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ON REDUCED MODULES

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Abstract. Let α be an endomorphism of an arbitrary ring R with identity. In this note, we concern the relations between polynomial and power series extensions of a reduced module. Among others we prove that a ring R is α-reduced if and only if every flat right R-module is α-reduced, and for a module M, $M[x]$ is α-reduced if and only if $M[x, x^{-1}]$ is α-reduced.

1. Introduction

Throughout all rings have an identity 1 and all modules are unital and α denotes a nonzero endomorphism of a given ring with $\alpha(1) = 1$, and $\mathbf{1}$ is the identity endomorphism, unless specified otherwise. Let R be a ring and M be a right R-module. Recall that R is *reduced* if it has no nonzero nilpotent elements and M is called *α-reduced* if, for any $m \in M$ and any $a \in R$,

(1) $ma = 0$ implies $mR \cap Ma = 0$,
(2) $ma = 0$ if and only if $m\alpha(a) = 0$.

The module M is called *reduced* if it is $\mathbf{1}$-reduced. Hence R is a reduced ring if and only if R_R is a reduced module. The module M_R is α-reduced if and only if $M[x; \alpha]_{R[x;\alpha]}$ is reduced [6].

We write $R[x], R[[x]], R[x, x^{-1}]$ and $R[[x, x^{-1}]]$ for the polynomial ring, the power series ring, the Laurent polynomial ring and the Laurent power series ring over R, respectively.

For a module M, we consider

$$M[x; \alpha] = \left\{ \sum_{i=0}^{s} m_i x^i \ : \ s \geq 0, m_i \in M \right\},$$

$$M[[x; \alpha]] = \left\{ \sum_{i=0}^{\infty} m_i x^i \ : m_i \in M \right\},$$

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$$M[x,x^{-1};\alpha]=\left\{\sum_{i=-s}^{t} m_i x^i \ :\ s\geq 0, t\geq 0, m_i\in M\right\},$$
$$M[[x,x^{-1};\alpha]]=\left\{\sum_{i=-s}^{\infty} m_i x^i \ :\ s\geq 0, m_i\in M\right\}.$$

Each of these is an abelian group under obvious addition operation. Moreover $M[x;\alpha]$ becomes a module over $R[x;\alpha]$ under the following scalar product operation: For $m(x)=\sum_{i=0}^{s} m_i x^i \in M[x;\alpha]$ and $f(x)=\sum_{i=0}^{t} a_i x^i \in R[x;\alpha]$

$$m(x)f(x)=\sum_{k=0}^{s+t}\left(\sum_{i+j=k} m_i\alpha^i(a_j)\right)x^k.$$

Similarly, $M[[x;\alpha]]$ is a module over $R[[x;\alpha]]$. The modules $M[x;\alpha]$ and $M[[x;\alpha]]$ are called the *skew polynomial extension* and the *skew power series extension of* M, respectively. If $\alpha\in Aut(R)$, then with a similar scalar product, $M[[x,x^{-1};\alpha]]$ (resp. $M[x,x^{-1};\alpha]$) becomes a module over $R[[x,x^{-1};\alpha]]$ (resp.$R[x,x^{-1};\alpha]$). The modules $M[x,x^{-1};\alpha]$ and $M[[x,x^{-1};\alpha]]$ are called the *skew Laurent polynomial extension* and the *skew Laurent power series extension of* M, respectively. Background material can be found in [2].

2. Reduced Modules

In [6], Lee and Zhou introduced reduced modules as the generalization of reduced rings. So far, various results of reduced rings are extended to reduced modules. We now continue to investigate further properties of reduced modules.

We begin with a simple observation.

Theorem 2.1. *Let M be a module. For any $m\in M$ and any $a,b\in R$, the following are equivalent:*
(1) *M is α-reduced module.*
(2) (*i*) *$ma=0$ implies $mR\alpha(a)=0$.*
(*ii*) *$ma\alpha(a)=0$ implies $ma=0$.*
(3) (*i*) *$mab=0$ implies $(mbR)\cap(Ma)=0$.*
(ii) $ma=0$ if and only if $m\alpha(a)=0$.
(4) (*i*) *$mab=0$ implies $(maR)\cap(Mb)=0$.*
(ii) $ma=0$ if and only if $m\alpha(a)=0$.

Proof. It is straightforward. □

Corollary 2.2. *Let M be an α-reduced module. Let $m\in M$ and $a\in R$. Then $ma=0$ if and only if $ma^2=0$. In this case $mRa=0$.*

In [5], a ring R is called *α-rigid* if $a\alpha(a)=0$ implies $a=0$, for any $a\in R$. A module M is called α-rigid if $ma\alpha(a)=0$ implies $ma=0$, for any $m\in M$ and

$a \in R$. The module M is called *rigid* if it is $\mathbf{1}$-rigid. A ring R is α-rigid if and only if R_R is an α-rigid module.

