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ON THE EMBEDDING OF COMPLEMENTS OF SOME HYPERBOLIC PLANES II

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Abstract. In this paper, we studied that a linear space, which is the complement of linear space whose points are not on a pentagon, hexagon or a heptagon in a projective subplane of order m, is embeddable in an unique way in a projective plane of order n. In addition, we showed that this linear space is the complement of certain regular hyperbolic plane in the sense of Graves [5] with respect to a finite projective plane..

1. Introduction

The complementation problem with respect to a projective plane is the following : Remove a certain configuration of points and lines from the plane, determine the parameters of the resulting space. The problem of embedding the " *complements* " of various configuration in the projective planes has been studied by various authors ([1],[2],[3],[4],[5],...). In 1970, Dickey solved the problem for the case where the configuration removed was a unital [5]. In 1987, L. M. Batten characterized linear spaces which are the complements of affine or projective subplanes of finite projective planes and showed that these spaces can be embeddable in an unique way in a projective plane of order n [1]. A generalized of Batten's Theorem [1] was given by Günaltılı and Olgun [7]. In [8], Günaltılı, Anapa and Olgun showed that a linear space, which is the complement of a linear space having points are not on a trilateral or a quadrilateral in a projective subplane of order m, is embeddable in an unique way in a projective plane of order n. In addition, it was determined that this linear space is the complement of certain regular hyperbolic plane in the sense of Graves [6] with respect to a finite projective plane.

In this study, we showed that a linear space, which is the complement of a linear space whose points are not on a pentagon, hexagon or a heptagon in a projective subplane of order m, is embeddable in a unique way in a projective plane of order n. In addition, we determined that this linear space is the complement of certain

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regular hyperbolic plane in the sense of Graves [6] with respect to a finite projective plane.

Now, we give some definitions required.

Definition 1.1 : A *non-degenerate finite linear space* is a pair $\mathcal{S} = (\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a finite set of points and $\mathcal{L}$ is a family of proper subsets of $\mathcal{P}$, which are called lines, such that: any two points are on unique line, each line has at least two points and there are at least two lines.

If $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a finite linear space, that is, $|\mathcal{P}| < \infty$. The number of lines passing through a point P is called the degree of P and denoted by $b(P)$. The number of points on a line l is called the degree of l and denoted by $v(l)$. v and b denote the total number of points and lines of $\mathcal{S}$, respectively. The number $v, b, b(P)$ and $v(l)$ are called the parameters of $\mathcal{S}$.

The parameters of finite linear space k_m, k_M, r_m, r_M are defined as stated below.

$k_m = \min\{v(l) : l \in \mathcal{L}\}$

$k_M = \max\{v(l) : l \in \mathcal{L}\}$

$r_m = \min\{b(P) : P \in \mathcal{P}\}$

$r_M = \max\{b(P) : P \in \mathcal{P}\}$

The term $i-$point or $i-$line may also used to refer respectively to a point or a line of degree i. Moreover; b_k denotes the total number of $k-$lines, v_k denotes the total number of $k-$points and $b_k(P)$ denotes the total number of $k-$lines passing through a point P.

The integer n defined by $n + 1 = \max b(P) : P \in \mathcal{P}$ } is the " *order* " of the linear space.

Definition 1.2 : A finite $(m+1)-$regular *hyperbolic plane* $(\mathcal{P}, \mathcal{L})$, in the sense of Graves, is a non-trivial $(m+1)-$regular linear space such that :

- H1 There are four points, no three of which are collinear
- H2 If P is a point not on a line l, then there exist at least two lines, not meeting l and through P.
- H3 If a subset $\mathcal{P}^{'}$ of the points of $\mathcal{P}$ contains three non-collinear points and contains all points on the lines through pairs of distinct points of $\mathcal{P}^{'}$ contains all points of $\mathcal{P}$.

Proposition 1.1 : (Bumcrot, [4]) Any finite linear space satisfying the following condition :

- I $r_m \geq k_M + 2$
- II $k_m(k_m - 1) \geq r_M$

is a hyperbolic plane in the sense of Graves [6].

A linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is said to be embeddable in a linear space $\mathcal{S}' = (\mathcal{P}', \mathcal{L}')$ if $\mathcal{S}'$ can be obtained from $\mathcal{S}$ by addition of some points called as ideal points and some lines called as ideal lines.

