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BISHOP FRAME OF THE SPACELIKE CURVE WITH A SPACELIKE PRINCIPAL NORMAL IN MINKOWSKI 3-SPACE

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Abstract. In this study, we generalize for a spacelike curve with a spacelike principal normal which was studied by Bishop [1] to Minkowski 3-Space. In addition, the Bishop Darboux vector (matrix) for spacelike curve is found. Furthermore, using the derivative of the tangent vector T of the spacelike curve, the relations between the curvature funtions κ, τ and k_1, k_2 is found.

1. Preliminaries

Let $R^3 = \{(x_1, x_2, x_3) | \, x_1, x_2, x_3 \in R\}$ be a 3-dimensional vector space, and let $x = (x_1,\ x_2,\ x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in R^3. The Lorentz scalar product of x and y is defined by

$$\langle x, y\rangle_L = x_1y_1 + x_2y_2 - x_3y_3.$$

$E_1^3 = \left(R^3, \langle x, y\rangle_L\right)$ is called 3-dimensional Lorentzian space, Minkowski 3-space or 3- dimensional semi-Euclidean space. The vector x in E_1^3 is called a spacelike vector, null vector or a timelike vector if $\langle x, x\rangle_L > 0$ or $x = 0$, $\langle x, x\rangle_L = 0$ or $\langle x, x\rangle_L < 0$, respectively. For $x \in E_1^3$, the norm of the vector x defined by $\|x\|_L = \sqrt{|\langle x, x\rangle_L|}$, and x is called a unit vector if $\|x\|_L = 1$. For any $x,\ y \in E_1^3$, Lorentzian vectoral product of x and y is defined by

$$x \wedge_L y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_2y_1 - x_1y_2)\,.$$

The Lorentzian sphere of center $m = (m_1,\ m_2,\ m_3)$ and radius $r \in R^+$ in the Minkowski 3-space is defined by $S_1^2 = \left\{a = (a_1,\ a_2,\ a_3) \in E_1^3 \,\middle|\, \langle a - m,\ a - m\rangle_L = r^2\right\}$. Denote by $\{T,\ N,\ B\}$ the moving Frenet frame along the curve α. Then $T,\ N$ and B are the tangent, the principal normal and vector binormal of the curve α respectively. If α is a spacelike curve with a spacelike principal normal, then this set of orthogonal unit vectors, known as the Frenet-Serret frame, have properties

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$T^{'} = \kappa N$, $N^{'} = -\kappa T + \tau B$, $B^{'} = \tau N$ and $\langle T, T\rangle_L = \langle N, N\rangle_L = 1$, $\langle B, B\rangle_L = -1$ [2].

2. Introduction

The Frenet frame of a 3-times continuously differentiable non-degenerate spacelike curve with a spacelike principal normal invaryant in semi-Euclidean space has long been the standart vehicle for analysing properties of the spacelike curve invariant under semi-Euclidean motions. For arbitrary moving frames that is, orthonormal basis fields, we can express the derivatives of the frame with respect to the spacelike curve with a spacelike principal normal parameter in term of the frame its self, and due to semi-ortonormality the coefficient matrix is always semi- skew symmetric.Thus it generally has three nonzero entries.The Frenet frame gains part of its special significance from the fact that one of the three derivatives is zero. Another feature of the Frenet frame is that it is adapted to the spacelike curve with a spacelike principal normal: the members are either tangent to or perpendicular to the spacelike curve with a spacelike principal normal. It is the purpose of this paper to show that there are other frames which have these same advantages and to compare them with the Frenet frame.

3. Parallel Fields

3.1. **Relatively Parallel Fields.** We say that a normal vector field N along a curve α is relatively parallel if its derivative tangential. Such a vector field turns only whatever amount is necessary for it to remain normal, so it is as close to being paralel as possible without losing normality. Since its derivative is perpendicular to it, a relatively parallel normal vector field has constant length. Such fields occur classically in the discussion of curves which are said to be parallel to given spacelike curve with a spacelike principal normal.Indeed, if a curve α considered as a displacement vector function of a parameter t, then if N is relatively parallel, the spacelike curve with a spacelike principal normal with displacement vector $\alpha + N$ has velocity $(\alpha + N)^{'} = (v + f)T$, where T is the unit tangential vector field of α, v is the speed of α, and $N^{'} = fT$. Thus the segment between two curve is locally perpendicular to both. Whether or not this segment is locally a segment of minimum length between the two curves depends on the curvature and the length of N. It is easily verified that the segment local minimizes length if N is short enough. Conversly, a spacelike curve with a spacelike principal normal which runs at constant distance from α must be given $\alpha + N$, where N relatively parallel.