Corollary 2.3. *If the module M is α-reduced , then it is rigid and α-rigid.*

Proposition 2.4. *The class of α-reduced modules is closed under submodules, direct products and so direct sums.*

The class of reduced modules need not be closed under homomorphic images:

Example 2.5. *Let $R = \mathbb{Z}$ denote the ring of integers and consider $M = \mathbb{Z}$ as a $\mathbb{Z}$-module and submodule $N = 8\mathbb{Z}$ in M. Then M/N is not a reduced R-module.*

Proof. It is evident that M is a reduced R-module. Let $m = 4 + N \in M/N$ and $a = 2 \in R$. Then $ma = 0$. However $m = 4 + N = (2 + N)a \in (mR) \cap (M/N)a \neq 0$. So $(mR) \cap (M/N)a \neq 0$. □

Recall that a module M is called *cogenerated* by R if it is embedded in a direct product of copies of R. A module M is *faithful* if the only $a \in R$ such that $Ma = 0$ is $a = 0$.

Proposition 2.6. *The following conditions are equivalent:*
(1) *R is an α-reduced ring.*
(2) *Every cogenerated R-module is α-reduced.*
(3) *Every submodule of a free R-module is α-reduced.*
(4) *There exists a faithful α-reduced R-module.*

Proof. (1) $\Rightarrow$ (2) Let M be a cogenerated R-module. Then M is isomorphic to a direct product of copies of R. Any submodule of a direct product of copies of R is α-reduced R-module from (1) and Proposition 2.4. Hence M is α-reduced.
(2) $\Rightarrow$ (3) Clear since every submodule of a free R-module is cogenerated by R.
(3) $\Rightarrow$ (4) R itself as a right R-module is faithful free R-module. So it is α-reduced from (3).
(4) $\Rightarrow$ (1) Let M be a faithful α-reduced R-module. Assume that $rs = 0$ for some r, $s \in R$. To prove (1) we show that $rR \cap Rs = 0$, and $rs = 0$ if and only if $r\alpha(s) = 0$. So for any $m \in M$, we have $mrs = 0$. Then $mrR \cap Ms = 0$ by (4). Let $rr_1 = r_2 s \in rR \cap Rs$ for some r_1, $r_2 \in R$. Then $mrr_1 = mr_2 s \in mrR \cap Ms = 0$. Hence $mrr_1 = 0$ for all $m \in M$. Since M is faithful $rr_1 = 0$. Thus $rR \cap Rs = 0$. In the same way $rs = 0$ if and only if $mrs = 0$ for every $m \in M$. Since M is α-reduced, $mrs = 0$ if and only if $mr\alpha(s) = 0$ for every $m \in M$. Being M faithful implies that $mr\alpha(s) = 0$ for every $m \in M$ if and only if $r\alpha(s) = 0$. This completes the proof. □

Let $T(M) = \{m \in M \mid ma = 0 \text{ for some nonzero } a \in R\}$ be the set of all torsion elements of a module M.

Theorem 2.7. *Let R be a ring with no non-zero divisors of zero and M an α-reduced module. Then $T(M)$ is an α-reduced submodule of M.*

Proof. We first prove that $T(M)$ is a submodule of M. Let m_1, $m_2 \in T(M)$ and $r \in R$. We prove that $m_1 - m_2$ and $m_1 r$ belong to $T(M)$. There exist non-zero t_1, $t_2 \in R$ with $m_1 t_1 = 0$ and $m_2 t_2 = 0$. By Corollary 2.2, $m_1 R t_1 = 0$. In particular $m_1 t_2 t_1 = 0$ and $m_2 t_2 t_1 = 0$. Then $(m_1 - m_2) t_2 t_1 = 0$ and so $m_1 - m_2 \in T(M)$. Assume that $m_1 t_1 = 0$. Then $m_1 R t_1 = 0$. Hence $m_1 r \in T(M)$ for all $r \in R$. Since α-reduced modules are closed under submodules, $T(M)$ is also an α-reduced module. □

Theorem 2.8. *Let R be a reduced ring with no non-zero divisors of zero. Then M is an α-reduced module if and only if $T(M)$ is an α-reduced module.*