2. Main Results

In this section, we showed that a linear space, which is the complement of linear space whose points are not on a pentagon, hexagon or a heptagon in a projective subplane of order m, is embeddable in an unique way in a projective plane of order n. In addition, we determined that this linear space is the complement of certain regular hyperbolic plane in the sense of Graves [6] with respect to a finite projective plane.

Proposition 2.1 : Any $(m+1)-$regular linear space with line degree $m-2$, $m-3$ or $m-4, m \geq 7$, is a hyperbolic plane in the sense of Graves [6] .

Proof : Let $\mathcal{S}$ be an $(m+1)-$regular linear space with line degree $m-2$, $m-3$ or $m-4$. It is clear that $r_m \geq k_M + 2$ and
$k_m(k_m-1) = (m-4)(m-5) \geq m+1$, since $k_m = m-4$, $k_M = m-2$ and $r_m = r_M = m+1$, $m \geq 7$. By the Proposition 1.1, $\mathcal{S}$ is a hyperbolic plane and it is called a hyperbolic plane of $(5,m)-$type.

Proposition 2.2 : A real complement of linear space whose points are on a pentagon in a projective of order m is a hyperbolic plane, $m \geq 7$.

Proof : Let π be a projective plane of order m and $\mathcal{S}$ be a real complement of a pentagon in a projective plane of order m. The total number of points on a pentagon is $5(m-1)$. Thus, $\mathcal{S}$ is an $(m+1)-$regular linear space with m^2-4m+6 points, m^2+m-4 lines and it's line degree is $m-2, m-3$, or $m-4$. Also, $\mathcal{S}$ is a hyperbolic plane of $(5,m)-$type by the Proposition 2.1.

Proposition 2.3 : Any $(m+1)-$regular linear space with line degree $m-2, m-3, m-4$ or $m-5, m \geq 9$, is a hyperbolic plane in the sense of Graves [6].

Proof : Let $\mathcal{S}$ be an $(m+1)-$regular linear space with line degree $m-2, m-3, m-4$ or $m-5$. It is clear that $r_m \geq k_M + 2$ and
$k_m(k_m-1) = (m-5)(m-6) \geq m+1$, since $k_m = (m-5)$, $k_M = (m-2)$ and$r_m = r_M = m+1$, $m \geq 9$. By the Proposition 1.1, $\mathcal{S}$ is a hyperbolic plane and it is called a hyperbolic plane of $(6,m)-$type.

Proposition 2.4 : A real complement of a linear space whose points are on a hexagon in a projective plane of order m is a hyperbolic plane, $m \geq 9$.

Proof : Let π be a projective plane of order m and $\mathcal{S}$ be a real complement of a hexagon in π, $m \geq 9$. The total number of points on a hexagon is $6m-9$. Thus; $\mathcal{S}$ is an $(m+1)-$regular linear space with $m^2-5m+10$ points, m^2+m-5 lines and it's line degree is $m-2$, $m-3$, $m-4$ or $m-5$. Also, $\mathcal{S}$ is a hyperbolic plane of $(6,m)-$type by the Proposition 2.3.

Proposition 2.5 : Any $(m+1)-$regular linear space with line degree $m-3, m-4, m-5$ or $m-6, m \geq 10$, is a hyperbolic plane in the sense of Graves [6].

Proof : Let $\mathcal{S}$ be an $(m+1)-$regular linear space with line degree $m-3, m-4, m-5$ or $m-6$. It is clear that $r_m \geq k_M + 2$ and
$k_m(k_m-1) = (m-6)(m-7) \geq m+1$, since $k_m = (m-6)$, $k_M = m-3$ and $r_m = r_M = m+1$, $m \geq 10$. By the Proposition 1.1, $\mathcal{S}$ is a hyperbolic plane and it is called a hyperbolic plane of $(7,m)-$type.

Proposition 2.6 : A real complement of a linear space whose points are on a heptagon in a projective plane of order m is a hyperbolic plane, $m \geq 10$.

Proof : Let π be a projective plane of order m and $\mathcal{S}$ be a real complement of a heptagon in π, $m \geq 10$. The total number of points on a heptagon is $7m-14$. Thus; $\mathcal{S}$ is an $(m+1)-$regular linear space with $m^2-6m+15$ points, m^2+m-6 lines and it's line degree is $m-3$, $m-4$, $m-5$ or $m-6$. Also, $\mathcal{S}$ is a hyperbolic plane of $(7,m)-$type, by the Proposition 2.5.