A single normal vector field N_0 at a point $\alpha(t_0)$ generates a unique relatively paralel field N such that $N(t_0) = N_0$. The uniqueness is trivial: the difference of two relatively parallel fields is obviously relatively parallel, so if two such coincide at one point, their difference has constant length 0. To show existence one takes auxiliary adapted frames; the Frenet frame would do if it exists, but we want existence even for degenerate curves, that is, those which have curvature vanishing at some points.

Such frames can be constructed locally by appling the Gram-Schmidt process to T two parallel fields.

Theorem 3.1. *Let α be spacelike curve with a spacelike principal normal with unit speed.If T, N_1, N_2 is adapted frame, then we have*

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & \rho_{01} & -\rho_{02} \\ -\rho_{01} & 0 & -\rho_{12} \\ -\rho_{02} & -\rho_{12} & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}. \tag{3.1}$$

Proof. If T , N_1 are spacelike vectors but N_2 timelike vector then we can write

$$T' = \rho_{00} T + \rho_{01} N_1 + \rho_{02} N_2 \tag{3.2}$$

for some the functions ρ_{00}, ρ_{01} and ρ_{02}. From the equation (3.2), we find $\rho_{00} = 0$, $\rho_{01} = \left\langle T', N_1 \right\rangle_L$ and $\rho_{02} = -\left\langle T', N_2 \right\rangle_L$

So we get

$$T' = 0T + \rho_{01} N_1 - \rho_{02} N_2.$$

Similary, we can write

$$N_1' = -\rho_{01} T + 0N_1 - \rho_{12} N_2$$

and

$$N_2' = -\rho_{02} T - \rho_{12} N_1 + 0N_2.$$

Thus we have eq. (3.1) or shortly, $X' = KX$. Moreover K is semi skew-matrix for satisfying $K^T = -\epsilon K \epsilon$, where ϵ is $\operatorname{diag}(1, 1, -1)$ matrix. Now we find the condition for a normal field of constant L to relatively paralel. There is a smooth function θ such that $N = L\,[N_1 \cosh\theta + N_2 \sinh\theta]$. Differentiating, we have

$$N' = L\left[\left(\theta' - \rho_{12}\right)(N_1 \sinh\theta + N_2 \cosh\theta) - (\rho_{01} \cosh\theta + \rho_{02} \sinh\theta)\, T\right].$$

From this we see that N is relatively paralel if and only if $\theta' = \rho_{12}$.

Since there is a solution for θ satifying any initial condition, this shows that localy relatively parallel normal fields exist. To get global existence we can patch to gether local ones, which exist on a covering by interval. Smoothness at the points where they link together is consequence of the uniqueness part.

We define a tangential field to be relatively parallel if it is a constant multiple of the unit tangent field T. An arbitrary field is relatively parallel if its tangential and normal components are relatively parallel. We spell out the complete hypotheses for the existence and uniqueness of these fields as follows. □

Theorem 3.2. *Let α be a C^k spacelike curve with a spacelike principal normal in Minkowski 3-space which is regular, that is, the velocity never vanishes ($k \geq 2$). Then for any vector X_0 at $\alpha(t_0)$ there is a unique C^{k-1} relatively parallel field X along α such that $X(t_0) = X_0$ and the scalar product of two relatively is constant.*

Proof. To prove that the scalar product $\langle X, Y\rangle_L$ of two relatively paralel fileds X, Y is constant, we observe that it is trivial for tangential ones and maybe verified for the tangential and normal parts separately. Thus we assume X and Y are normal, with derivatives $f\ T$ and gT. Then the derivative of $\langle X, Y\rangle_L$ is

$$\begin{aligned}\frac{d}{dt}\langle X, Y\rangle_L &= \left\langle X', Y\right\rangle_L + \left\langle X, Y'\right\rangle_L \\ &= \langle fT, Y\rangle_L + \langle X, gT\rangle_L \\ &= f\langle T, Y\rangle_L + g\langle X, T\rangle_L \\ &= f.0 + g.0 = 0\end{aligned}$$

as desired. Thus, $\langle X, Y\rangle_L$ is constant. □

3.2. **Special Adapted Frames.** It should be clear that the relatively parallel fields on C^2 regular form a 3-dimensional vector over R with distinguised subspaces consisting of an oriented 1-dimensional tangential part and a 2-dimensional normal part, and there is a Lorentzian scalar product inherited from the pointwise scalar product on the ambient semi-Euclidean.We call an semi-orthonormal basis of this vector space which fits the two subspaces a relatively paralel adapted frame or RPAF. If we assume that the ambient semi-Euclidean space has a preferred orientation, then so does the normal space of the spacelike curve with a spacelike principal normal, and we may refer to properly oriented RPAF. The totality of RPAF's are in the form of two circles(in the Lorentzian mean), one in each orientation class, since they can be parametrized by the 2-dimensional semi-orthogonal group, according to the following obvious result.

