also mention some of the important papers on similar type of operators are due to Zeng and Cheng [10],

Received by the editors April 09, 2007; Rev: Feb. 13, 2008 Accepted: March 20, 2008.

2000 *Mathematics Subject Classification*. Primary 41A25.; Secondary 41A36.

Key words and phrases. Rate of convergence, Approximation, MKZD operators, Chlodowsky-type MKZD operators, Bounded variation.

Bojanic and Vuilleumier [11], Cheng [12], Gupta and Gupta et al. [13,14]. It is useful to mention very recent papers by Karsli and Ibikli [15-16], which were dealt with the rate of pointwise convergence of the Chlodowsky-type Bernstein operators and its Bézier variant in the space $BV[0, b_n]$ respectively.

Recently, Ibikli and Karsli [17] have defined Chlodowsky type Durrmeyer operators D_n , for a function defined on the interval $[0, b_n]$, as: $D_n : BV[0, \infty) \rightarrow \mathcal{P}$,

$$D_n(f; x) = \frac{(n+1)}{b_n} \sum_{k=0}^n P_{n,k} \left(\frac{x}{b_n} \right) \int_0^{b_n} f(t) P_{n,k} \left(\frac{t}{b_n} \right) dt, \quad 0 \leq x \leq b_n \quad (1.3)$$

where $\mathcal{P} := \{P : [0, \infty) \rightarrow R\}$, is a polynomial function, (b_n) is a positive increasing sequence with the properties

$$\lim_{n \rightarrow \infty} b_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{b_n}{n} = 0$$

and $P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis. They also estimated in the same paper, the rate of convergence of the operators (1.3) for functions in $BV[0, \infty)$.

In the present paper, we define a new kind of MKZD operators for functions defined on $[0, b_n]$, named Chlodowsky-type MKZD operators as

$$M_n^*(f; x) = \sum_{k=0}^{\infty} \frac{n+k}{b_n} m_{n,k} \left(\frac{x}{b_n} \right) \int_0^{b_n} f(t) b_{n,k} \left(\frac{t}{b_n} \right) dt, \quad 0 \leq x \leq b_n, \quad (1.4)$$

where (b_n) is a positive increasing sequence with the properties

$$\lim_{n \rightarrow \infty} b_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{b_n}{n} = 0$$

and $b_{n,k}(t) = n \binom{n+k}{k} t^k (1-t)^{n-1}$.

The aim of this paper is to study the behavior of the M_n^* operators defined by (1.4) for functions of bounded variation and give an estimate, by means of the techniques of probability theory, on the rate of convergence of the operators (1.4) on the interval $[0, b_n]$, $(n \rightarrow \infty)$ extending infinity.

For the sake of brevity, let the auxiliary function f_x be defined by

$$f_x(t) = \begin{cases} f(t) - f(x+) & , \quad x < t \leq b_n \\ 0 & , \quad t = x \\ f(t) - f(x-) & , \quad 0 \leq t < x . \end{cases} \quad (1.5)$$

The main theorem of this paper is as follows.

Theorem. Let f be a function of bounded variation on every finite subinterval of $[0, \infty)$. Then for every $x \in (0, \infty)$, and n sufficiently large, we have,

$$\begin{aligned} \left| M_n^*(f; x) - \frac{1}{2} (f(x+) + f(x-)) \right| &\leq \frac{3A_n(x)b_n^2}{x^2(b_n - x)^2} \left(\sum_{k=1}^n \bigvee_{x - \frac{x}{\sqrt{k}}}^{x + \frac{b_n - x}{\sqrt{k}}} (f_x) \right) \\ &\quad + |f(x+) - f(x-)| \frac{1}{\sqrt{2e}} \frac{1}{\sqrt{n \frac{x}{b_n}}} \end{aligned} \quad (1.6)$$

where

$$A_n(x) = \frac{4x b_n}{n-1} + \frac{2b_n^2}{(n-1)(n-2)} + \frac{4x^2}{n+1}$$

and $\bigvee_a^b(f_x)$ is the total variation of f_x on $[a, b]$.

2. Auxiliary Results

In this section we give certain results, which are necessary to prove our main theorem.