Theorem 2.1 : Let $\mathcal{S}=(\mathcal{P},\mathcal{L})$ be an $(n+1)-$regular linear space such that:

(i) $v=n^2+n-(m^2-4m+5)$, $b=n^2+n+1$, $m\geq 7$, $n\geq m^2$, $n,m\in\mathbb{Z}$.
(ii) $b_{n+3-m}=15$ and $b_{n+4-m}=10(m-4)$
(iii) every line has $n+1, n, n+3-m, n+4-m, n+5-m$ points.

Then $\mathcal{S}$ is embeddable in a unique way in a projective plane of order n and is complement of a hyperbolic plane $(5,m)-$type.

Proof : Let $\mathcal{P}_{ijk}$ be the set of points of $\mathcal{S}$ such that there are i lines of degree $n+3-m$, j lines of degree $n+4-m$, k lines of degree $n+5-m$, h lines of degree n and w lines of degree $n+1$ through every point P of it. Then;

$$(n+2-m)i+(n+3-m)j+(n+4-m)k+(n-1)h+wn=v-1$$

$$i+j+k+h+w=n+1$$

From the above equalities, the following results are obtained.

$$\begin{aligned} h &= (m^2-4m+6)-(m-2)i-(m-3)j-(m-4)k \\ w &= n+1-(m^2-4m+6)+(m-3)i+(m-4)j+(m-5)k \end{aligned}$$

Also; by the simple counting methods,

$$\sum_i |\mathcal{P}_i| = v, \quad \sum_t b_t = b, \quad t\in\{n+1, n, n+3-m, n+4-m, n+5-m\}$$

$$\begin{aligned} \sum_{i,j,k} |\mathcal{P}_{ijk}|\, i &= 15(n+3-m) \\ \sum_{i,j,k} |\mathcal{P}_{ijk}|\, j &= 10(m-4)(n+4-m) \\ \sum_{i,j,k} |\mathcal{P}_{ijk}|\, k &= b_{n+5-m}(n+4-m) \\ \sum_{i,j,k} |\mathcal{P}_{ijk}|\, h &= nb_n \text{ and } \sum_{i,j,k} |\mathcal{P}_{ijk}|\, w = (n+1)b_{n+1}. \end{aligned}$$

Then; the following results are obtained.

$$\begin{aligned} b_n &= (m^2-4m+6)(n-m) \\ b_{n+1} &= n^2-m^2(n+5-m)+m(4n+5)-5(n+1) \\ b_{n+3-m} &= 15 \\ b_{n+4-m} &= 10(m-4) \\ b_{n+5-m} &= (m^2-9m+21) \end{aligned}$$

Since $b_n = (m^2-4m+6)(n-m)$, $n \geq m^2$ and $m \geq 7$, there exists at least one $n-$line. For every $n-$line l, we define

$$\Pi_l = \{l\} \cup \{x : x \text{ is a line disjoint to } l\ \}$$

Since each point has degree $n+1 = v(l)+1$, each point outside l lies on exactly one line which is parallel to l. This shows that Π_l is a partition of the points of $\mathcal{S}$ into disjoint lines, and Π_l induces an equivalence relation among the lines in $\mathcal{S}$ of size n. This equivalent relation on the lines of size n is refered as parallelism. Since l meets n^2 other lines, $|\Pi_l| = n+1$. Hence; each $n-$line induces a partition of the points into $n+1$ lines refered as the parallel class associated with that $n-$line.

Suppose that l and l' are two different $n-$lines which meet. Then l' meets n lines of Π_l, so $|\Pi_l \cap \Pi_{l'}| = 1$.

Let each such parallel class corresponds to a " *new point* ". Consider the structure $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*)$, where $\mathcal{P}^*$ is $\mathcal{P}$ along with the new points, and $\mathcal{L}^*$ consists of the lines of $\mathcal{L}$ " *extended* " by those parallel classes to which they belong. We first of all prove that $\mathcal{S}^*$ is a linear space. It is clear that two old points (points of $\mathcal{P}$) are on a unique line of $\mathcal{L}^*$. Let X and Y be distinct new points. We show that they determine a unique line of $\mathcal{L}^*$. Let l_X and l_Y be $n-$lines which determine the parallel classes corresponding to X and Y. If l_X and l_Y do not meet, then $X = Y$ which is a contradiction. So l_X and l_Y meet. Since the point degree of $\mathcal{S}$ is $n+1$, each point of l_Y is on a unique line of the parallel class determined by l_X. This leaves precisely one line of the parallel class parallel to both l_X and l_Y. This is the required line. It follows from our method of construction that each point of $\mathcal{S}^*$ is on $n+1$ lines.