Theorem 3.3. *If $\{T, N_1, N_2\}$ is a relatively paralel adapted frame, then RPAF's consists of frames the form $\{T, aN_1 + bN_2, cN_1 + dN_2\}$, where*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

runs through semi-orthogonal matrices having entries.

Proof. Now if $\{T, N_1, N_2\}$ is a RPAF**,** denoting derivatives with respect to arc length by a dot, we have,

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} \begin{bmatrix} 0 & k_1 & -k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}. \tag{3.3}$$

This shows that we accomplished our original goal showing that there are other adapted frames which have only two nonzero entires in their Cartan matrices.(for a more general) discussion of Cartan matrices. In fact, given some one such RPAF, The Theorem 3.3 tells us that possible Cartan matrices for RPAF's are

$$K = \begin{bmatrix} 0 & ak_1 + bk_2 & ck_1 + dk_2 \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix}$$

where $*$ denotes an entry which can be determined by using semi skew-symmetry. The Frenet frame has Cartan matrix

$$\begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix},$$

and is unique once the orientation of the ambient space and a convention on the sign of torsion τ have been chosen. The only other possibilities for Cartan matrix with one entry vanishing would be

$$\begin{bmatrix} 0 & 0 & f \\ 0 & 0 & g \\ f & g & 0 \end{bmatrix}, \begin{bmatrix} 0 & f & g \\ -f & 0 & 0 \\ g & 0 & 0 \end{bmatrix}.$$

It is simple to relate the entries of the variuos Cartan matrices. Indeed,

$$\kappa = \left\| T^{'} \right\|_L = \| k_1 N_1 + k_2 N_2 \|_L = \sqrt{|k_1^2 - k_2^2|}.$$

Writing the principal normal as

$$N = N_1 \cosh\theta + N_2 \sinh\theta = \left(\frac{k_1}{\kappa}\right) N_1 + \left(\frac{k_2}{\kappa}\right) N_2, \tag{3.4}$$

and differentiating we obtain $N^{'} = -\kappa T + \tau B = -\kappa T + \theta' \left(N_1 \sinh\theta + N_2 \cosh\theta\right).$ If $\{T, N_1, N_2\}$ is properly oriented, we conclude that $B = N_1 \sinh\theta + N_2 \cosh\theta$ and hence $\theta' = \tau$. Thus κ and indefined integral $\int \tau(s)ds$ are polar coordinates for the curve (k_1, k_2). □

4. The Normal Development of a Spacelike Curve with a Spacelike Principal Normal

We want to view (k_1, k_2) as a sort of invariant of spacelike curve α with a spacelike principal normal . This slightly more difficult to conceive than in a case of (κ, τ), since the RPAF is not unique. However, we have spelled out what degree of freedom there is Theorem 3.3. (k_1, k_2) is determined up to an semi-orthogonal transformation in the non oriented case and up to a semi-rotation about the origin in the oriented case. Thus we must think of (k_1, k_2) as a parametrized (by an arc length for α) continuous spacelike curve with a spacelike principal normal in a centro semi-Euclidean plane, that is a semi-Euclidean plane having distiguished point. When conceived of in this way we call (k_1, k_2) the normal development of spacelike curve α with a spacelike principal normal. This situation is not really so different from the case of the Frenet invariants (κ, τ), because in the non oriented case (κ, τ) and the Frenet frame are determined only up to an action by the two-element group, with the non identity changing the sign of τ and B. That is, cannot be distinguished from $(\kappa, -\tau)$. The standart facts about the relation (k_1, k_2) and spacelike curve α spacelike curve with a spacelike principal normal as an object of semi-Euclidean geometry correspond to similary facts about (k_1, k_2) and α.The proofs are identical with the Frenet case, and in fact are partly given in unified form in [3] .

Theorem 4.1. *Two C^2 regulary spacelike curve with a spacelike principal normal in semi-Euclidean space are congurent if and only if they have the same normal devolepment. For any parametrized continous curve in centro semi-Euclidean plane there is a C^2 regular curve in semi-Euclidean space having the given curve as its normal development, i.e., Two curves are congruent if and only if they have the same arc length parametrization of their curvature and torsion.*

The modifications for the oriented case are clear: make both semi-Euclidean space and the centro semi-Euclidean plane be oriented and congurences be proper.