Lemma 1. For $s \in \mathbb{N}$

$$M_n^*(t^s; x) = b_n^s \sum_{k=0}^{\infty} m_{n,k} \left(\frac{x}{b_n} \right) \frac{(n+k)!}{k!} \frac{(k+s)!}{(k+s+n)!}. \quad (2.1)$$

Proof. We have

$$\begin{aligned} M_n^*(t^s; x) &= \sum_{k=0}^{\infty} \frac{n+k}{b_n} m_{n,k} \left(\frac{x}{b_n} \right) \left[\int_0^{b_n} b_{n,k} \left(\frac{t}{b_n} \right) t^s dt \right] \\ &= \sum_{k=0}^{\infty} \frac{n+k}{b_n} m_{n,k} \left(\frac{x}{b_n} \right) \left[\int_0^{b_n} \binom{n+k-1}{k} \left(\frac{t}{b_n} \right)^k \left(1 - \frac{t}{b_n} \right)^{n-1} t^s dt \right]. \end{aligned}$$

Setting $u = \frac{t}{b_n}$, we see that

$$\begin{aligned}
M_n^*(t^s; x) &= \sum_{k=0}^{\infty} \frac{n+k}{b_n} m_{n,k} \left(\frac{x}{b_n}\right) b_n^{s+1} \binom{n+k-1}{k} \int_0^1 u^{k+s} (1-u)^{n-1} du \\
&= \sum_{k=0}^{\infty} (n+k) m_{n,k} \left(\frac{x}{b_n}\right) b_n^s \binom{n+k-1}{k} B(k+s+1, n) \\
&= b_n^s \sum_{k=0}^{\infty} (n+k) m_{n,k} \left(\frac{x}{b_n}\right) \frac{(n+k-1)! (k+s)! (n-1)!}{(n-1)! k! (k+s+n)!} \\
&= b_n^s \sum_{k=0}^{\infty} m_{n,k} \left(\frac{x}{b_n}\right) \frac{(n+k)! (k+s)!}{k! (k+s+n)!}.
\end{aligned}$$

For $s = 0, 1$ and 2 in (2.1), we get respectively

$$M_n^*(1; x) = \sum_{k=0}^{\infty} m_{n,k} \left(\frac{x}{b_n}\right) = 1, \quad (2.2)$$

$$\begin{aligned}
M_n^*(t; x) &= b_n \sum_{k=0}^{\infty} m_{n,k} \left(\frac{x}{b_n}\right) \frac{(n+k)!}{k!} \frac{(k+1)!}{(k+1+n)!} \\
&= \left(1 - \frac{x}{b_n}\right)^n b_n \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)! k!} \left(\frac{x}{b_n}\right)^k \frac{k+1}{k+1+n} \\
&= \left(1 - \frac{x}{b_n}\right)^n b_n \sum_{k=1}^{\infty} \frac{(n+k-2)!}{(n-1)! (k-1)!} \left(\frac{x}{b_n}\right)^k \frac{k+1}{k+1+n} \frac{n+k-1}{k} \\
&= \left(1 - \frac{x}{b_n}\right)^n b_n \sum_{k=1}^{\infty} \frac{(n+k-2)!}{(n-1)! (k-1)!} \left(\frac{x}{b_n}\right)^k \frac{k+1}{k+1+n} \left[1 - \frac{2}{n+k+1}\right] \\
&= \left(1 - \frac{x}{b_n}\right)^n b_n \sum_{k=1}^{\infty} \frac{(n+k-2)!}{(n-1)! (k-1)!} \left(\frac{x}{b_n}\right)^k \frac{k+1}{k+1+n} \left[1 - \frac{2}{n+k+1}\right] \\
&\geq \left(1 - \frac{x}{b_n}\right)^n b_n \sum_{k=1}^{\infty} \frac{(n+k-2)!}{(n-1)! (k-1)!} \left(\frac{x}{b_n}\right)^k \frac{k+1}{k+1+n} \left[1 - \frac{2}{n+1}\right] \\
&\geq \left[1 - \frac{2}{n+1}\right] \left(1 - \frac{x}{b_n}\right)^n b_n \sum_{k=1}^{\infty} \frac{(n+k-2)!}{(n-1)! (k-1)!} \left(\frac{x}{b_n}\right)^k
\end{aligned}$$