Finally, it is shown that any two lines of $\mathcal{S}^*$ always meet. Let l and l' be lines of $\mathcal{S}^*$ which do not meet in $\mathcal{S}$. Then neither l or l' are $(n+1)-$lines. To prove that they meet in $\mathcal{S}^*$, it suffices to find an $n-$line parallel to both.

Let l and l' be two disjoint lines which have size less than n in $\mathcal{S}$. It is clear that $d(l) = n+1-v(l) \geq m-4$ and $d(l') = n+1-v(l') \geq m-4$ since $v(l) \in \{n+1, n, n+3-m, n+4-m, n+5-m\}$, for all $l \in \mathcal{L}$.

Let x be the number of lines meeting l (excluding l itself); let y be the number of lines meeting l' (excluding l' itself); and z be the number of lines meeting both

l and l'. The following three equations are obtained by a simple counting method:

$$\begin{aligned} x &= n(n+1-d(l)) \\ y &= n(n+1-d(l')) \end{aligned}$$

and

$$z = (n+1-d(l))(n+1-d(l')).$$

Therefore; $x+y-z = n^2 - (d(l)-1)(d(l')-1)$.

Let $m(l,l')$ and $m_n(l,l')$ be the total number of lines and $n-$lines, respectively, meeting l or l' excluding l and l' themselves. Since;
$m(l,l') = x+y-z$, the following result is obtained.

$$m(l,l') = n^2 - (d(l)-1)(d(l')-1) \leq n^2 - (m-5)^2.$$

Since any line of size $n+1$ meets every other line, all the lines of size $n+1$ meet both l and l'. Therefore; since $b_{n+1} \geq 0$, $d(l) \geq m-4$ and $d(l') \geq m-4$, it is clear that $m_n(l,l') \leq n^2 - (m-5)^2 - b_{n+1} \leq b_n$. Thus; there is at least one $n-$line parallel to both. Consequently; $\mathcal{S}^*$ is a projective plane of order n.

Consider the complement of $\mathcal{S}$ in $\mathcal{S}^*$. $\mathcal{S}^* \setminus \mathcal{S}$ is an $(m+1)-$regular linear space whose lines are set of $\{m-2\}, \{m-3\}$ or $\{m-4\}$ points, which are extensions of $(n+3-m)-$lines, $(n+4-m)-$lines and $(n+5-m)-$lines of $\mathcal{S}$, respectively. Therefore; $\mathcal{S}^* \setminus \mathcal{S}$ is a hyperbolic plane of $(5,m)-$type, by the Proposition 2.1.

Theorem 2.2 : Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be an $(n+1)-$regular linear space such that:

(i) $v = n^2+n-(m^2-5m+9)$, $b = n^2+n+1$, $m \geq 9$, $n \geq m^2$, $n, m \in \mathbb{Z}$.
(ii) $b_{n+4-m} = 45 - 3b_{n+3-m}$ and $b_{n+5-m} = 15(m-7) + 3b_{n+3-m}$
(iii) every line has $n+1, n, n+3-m, n+4-m, n+5-m, n+6-m$ points.

Then $\mathcal{S}$ is embeddable in a unique way in a projective plane of order n and is complement of a hyperbolic plane $(6,m)-$type.

Proof : Let $\mathcal{P}_{ijkt}$ be the set of points of $\mathcal{S}$ such that there are i lines of degree $n+3-m$, j lines of degree $n+4-m$, k lines of degree $n+5-m$, t lines of degree $n+6-m$, h lines of degree n and w lines of degree $n+1$ through every point P of it. Then;

$$(n+2-m)i + (n+3-m)j + (n+4-m)k + (n+5-m)t + (n-1)h + wn = v-1$$

$$i+j+k+t+h+w = n+1$$

From the above equalities, the following results are obtained.

$$\begin{aligned} h &= (m^2-5m+10) - (m-2)i - (m-3)j - (m-4)k - (m-5)t \\ w &= n+1-(m^2-5m+10) + (m-3)i + (m-4)j + (m-5)k + (m-6)t \end{aligned}$$