Proof. One way follows by Proposition 2.4. Conversely, assume that $T(M)$ is an α-reduced module. Let $0 \neq m \in M$ and $0 \neq a \in R$ with $ma = 0$. We prove $mR \cap Ma = 0$. Let $mr = m'a \in mR \cap Ma$. Multiplying by a from right we have $mra = m'a^2$. Since $m \in T(M)$ and $T(M)$ is α-reduced, by Corollary 2.2, $mRa = 0$. So $0 = mra = m'a^2$. Hence $m' \in T(M)$. Again by Corollary 2.2, $m'a^2 = 0$ implies $m'a = 0$. This completes the proof. □

Proposition 2.9. *A ring R is α-reduced if and only if every flat module M is α-reduced.*

Proof. Sufficiency is clear since the module R_R is an α-reduced module. Conversely, let M be a flat module over an α-reduced ring R and $0 \to K \to F \to M \to 0$ a short exact sequence with F free right R-module. Assume that R is an α-reduced ring. Then R_R is a flat module. By Proposition 2.4, F is an α-reduced module. We may write $M = F/K$ and any element $\overline{y} = y + K \in M$ for $y \in F$. Let $\overline{y}a = 0$, where $\overline{y} \in M$ and $a,b \in R$. We prove $(\overline{y}R) \cap (Ma) = 0$. Let $\overline{y}r = \overline{y_1}a \in (\overline{y}R) \cap (Ma)$ where $\overline{y_1} \in M$ and $r \in R$. Then $ya \in K$ and $yr - y_1 a \in K$. By hypothesis there exists a homomorphism $\theta : F \to K$ with $\theta(ya) = ya$ and $\theta(yr - y_1 a) = yr - y_1 a$. Set $u = \theta(y) - y$. Then $ua = 0$. Since F is an α-reduced R-module, $uR \cap Fa = 0$. From $\theta(yr - y_1 a) = yr - y_1 a$ we have $(\theta(y) - y)r = (\theta(y_1) - y_1)a \in uR \cap Fa = 0$. So $\theta(y)r = yr$, $\theta(y_1)a = y_1 a$. Since $\theta(y)r = yr \in K$, $\overline{y}r = 0$. Hence $(\overline{y}\ R) \cap (Ma) = 0$. □

A *regular element* of a ring R means a nonzero element which is not zero divisor. Let S be a multiplicatively closed subset of R consisting of regular central elements. We may localize R and M at S and we may seek when the localization $S^{-1}M_{S^{-1}R}$ is α-reduced. If $\alpha : R \to R$ is a homomorphism of the ring R, then $S^{-1}\alpha : S^{-1}R \to S^{-1}R$ defined by $S^{-1}\alpha(a/s) = \alpha(a)/s$ is a homomorphism of the ring $S^{-1}R$. Clearly this map extends α and we shall also denote this map by α.

Proposition 2.10. *Let S be a multiplicatively closed subset of R consisting of regular central elements. A module M_R is α-reduced if and only if $S^{-1}M_{S^{-1}R}$ is α-reduced.*

Proof. Assume that M_R is α-reduced and $(m/s)(a/t) = 0$ in $S^{-1}M$ where $m/s \in S^{-1}M$, $a/t \in S^{-1}R$. Hence $ma = 0$. By assumption $mR \cap Ma = 0$. To complete the proof it is enough to prove $(m/s)(S^{-1}R) \cap (S^{-1}M)(a/t) = 0$. Let $(m/s)(c/u) = (m'/s'^{-1}R) \cap (S^{-1}M)(a/t)$. Then $mcs'r = m'astu \in mR \cap Ma = 0$. So $(m/s)(c/u) = (m'/s')(a/t) = 0$. The rest of the proof is clear. □

Corollary 2.11. *For a module M, $M[x]_{R[x]}$ is α-reduced if and only if $M[x, x^{-1}]_{R[x,x^{-1}]}$ is α-reduced.*

Proof. Let $S = \{1, x, x^2, ...\}$. Then S is a multiplicatively closed subset of $R[x]$ consisting of regular central elements. Since $S^{-1}M[x] = M[x, x^{-1}]$ and $S^{-1}R[x] = R[x, x^{-1}]$, the result is clear from Proposition 2.10. □

Lemma 2.12. *Let M be an α-reduced module. For any $n \in \mathbb{N}$ and any permutation $\sigma \in S_n$, $ma_1 \dots a_n = 0$ if and only if $ma_{\sigma(1)} \dots a_{\sigma(n)} = 0$, where $m \in M$, for $i = 1, 2, \dots n$, $a_i \in R$.*