$$\begin{aligned}
&= \left[1 - \frac{2}{n+1}\right] \left(1 - \frac{x}{b_n}\right)^n b_n \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} \left(\frac{x}{b_n}\right)^{k+1} \\
&= \left[1 - \frac{2}{n+1}\right] \frac{x}{b_n} b_n \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^n \\
&= \left[1 - \frac{2}{n+1}\right] x \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^n \\
&= \left[1 - \frac{2}{n+1}\right] x
\end{aligned}$$

and

$$\begin{aligned}
M_n^*(t^2; x) &= b_n^2 \sum_{k=0}^{\infty} m_{n,k} \left(\frac{x}{b_n}\right) \frac{(n+k)!}{k!} \frac{(k+2)!}{(k+2+n)!} \\
&= \left(1 - \frac{x}{b_n}\right)^n b_n^2 \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} \left(\frac{x}{b_n}\right)^k \frac{(n+k)!}{k!} \frac{(k+2)!}{(k+2+n)!} \\
&= \left(1 - \frac{x}{b_n}\right)^n b_n^2 \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} \left(\frac{x}{b_n}\right)^k \frac{(k+2)(k+1)}{(k+2+n)(k+1+n)} \\
&= \left(1 - \frac{x}{b_n}\right)^n b_n^2 \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} \left(\frac{x}{b_n}\right)^k \frac{k(k-1) + 4k + 2}{(k+2+n)(k+1+n)} \\
&\leq \left(1 - \frac{x}{b_n}\right)^n b_n^2 \frac{1}{(n-1)!} \sum_{k=2}^{\infty} \frac{(n+k-3)!}{(k-2)!} \left(\frac{x}{b_n}\right)^k \\
&\quad + 4 \left(1 - \frac{x}{b_n}\right)^n b_n^2 \sum_{k=1}^{\infty} \frac{(n+k-3)!}{(k-1)!} \left(\frac{x}{b_n}\right)^k \\
&\quad + 2 \left(1 - \frac{x}{b_n}\right)^n b_n^2 \sum_{k=0}^{\infty} \frac{(n+k-3)!}{k!} \left(\frac{x}{b_n}\right)^k \\
&\leq \left(1 - \frac{x}{b_n}\right)^n b_n^2 \left[\sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} \left(\frac{x}{b_n}\right)^{k+2} + \frac{4}{n-1} \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} \left(\frac{x}{b_n}\right)^{k+1} \right. \\
&\quad \left. + \frac{2}{(n-1)(n-2)} \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} \left(\frac{x}{b_n}\right)^k \right]
\end{aligned}$$

$$\begin{aligned}
&= x^2 \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^n + \frac{4xb_n}{n-1} \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^n \\
&\quad + \frac{2b_n^2}{(n-1)(n-2)} \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^n \\
&= x^2 + \frac{4xb_n}{n-1} + \frac{2b_n^2}{(n-1)(n-2)}.
\end{aligned} \tag{2.3}$$

From (2.2), (??) and (2.3), an easy computation gives

$$|M_n^*((t-x); x)| = \frac{2x}{n+1}$$

and

$$\begin{aligned}
M_n^*((t-x)^2; x) &\leq x^2 + \frac{4xb_n}{n-1} + \frac{2b_n^2}{(n-1)(n-2)} - 2\left[1 - \frac{2}{n+1}\right]x^2 + x^2 \\
&= \frac{4xb_n}{n-1} + \frac{2b_n^2}{(n-1)(n-2)} + \frac{4x^2}{n+1} := A_n(x).
\end{aligned} \tag{2.4}$$

Recently Zeng [9] estimated the exact bounds for Bernstein basis functions and Meyer Konig Zeller basis functions. For $k \in \mathbb{N}$ and $t \in (0, 1]$, the author [9] obtained the inequality:

$$\binom{n+k-1}{k} t^k (1-t)^n < \frac{1}{\sqrt{2e}} \frac{1}{\sqrt{nt}}. \tag{2.5}$$

Replacing the variable t with $\frac{t}{b_n}$, in eq.(2.5) we get the following result:

Lemma 2. For all $t \in (0, b_n]$ and $k \in \mathbb{N}$, we have

$$\binom{n+k-1}{k} \left(\frac{t}{b_n}\right)^k \left(1 - \frac{t}{b_n}\right)^n < \frac{1}{\sqrt{2e}} \frac{1}{\sqrt{n\frac{t}{b_n}}}. \tag{2.6}$$

In the following statements (and throughout the paper), we use the notations;

$$K_n\left(\frac{x}{b_n}, \frac{u}{b_n}\right) := \sum_{k=0}^{\infty} \frac{n+k}{b_n} m_{n,k}\left(\frac{x}{b_n}\right) b_{n,k}\left(\frac{u}{b_n}\right)$$

and

$$\lambda_n\left(\frac{x}{b_n}, \frac{t}{b_n}\right) := \int_0^t K_n\left(\frac{x}{b_n}, \frac{u}{b_n}\right) du. \tag{2.7}$$

Here we note that

$$\lambda_n\left(\frac{x}{b_n}, 1\right) = \int_0^{b_n} K_n\left(\frac{x}{b_n}, \frac{u}{b_n}\right) du = 1.$$

Lemma 3. For all $x \in (0, b_n)$, we have

$$\lambda_n\left(\frac{x}{b_n}, \frac{y}{b_n}\right) = \int_0^y K_n\left(\frac{x}{b_n}, \frac{u}{b_n}\right) du \leq \frac{A_n(x)}{(x-y)^2}, \quad 0 \leq y < x, \quad (2.8)$$

$$1 - \lambda_n\left(\frac{x}{b_n}, \frac{z}{b_n}\right) = \int_z^1 K_n\left(\frac{x}{b_n}, \frac{u}{b_n}\right) du \leq \frac{A_n(x)}{(z-x)^2}, \quad x < z \leq 1 \quad (2.9)$$

where $A_n(x)$ is defined in Theorem.

Proof. We first prove (2.8) as follows:

$$\begin{aligned} \lambda_n\left(\frac{x}{b_n}, \frac{y}{b_n}\right) &= \int_0^y K_n\left(\frac{x}{b_n}, \frac{u}{b_n}\right) du \leq \int_0^y K_n\left(\frac{x}{b_n}, \frac{u}{b_n}\right) \left(\frac{x-u}{x-y}\right)^2 du \\ &= \frac{1}{(x-y)^2} M_n^*((u-x)^2; x). \end{aligned}$$

By (2.4), we have

$$\lambda_n\left(\frac{x}{b_n}, \frac{y}{b_n}\right) \leq \frac{A_n(x)}{(x-y)^2}.$$

The proof of (2.9) is analogous.

3. Main Result

Proof. For any $f(t) \in [0, b_n]$, from (1.5) it is clear that

$$\begin{aligned} f(t) &= \frac{1}{2} (f(x+) + f(x-)) + f_x(t) + \frac{f(x+) - f(x-)}{2} \operatorname{sgn}(t-x) \\ &\quad + \delta_x(t) \left[f(x) - \frac{1}{2} (f(x+) + f(x-)) \right]. \end{aligned} \quad (3.1)$$

Applying the operator (1.4) to (3.1), we have

$$\begin{aligned} M_n^*(f; x) &= \frac{1}{2} (f(x+) + f(x-)) M_n^*(1; x) + M_n^*(f_x; x) \\ &\quad + \frac{f(x+) - f(x-)}{2} M_n^*(\operatorname{sgn}(t-x); x) + \left[f(x) - \frac{1}{2} (f(x+) + f(x-)) \right] M_n^*(\delta_x; x). \end{aligned}$$

Hence

$$\begin{aligned} &\left| M_n^*(f; x) - \frac{1}{2} (f(x+) + f(x-)) M_n^*(1; x) \right| \leq |M_n^*(f_x; x)| \\ &+ \left| \frac{f(x+) - f(x-)}{2} \right| |M_n^*(\operatorname{sgn}(t-x); x)| + \left| f(x) - \frac{1}{2} (f(x+) + f(x-)) \right| |M_n^*(\delta_x; x)|. \end{aligned}$$

For operators $M_n^*(f; x)$, it is obvious that $M_n^*(\delta_x; x) = 0$. From this fact, we can write

$$\left| M_n^*(f; x) - \frac{1}{2}(f(x+) + f(x-)) \right| \leq |M_n^*(f; x)| + \left| \frac{f(x+) - f(x-)}{2} \right| |M_n^*(\text{sgn}(t-x); x)|. \quad (3.2)$$

In order to prove the main theorem, we need the estimates for $M_n^*(f; x)$ and $M_n^*(\text{sgn}(t-x); x)$ in (3.2).