Also; by the simple counting methods,

$$\sum_i |\mathcal{P}_i| = v, \quad \sum_t b_t = b, \quad t \in \{n+1, n, n+3-m, n+4-m, n+5-m, n+6-m\}$$

$$\begin{aligned}
\sum_{i,j,k} |\mathcal{P}_{ijkt}|\, i &= (n+3-m)b_{n+3-m} \\
\sum_{i,j,k} |\mathcal{P}_{ijkt}|\, j &= (45-3b_{n+3-m})(n+4-m) \\
\sum_{i,j,k} |\mathcal{P}_{ijkt}|\, k &= (15(m-7)+3b_{n+3-m})(n+5-m) \\
\sum_{i,j,k} |\mathcal{P}_{ijkt}|\, t &= (n+6-m)b_{n+6-m} \\
\sum_{i,j,k} |\mathcal{P}_{ijk}|\, h &= nb_n \text{ and } \sum_{i,j,k} |\mathcal{P}_{ijk}|\, w = (n+1)b_{n+1}.
\end{aligned}$$

Then; the following results are obtained.

$$\begin{aligned}
b_n &= (m^2-5m+10)(n-m) \\
b_{n+1} &= n^2 - m^2(n+6-m) + m(5n+9) - 9n + 6 \\
b_{n+4-m} &= 45 - 3b_{n+3-m} \\
b_{n+5-m} &= 15(m-7) + 3b_{n+3-m} \\
b_{n+6-m} &= (m^2-14m+55) - b_{n+3-m}
\end{aligned}$$

Since $b_n = (m^2-5m+10)(n-m)$, $n \geq m^2$ and $m \geq 9$, there exists at least one $n-$line. For every $n-$line l, it can be defined Π_l parallel classes and constructed $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*)$ as in the proof of Theorem 2.1. Using the technique in the Theorem 2.1, it is easily shown any two points of $\mathcal{S}^*$ are on exactly one line.

Thus, we must show that any two lines of $\mathcal{S}^*$ always meet. Let l and l' be lines of $\mathcal{S}^*$ which do not meet in $\mathcal{S}$. Then neither l or l' are $(n+1)-$lines. To prove that they meet in $\mathcal{S}^*$, it suffices to find an $n-$line parallel to both.

Let l and l' be lines of $\mathcal{S}$ such that $v(l) < n$ and $v(l') < n$. It is clear that $d(l) = n+1-v(l) \geq m-5$ and $d(l') = n+1-v(l') \geq m-5$ since $v(l) \in \{n+1, n, n+3-m, n+4-m, n+5-m, n+6-m\}$, for all $l \in \mathcal{L}$. Again using the technique in the Theorem 1.2, it is easily calculated that $m_n(l,l') \leq n^2-(m-6)^2-b_{n+1} \leq b_n$, since $\min\limits_{l\in\mathcal{L}}(n+1-v(l)) = m-5$ and $n \geq m^2$. There is at least one $n-$line parallel to both. Thus; $\mathcal{S}^*$ is a projective plane of order n.

Consider the complement of $\mathcal{S}$ in $\mathcal{S}^*$. $\mathcal{S}^* \setminus \mathcal{S}$ is an $(m+1)-$regular linear space whose lines are set of $\{m-2\}, \{m-3\}, \{m-4\}$ or $\{m-5\}$ points, which are extensions of$(n+3-m)-$lines, $(n+4-m)-$lines, $(n+5-m)-$lines and $(n+6-m)-$lines of $\mathcal{S}$, respectively. Therefore; $\mathcal{S}^* \setminus \mathcal{S}$ is a hyperbolic plane of $(6,m)-$type, by the Proposition 2.3.

Theorem 2.3 :Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be an $(n+1)-$regular linear space such that:

(i) $v = n^2+n-(m^2-6m+14)$, $b = n^2+n+1$, $m \geq 10$, $n \geq m^2$, $n, m \in \mathbb{Z}$.

(ii) $b_{n+5-m} = 105 - 3b_{n+4-m}$ and $b_{n+6-m} = 21(m-11) + 3b_{n+4-m}$

(iii) every line has $n+1, n, n+4-m, n+5-m, n+6-m, n+7-m$ points.

Then $\mathcal{S}$ is embeddable in a unique way in a projective plane of order n and is complement of a hyperbolic plane $(7,m)-$type.