Theorem 4.2. *Let α be spacelike curve with a spacelike principal normal. A C^2 regular curve α lies on a Lorentzian sphere if and only if its normal devolepment lies on a line not the origin. The distance of this line from the origin and the radius of the Lorentzian sphere are reciprocals.*

Proof. **⇒:** If α lies on a Lorentzian sphere with center p and radius r, then

$$\langle \alpha - p, \, \alpha - p \rangle_L = r^2. \tag{4.1}$$

Differentiating with respect to arc length gives

$$\langle T, \, \alpha - p \rangle_L = 0, \tag{4.2}$$

so

$$\alpha - p = fN_1 + gN_2; \tag{4.3}$$

for some functions f, g. From equation (4.3), we get

$$f = \langle \alpha - p, N_1 \rangle_L \, , \, g = - \langle \alpha - p, N_2 \rangle_L \, . \tag{4.4}$$

On deriving for equation (4.4), we have

$$f' = \frac{d}{ds} \langle \alpha - p, N_1 \rangle_L = \langle T, N_1 \rangle_L + \left\langle \alpha - p, N_1' \right\rangle_L = 0 - \langle \alpha - p, k_1 T \rangle_L = 0.$$

Thus f is constant. Similary, g is constant. Then differentiating equation (4.2), we get

$$\begin{aligned}
\langle T, \alpha - p\rangle_L &= 0 \\
\left\langle T^{'}, \alpha - p\right\rangle_L + \langle T, T\rangle_L &= 0 \\
\langle k_1 N_1 - k_2 N_2, \alpha - p\rangle_L + 1 &= 0 \\
k_1 \underbrace{\langle N_1, \alpha - p\rangle_L}_{f} - \underbrace{k_2 \langle N_2, \alpha - p\rangle_L}_{-g} + 1 &= 0 \\
f\, k_1 + g\, k_2 + 1 &= 0.
\end{aligned}$$

That is, (k_1, k_2) is on the line

$$f\, x + g\, y + 1 = 0. \tag{4.5}$$

Moreover, distance of line l from the origin is

$$\frac{1}{f^2 - g^2} = \frac{1}{r^2} = d, \quad f^2 - g^2 > 0,\ r > 0$$

$\Leftarrow$: Conversly, suppose that

$$fx + gy + 1 = 0.$$

where f and g are constant. Let

$$\overrightarrow{p\alpha} = fN_1 + gN_2,$$

then

$$\begin{aligned}
-p^{'} &= -\alpha^{'} + fN_1^{'} + gN_2^{'} \\
&= -T - fk_1 T - gk_2 T \\
&= (1 + fk_1 + gk_2)\, T \\
&= 0
\end{aligned}$$

so p is constant. Moreover, let

$$\|\overrightarrow{p\alpha}\|_L^2 = \langle \alpha - p, \alpha - p\rangle_L\,,$$

then

$$\frac{d}{ds}\langle \alpha - p, \alpha - p\rangle_L = \langle T, \alpha - p\rangle_L = 0.$$

Thus $\langle \alpha - p, \alpha - p\rangle_L = r^2$ is constant and spacelike curve with a spacelike principal normal lies on a Lorentzian sphere of radius r and center p. □

Definition 4.3. If a rigid body moves along a spacelike curve α with a spacelike principal normal(which we suppose is unit speed), then the motion of body consists of translation along spacelike curve α with a spacelike principal normal and rotation about spacelike curve α with a spacelike principal normal. The rotation is determined by an angular velocity vector ω which satifies

$$T^{'} = \omega \wedge_L T,\ N_1^{'} = \omega \wedge_L N_1,\ N_2^{'} = \omega \wedge_L N_2.$$

The vector ω is called the Darboux vector.