Firstly, we evaluate $M_n^*(f; x)$ as follows:

$$\begin{aligned} |M_n^*(f; x)| &= \left| \sum_{k=0}^{\infty} \frac{n+k}{b_n} m_{n,k} \left(\frac{x}{b_n} \right) \left[\int_0^{x-\frac{x}{\sqrt{n}}} + \int_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{n}}} + \int_{x+\frac{b_n-x}{\sqrt{n}}}^{b_n} \right] b_{n,k} \left(\frac{t}{b_n} \right) f_x(t) dt \right| \\ &\leq \left| \sum_{k=0}^{\infty} \frac{n+k}{b_n} m_{n,k} \left(\frac{x}{b_n} \right) \int_0^{x-\frac{x}{\sqrt{n}}} b_{n,k} \left(\frac{t}{b_n} \right) f_x(t) dt \right| \\ &\quad + \left| \sum_{k=0}^{\infty} \frac{n+k}{b_n} m_{n,k} \left(\frac{x}{b_n} \right) \int_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{n}}} b_{n,k} \left(\frac{t}{b_n} \right) f_x(t) dt \right| \\ &\quad + \left| \sum_{k=0}^{\infty} \frac{n+k}{b_n} m_{n,k} \left(\frac{x}{b_n} \right) \int_{x+\frac{b_n-x}{\sqrt{n}}}^{b_n} b_{n,k} \left(\frac{t}{b_n} \right) f_x(t) dt \right| \\ &=: |I_1(n, x)| + |I_2(n, x)| + |I_3(n, x)|. \end{aligned} \quad (3.3)$$

We shall evaluate $I_1(n, x)$, $I_2(n, x)$ and $I_3(n, x)$.

First we estimate $I_2(n, x)$. For $t \in \left[x - \frac{x}{\sqrt{n}}, x + \frac{b_n-x}{\sqrt{n}} \right]$, we have from (2.7)

$$\begin{aligned} |I_2(n, x)| &\leq \int_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{n}}} |f_x(t) - f_x(x)| \left| dt \left(\lambda_n \left(\frac{x}{b_n}, \frac{t}{b_n} \right) \right) \right| \leq \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}} (f_x) \\ &\leq \frac{1}{n-1} \sum_{k=2}^n \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}} (f_x). \end{aligned} \quad (3.4)$$

Next, we estimate $I_1(n, x)$. Using partial Lebesgue-Stieltjes integration, we have again from (2.7)

$$I_1(n, x) = f_x(x - \frac{x}{\sqrt{n}})\lambda_n(\frac{x}{b_n}, \frac{x - \frac{x}{\sqrt{n}}}{b_n}) - f_x(0)\lambda_n(\frac{x}{b_n}, \frac{0}{b_n}) - \int_0^{x - \frac{x}{\sqrt{n}}} \lambda_n(\frac{x}{b_n}, \frac{t}{b_n}) dt (f_x(t)).$$

Since $\left|f_x(x - \frac{x}{\sqrt{n}})\right| = \left|f_x(x - \frac{x}{\sqrt{n}}) - f_x(x)\right| \leq \bigvee_{x - \frac{x}{\sqrt{n}}}^x (f_x)$, it follows that

$$|I_1(n, x)| \leq \bigvee_{x - \frac{x}{\sqrt{n}}}^x (f_x) \left| \lambda_n(\frac{x}{b_n}, \frac{x - \frac{x}{\sqrt{n}}}{b_n}) \right| + \int_0^{x - \frac{x}{\sqrt{n}}} \lambda_n(\frac{x}{b_n}, \frac{t}{b_n}) dt \left(- \bigvee_t^x (f_x) \right).$$