Proof : Let $\mathcal{P}_{ijkt}$ be the set of points of $\mathcal{S}$ such that there are i lines of degree $n+4-m$, j lines of degree $n+5-m$, k lines of degree $n+6-m$, t lines of degree $n+7-m$, h lines of degree n and w lines of degree $n+1$ through every point P of it. Then;

$$(n+3-m)i+(n+4-m)j+(n+5-m)k+(n+6-m)t+(n-1)h+wn=v-1$$

$$i+j+k+t+h+w=n+1$$

From the above equalities, the following results are obtained.

$$\begin{aligned} h &= (m^2-6m+15)-(m-3)i-(m-4)j-(m-5)k-(m-6)t \\ w &= n+1-(m^2-6m+15)+(m-4)i+(m-5)j+(m-6)k+(m-7)t \end{aligned}$$

Also; by the simple counting methods,

$$\sum_i |\mathcal{P}_i| = v, \quad \sum_t b_t = b, \quad t \in \{n+1, n, n+4-m, n+5-m, n+6-m, n+7-m\}$$

$$\begin{aligned} \sum_{i,j,k,t} |\mathcal{P}_{ijkt}|\, i &= (n+4-m)b_{n+4-m} \\ \sum_{i,j,k,t} |\mathcal{P}_{ijkt}|\, j &= (105-3b_{n+4-m})(n+5-m) \\ \sum_{i,j,k,t} |\mathcal{P}_{ijkt}|\, k &= (21(m-11)+3b_{n+4-m})(n+6-m) \\ \sum_{i,j,k,t} |\mathcal{P}_{ijkt}|\, t &= (n+7-m)b_{n+7-m} \\ \sum_{i,j,k} |\mathcal{P}_{ijk}|\, h &= nb_n \text{ and } \sum_{i,j,k} |\mathcal{P}_{ijk}|\, w = (n+1)b_{n+1}. \end{aligned}$$

Then; the following results are obtained.

$$\begin{aligned} b_n &= (m^2-6m+15)(n-m) \\ b_{n+1} &= n^2-m^2(n+7-m)+m(6n+14)-(14n-7) \\ b_{n+5-m} &= 105-3b_{n+4-m} \\ b_{n+6-m} &= 21(m-11)+3b_{n+4-m} \\ b_{n+7-m} &= (m^2-20m+120-3b_{n+4-m}) \end{aligned}$$

Since $b_n = (m^2-6m+15)(n-m)$, $n \geq m^2$ and $m \geq 10$ there exists at least one $n-$line. For every $n-$line l, it can be defined Π_l parallel classes and constructed $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*)$ as in the proof of Theorem 2.1. Using the technique in the Theorem 2.1, it is easily shown any two points of $\mathcal{S}^*$ are on exactly one line.

Thus, we must show that any two lines of $\mathcal{S}^*$ always meet. Let l and l' be lines of $\mathcal{S}^*$ which do not meet in $\mathcal{S}$. Then neither l or l' are $(n+1)-$lines. To prove that they meet in $\mathcal{S}^*$, it suffices to find an $n-$line parallel to both.

Let l and l' be lines of $\mathcal{S}$ such that $v(l) < n$ and $v(l') < n$. It is clear that $d(l) = n+1-v(l) \geq m-6$ and $d(l') = n+1-v(l') \geq m-6$ since $v(l) \in \{n+1, n, n+4-m, n+5-m, n+6-m, n+7-m\}$, for all $l \in \mathcal{L}$. Again using the technique in the Theorem 1.2, it is easily calculated that $m_n(l,l') \leq n^2-(m-7)^2-b_{n+1} \leq b_n$, since $\min_{l\in\mathcal{L}}(n+1-v(l)) = m-6$ and $n \geq m^2$. There is at least one $n-$line parallel to both. Thus; $\mathcal{S}^*$ is a projective plane of order n.

Consider the complement of $\mathcal{S}$ in $\mathcal{S}^*$. $\mathcal{S}^* \setminus \mathcal{S}$ is an $(m+1)-$regular linear space whose lines are set of $\{m-3\}, \{m-4\}, \{m-5\}$ or $\{m-6\}$ points, which are extensions of$(n+4-m)-$lines, $(n+5-m)-$lines, $(n+6-m)-$lines and $(n+7-m)-$lines of $\mathcal{S}$, respectively. Therefore; $\mathcal{S}^* \setminus \mathcal{S}$ is a hyperbolic plane of $(7,m)-$type, by the Proposition 2.5.

ÖZET: Bu çalışmada, m. mertebeden projektif altdüzlemdeki beşgen, altıgen ve yedigene ait olmayan noktaları noktalar kümesi olarak kabul eden bir lineer uzayın tümleyeni olan ve n mertebeden projektif düzleme tek olarak gömülebilen lineer uzaylar üzerinde çalışılmıştır. Ayrıca, bu lineer uzayın projektif düzlemlere göre Graves anlamında regular hiperbolik düzlemin tümleyeni olduğu gösterilmiştir.

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