Proof. The necessity is clear. Conversely, suppose that $ma_{\sigma(1)} \dots a_{\sigma(n)} = 0$, where $m \in M, a_i \in R$ and any $\sigma \in S_n$. Since M is α-reduced, from Theorem 2.1 (3)(i) we have, for any $m \in M$ and a, $b \in R$, $mab = 0$ implies $mba = 0$. Hence for $n = 1$ and $n = 2$ the claim is evident. Let $n = 3$ and $ma_1a_2a_3 = 0$. We prove that the claim holds in this case also. $ma_1a_2a_3 = m(a_1)(a_2a_3) = 0$ implies $m(a_2a_3)(a_1) = 0$. And $(ma_2)(a_3)(a_1) = 0$ implies $(ma_2)(a_1)(a_3) = 0$. Therefore, our claim holds for $\sigma_1 = (123)$ and $\sigma_2 = (12)$. Any other element of S_3 is a composition of cycles σ_1 and σ_2, so the case $n = 3$ is completed. For $n > 3$ to complete the proof it is enough to note that $S_n = \langle (12), (12 \dots n) \rangle$ and to apply associativity of multiplication in R. □

Lemma 2.13. *Let M be an α-reduced module. Then the following are equivalent:*
(1) $ma_1a_2 \dots a_n = 0$ *where* $m \in M, a_i \in R$.
(2) $m\alpha^{i_1}(a_1)\alpha^{i_2}(a_2) \dots \alpha^{i_n}(a_n) = 0$ *for any* $i_1, \dots, i_n \in \mathbb{N}$.

Proof. Note that, it is sufficient to show $ma_1 \dots a_{i-1}a_ia_{i+1} \dots a_n = 0$ if and only if $ma_1 \dots a_{i-1}\alpha(a_i)a_{i+1} \dots a_n = 0$ for any i. Since M is α-reduced module, using Lemma 2.12, it can be easily proved. □

The ring R is called *semicommutative* if $ab = 0$ implies $aRb = 0$, for any $a, b \in R$. Buhphang and Rege in [3] studied basic properties of semicommutative modules. In [1], Agayev and Harmanci focused on the semicommutativity of subrings of matrix rings, where the ring R is called *α-semicommutative* if $ab = 0$ implies $aR\alpha(b) = 0$, for any $a, b \in R$. A module M is called *α-semicommutative* if , for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR\alpha(a) = 0$. The module M is called *semicommutative* if it is $\mathbf{1}-$semicommutative.

Theorem 2.14. *A module M is α-reduced if and only if M is α-semicommutative and rigid.*

Proof. It is a direct result of definitions, Theorem 2.1 and Corollary 2.3. □

Lemma 2.15. *Let M be an α-reduced module. For $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x;\alpha]]$ and $f(x) = \sum_{j=0}^{\infty} a_j x^j, g(x) = \sum_{k=0}^{\infty} b_k x^k \in R[[x;\alpha]]$, if $m(x)f(x)g(x) = 0$, then $m_i\alpha^i(a_j)\alpha^{i+j}(b_k) = 0 = m_i a_j b_k$ for all i, j and k.*

Corollary 2.16. *Let M be an α-reduced module. Let $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x;\alpha]]$ and $f_1(x) = \sum_{j_1=0}^{\infty} a_{j_1} x^{j_1}, f_2(x) = \sum_{j_2=0}^{\infty} a_{j_2} x^{j_2}$, $\dots$, $f_n(x) = \sum_{j_n=0}^{\infty} a_{j_n} x^{j_n} \in R[[x;\alpha]]$. If $m(x)f_1(x)f_2(x)...f_n(x) = 0$, then $m_i\alpha^i(a_{j_1})\alpha^{i+j_1}(a_{j_2})...\alpha^{i+j_1+\dots+j_{n-1}}(a_{j_n}) = 0 = m_i a_{j_1} a_{j_2}...a_{j_n}$ for all $i, j_1, ..., j_n$.*

Remark 2.17. *Let S be a subring of a ring R with $1_R \in S$, $\alpha \in End(R)$ such that $\alpha(S) \subseteq S$ and $M_S \subseteq L_R$. If L_R is α-reduced, then M_S is also α-reduced.*

If $\alpha \in End(R)$, then the map $R[[x]] \to R[[x]]$ defined by

$$\sum_{j=0}^{\infty} a_j x^j \to \sum_{j=0}^{\infty} \alpha(a_j) x^j$$

is an endomorphism of the ring $R[[x]]$. Clearly this map extends α and we shall also denote this map by α.

We now determine when the skew(Laurent) polynomial extension and the skew (Laurent) power series extension of a module M are α-reduced.

Theorem 2.18. *The following are equivalent:*
(1) M_R is α-reduced.
(2) $M[x]_{R[x]}$ is α-reduced.
(3) $M[[x]]_{R[[x]]}$ is α-reduced.
(4) $M[x,x^{-1}]_{R[x,x^{-1}]}$ is α-reduced.
(5) $M[[x,x^{-1}]]_{R[[x,x^{-1}]]}$ is α-reduced.