By (2.4), it is clear that

$$\lambda_n(\frac{x}{b_n}, \frac{x - \frac{x}{\sqrt{n}}}{b_n}) \leq \frac{1}{(\frac{x}{\sqrt{n}})^2} A_n(x).$$

It follows that

$$\begin{aligned} |I_1(n, x)| &\leq \bigvee_{x - \frac{x}{\sqrt{n}}}^x (f_x) \frac{A_n(x)}{(\frac{x}{\sqrt{n}})^2} + \int_0^{x - \frac{x}{\sqrt{n}}} \frac{1}{(x-t)^2} A_n(x) dt \left(- \bigvee_t^x (f_x) \right) \\ &= \bigvee_{x - \frac{x}{\sqrt{n}}}^x (f_x) \frac{A_n(x)}{(\frac{x}{\sqrt{n}})^2} + A_n(x) \int_0^{x - \frac{x}{\sqrt{n}}} \frac{1}{(x-t)^2} dt \left(- \bigvee_t^x (f_x) \right). \end{aligned}$$

Furthermore, since

$$\begin{aligned} \int_0^{x - \frac{x}{\sqrt{n}}} \frac{1}{(x-t)^2} dt \left(- \bigvee_t^x (f_x) \right) &= - \frac{1}{(x-t)^2} \bigvee_t^x (f_x) \Big|_0^{x - \frac{x}{\sqrt{n}}} + \int_0^{x - \frac{x}{\sqrt{n}}} \frac{2}{(x-t)^3} \bigvee_t^x (f_x) dt \\ &= - \frac{1}{(\frac{x}{\sqrt{n}})^2} \bigvee_{x - \frac{x}{\sqrt{n}}}^x (f_x) + \frac{1}{x^2} \bigvee_0^x (f_x) + \int_0^{x - \frac{x}{\sqrt{n}}} \frac{2}{(x-t)^3} \bigvee_t^x (f_x) dt. \end{aligned}$$

Taking $t = x - \frac{x}{\sqrt{u}}$ in the last integral, we get

$$\int_0^{x - \frac{x}{\sqrt{n}}} \frac{2}{(x-t)^3} \bigvee_t^x (f_x) dt = \frac{1}{x^2} \int_1^n \bigvee_{x - \frac{x}{\sqrt{u}}}^x (f_x) du = \frac{1}{x^2} \sum_{k=1}^n \bigvee_{x - \frac{x}{\sqrt{k}}}^x (f_x).$$

Consequently, we obtain

$$|I_1(n, x)| \leq \frac{A_n(x)}{x^2} \left(\bigvee_0^x (f_x) + \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^x (f_x) \right). \quad (3.5)$$

Using a similar method, for evaluating $|I_3(n, x)|$, we find that

$$|I_3(n, x)| \leq \frac{A_n(x)}{(b_n - x)^2} \left(\bigvee_x^{b_n} (f_x) + \sum_{k=1}^n \bigvee_x^{x+\frac{b_n-x}{\sqrt{k}}} (f_x) \right). \quad (3.6)$$

Substituting (3.4), (3.5) and (3.6) in (3.3), we obtain

$$\begin{aligned} |M_n^*(f_x; x)| &\leq |I_1(n, x)| + |I_2(n, x)| + |I_3(n, x)| \\ &\leq \frac{A_n(x) b_n^2}{x^2 (b_n - x)^2} \left(\bigvee_0^{b_n} (f_x) + \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}} (f_x) \right) + \frac{1}{n-1} \sum_{k=2}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}} (f_x). \end{aligned}$$

On the other hand, note that

$$\bigvee_0^{b_n} (f_x) \leq \sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}} (f_x). \quad (3.7)$$

From (3.7), we can rewrite $|M_n^*(f_x; x)|$ as below;

$$|M_n^*(f_x; x)| \leq \frac{2A_n(x) b_n^2}{x^2 (b_n - x)^2} \left(\sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}} (f_x) \right) + \frac{1}{n-1} \sum_{k=2}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}} (f_x).$$

Note that $\frac{1}{n-1} \leq \frac{A_n(x) b_n^2}{x^2 (b_n - x)^2}$, for $n > 1$, $\frac{x}{b_n} \in [0, 1]$.