Theorem 4.4. *Darboux matrix have form:*

$$W=\begin{bmatrix} 0 & -k_1 & -k_2 \\ k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix}.$$

Proof. First of all let us find Darboux vector ω. Then we write

$$\omega = aT + bN_1 + cN_2 \tag{4.7}$$

and take cross products with T, N_1 and N_2 to determine a, b and c. After some calculations we can find as $a = 0$, $b = -k_2$ and $c = k_1$. Thus we can write Darboux vector as follows,

$$\omega = -k_2N_1 + k_1N_2 = (0, -k_2, k_1)_{\{T,\, N_1,\, N_2\}}.$$

Moreover, since $\vec{\omega} \wedge_L \vec{X} = W.\vec{X}$ for each X in E_1^3, we get

$$W=\begin{bmatrix} 0 & -k_1 & -k_2 \\ k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix}$$

□

Theorem 4.5. *If T is tangent vector of a space curve α with a spacelike principal normal, then the following the formulas hold:*

$$\begin{array}{lll} (a) & T^{'} \wedge_L T'' & = (k_2k_1' - k_2'k_1)\,T - \kappa^2\omega; \quad k_1 \neq 0 \\ (b) & \det\left(T, T^{'}, T''\right) & = k_2^{'}k_1 - k_2k_1^{'} \\ (c) & \dfrac{\det\left(T,T^{'},T''\right)}{\left\|T \ \wedge_L T^{'}\right\|_L^2} & = \theta^{'} \ \ or \ \pm\tau \end{array}$$

where ω is the Darboux vector of spacelike curve α with a spacelike principal normal.

Proof. (a) From $T^{'} = k_1N_1 - k_2N_2$ we get

$$\begin{aligned} T'' &= k_1^{'}N_1 - k_2^{'}N_2 + k_1(-k_1T) - k_2(-k_2T) \\ &= -\left(k_1^2 - k_2^2\right)T + k_1^{'}N_1 - k_2^{'}N_2 \\ &= -\kappa^2T + k_1^{'}N_1 - k_2^{'}N_2. \end{aligned}$$

Moreover one can easly find

$$\begin{aligned} \langle T'', T\rangle_L &= -\kappa^2, \\ \langle T'', N_1\rangle_L &= k_1' \ , \\ \langle T'', N_2\rangle_L &= k_2'. \end{aligned}$$

From definition of Darboux vector ω, we write $T^{'} = \omega \wedge_L T$. So we get

$$\begin{aligned} T' \wedge_L T'' &= -T^{''} \wedge_L (\omega \wedge_L T) \\ &= \langle T'', T\rangle_L \, \omega - \langle T'', \omega\rangle_L \, T \\ &= -\kappa^2 \omega - \langle T'', -k_2 N_1 + k_1 N_2\rangle_L \, T \\ &= -\kappa^2 \omega + k_2 \langle T'', N_1\rangle_L \, T - k_1 \langle T'', N_2\rangle_L \, T \\ &= (k_2 k_1' - k_1 k_2')\, T - \kappa^2 \omega \\ &= (k_2 k_1' - k_2' k_1)\, T - \kappa^2 \omega. \end{aligned}$$

(b) From the last equation we get

$$\begin{aligned} \langle T,\, T' \wedge_L T''\rangle_L &= \left\langle T,\ -\kappa^2 \omega + (k_2 k_1' - k_1 k_2')\, T\right\rangle_L \\ \det(T,\, T',\, T'') &= \kappa^2 \langle T, \omega\rangle_L + (k_2 k_1' - k_1 k_2') \langle T,\, T\rangle_L \\ &= -\kappa^2 \langle T,\ - k_2 N_1 + k_1 N_2\rangle + (k_2 k_1' - k_1 k_2')\,(+1) \\ &= k_2 k_1' - k_1 k_2'. \end{aligned}$$

(c) From the equality $T^{'} = \omega \wedge_L T$, we can write $-\omega = T \wedge_L T'$ and $\|T \wedge_L T'\|_L^2 = \left|k_1^2 - k_2^2\right|$. From the equality (3.4) we immediately see that $\tanh \theta = \frac{k_2}{k_1}$ or

$$\theta = \arg\tanh(\frac{k_2}{k_1}). \tag{4.8}$$

Differentiating from the equation (4.8), we find

$$\theta^{'} = \frac{(\frac{k_2}{k_1})^{'}}{1 - (\frac{k_2}{k_1})^2} = -\frac{k_1 k_2' - k_2 k_1'}{k_1^2 - k_2^2}.$$

Thus we have

$$\theta^{'} = \pm \frac{\det(T, T', T'')}{\|T \wedge_L T'\|_L^2}$$

or

$$\theta^{'} = \pm \tau.$$

□

ÖZET : Bu çalışmada, Bishop [1] tarafından yapılan çalışmayı, spacelike asli normalli spacelike eğriler için Minkowski 3-uzayına genelleştirdik. İlaveten, spacelike eğriler için Bishop Darboux vektörü (matrisi) bulundu. Ayrıca, spacelike eğrinin T teğet vektörünün türevi kullanılarak, (κ, τ) ve (k_1, k_2) eğrilik fonksiyonları arasındaki ilişkiler bulundu.

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