Consequently

$$|M_n^*(f_x; x)| \leq \frac{3A_n(x) b_n^2}{x^2 (b_n - x)^2} \left(\sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{b_n-x}{\sqrt{k}}} (f_x) \right). \quad (3.8)$$

Now we estimate $M_n^*(sgn(t-x); x)$.

By direct calculation, one has

$$\begin{aligned}
M_n^*(\operatorname{sgn}(t-x); x) &= \sum_{k=0}^{\infty} \frac{n+k}{b_n} m_{n,k}\left(\frac{x}{b_n}\right) \left[\int_x^{b_n} b_{n,k}\left(\frac{t}{a_n}\right) dt - \int_0^x b_{n,k}\left(\frac{t}{b_n}\right) dt \right] \\
&= \sum_{k=0}^{\infty} \frac{n+k}{b_n} m_{n,k}\left(\frac{x}{b_n}\right) \left[\int_0^{b_n} b_{n,k}\left(\frac{t}{b_n}\right) dt - 2 \int_0^x b_{n,k}\left(\frac{t}{b_n}\right) dt \right] \\
&= 1 - 2 \sum_{k=0}^{\infty} \frac{n+k}{b_n} m_{n,k}\left(\frac{x}{b_n}\right) \int_0^x b_{n,k}\left(\frac{t}{b_n}\right) dt.
\end{aligned}$$

Using the fact $\sum_{j=0}^k m_{n,j}\left(\frac{x}{b_n}\right) = \frac{n+k}{b_n} \int_x^{b_n} b_{n,k}\left(\frac{t}{b_n}\right) dt$, one has

$$M_n^*(\operatorname{sgn}(t-x); x) = 1 - 2 \sum_{k=0}^{\infty} m_{n,k}\left(\frac{x}{b_n}\right) \left(1 - \sum_{j=0}^k m_{n,j}\left(\frac{x}{b_n}\right) \right).$$

From (2.6), we obtain

$$\begin{aligned}
|M_n^*(\operatorname{sgn}(t-x); x)| &= \left| 1 - 2 \sum_{k=0}^{\infty} m_{n,k}\left(\frac{x}{b_n}\right) \left[1 - \sum_{j=0}^k m_{n,j}\left(\frac{x}{b_n}\right) \right] \right| \\
&= \left| -1 + 2 \sum_{k=0}^{\infty} m_{n,k}\left(\frac{x}{b_n}\right) \sum_{j=0}^k m_{n,j}\left(\frac{x}{b_n}\right) \right| \quad (3.9) \\
&= \left| \sum_{k=0}^{\infty} m_{n,k}\left(\frac{x}{b_n}\right) \sum_{j=0}^k m_{n,j}\left(\frac{x}{b_n}\right) - \sum_{k=0}^{\infty} \left(m_{n,k}\left(\frac{x}{b_n}\right) \right)^2 \right| \\
&\leq \left| m_{n,k}\left(\frac{x}{b_n}\right) \sum_{k=0}^{\infty} m_{n,k}\left(\frac{x}{b_n}\right) \right| \leq \frac{1}{\sqrt{2e}} \frac{1}{\sqrt{n \frac{x}{b_n}}}.
\end{aligned}$$

Finally writing the estimates of (3.8) and (3.9) in (3.2), we get the result, given by (1.6). Thus the proof of the theorem is completed.

Acknowledgments

The author is thankful to the referees for their valuable remarks and suggestions leading to a better presentation of this paper.

Özet: Bu makalede, $[0, b_n]$ aralığı üzerinde tanımlı fonksiyonlar için Chlodowsky tipi MKZD Operatörü olarak adlandırılan yeni bir tip MKZD Operatörü tanımlıyoruz ve olasılık teorisinin teknikleri vasıtasıyla, $[0, b_n]$ ($n \rightarrow \infty$) iken sonsuza genişleyen aralık üzerinde

sınırlı salımlı olan fonksiyonlar için M_n^* operatörünün yakınsaklık hızı üzerine bir tahmin veriyoruz.

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