Proof. (1) $\Leftrightarrow$ (2) In [6, Theorem 1.6] take $\alpha = \mathbf{1}$ to see that M_R is reduced if and only if $M[x]_{R[x]}$ is reduced. To prove $m(x)f(x) = 0$ if and only if $m(x)\alpha(f(x)) = 0$, where $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{k} a_j x^j \in R[x]$ for second condition, we must show that $m_i a_j = 0$ if and only if $m_i\alpha(a_j) = 0$. The last equality follows from Lemma 2.13.
(2) $\Leftrightarrow$ (3) From Corollary 2.11.
(1) $\Rightarrow$ (4) Take $\alpha = \mathbf{1}$ in [6, Theorem 1.6] and use Lemma 2.13 for second condition.
(4) $\Rightarrow$ (1) From Remark 2.17.
(4) $\Leftrightarrow$ (5) Since elements of $S^{-1}M[[x]]$ are in the form $\frac{1}{x^s}\sum_{i=0}^{\infty} a_i x^i$, which are exactly the elements of $M[[x,x^{-1}]]$. Proposition 2.10 completes the proof. □

Note that, for $\alpha = \mathbf{1}$ in Theorem 2.18, the following are equivalent:
(1) M is reduced.
(2) $M[x]$ is reduced.
(3) $M[[x]]$ is reduced.

(4) $M[x, x^{-1}]$ is reduced.
(5) $M[[x, x^{-1}]]$ is reduced.

Corollary 2.19. *The following are equivalent:*
(1) M is α-reduced.
(2) $M[x;\alpha]_{R[x;\ \alpha]}$ is reduced.
(3) $M[[x;\alpha]]_{R[[x;\ \alpha]]}$ is reduced.
(4) $M[x]_{R[x]}$ is α-reduced.
(5) $M[[x]]_{R[[x]]}$ is α-reduced.
(6) $M[x, x^{-1}]_{R[x,x^{-1}]}$ is α-reduced.
(7) $M[[x, x^{-1}]]_{R[[x,x^{-1}]]}$ is α-reduced.
(8) $M[x, x^{-1};\alpha]_{R[x,x^{-1};\alpha]}$ is reduced, where $\alpha \in Aut(R)$.
(9) $M[[x, x^{-1};\alpha]]_{R[[x,x^{-1};\alpha]]}$ is reduced, where $\alpha \in Aut(R)$.

Proof. Follows by [6, Theorem 1.6] and Theorem 2.18. □

In [6], a module M is called *α-Armendariz* if

(1) for any $m \in M$ and $a \in R$, $ma = 0$ if and only if $m\alpha(a) = 0$,
(2) for any $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{s} a_j x^j \in R[x;\alpha]$, $m(x)f(x) = 0$ implies $m_i\alpha^i(a_j) = 0$ for all i and j.

And M is Armendariz if it is **1**-Armendariz. Following [5], M is called *α-skew Armendariz* for any $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{s} a_j x^j \in R[x;\alpha]$, $m(x)f(x) = 0$ implies $m_i\alpha^i(a_j) = 0$ for all i and j. Every α-Armendariz module is α-skew Armendariz. For some positive integer n and an endomorphism α of a ring R with $\alpha^n = \mathbf{1}$, it is proven that R is α-skew Armendariz if and only if $R[x]$ is α-skew Armendariz in [4] and [5], and it is generalized as for any module M over the ring R, M is α-skew Armendariz if and only if $M[x]$ is α-skew Armendariz module over $R[x]$ in [7].

Theorem 2.20. *Let M be an α-reduced module. Then M is α-skew Armendariz. In particular, if M is reduced , then M is Armendariz.*

Proof. Let M be an α-reduced module, $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{s} a_j x^j \in R[x;\alpha]$. If $m(x)f(x) = 0$, then $m_i\alpha^i(a_j) = 0$ for all i and j by Lemma 2.15. Hence M is α-skew Armendariz. In particular, if $\alpha = \mathbf{1}$, then M is Armendariz. □

ÖZET: R birimli bir halka ve α da R halkasının bir endomorfizması olsun. Bu çalışmada, bir indirgenmiş modülün kuvvet serisi genişlemeleri ile polinom genişlemeleri arasındaki ilişkiler incelenmiştir. Ayrıca R halkasının α-indirgenmiş olması için gerek ve yeter şartın her düz sağ R-modülün α-indirgenmiş olması; bir M modülü için $M[x]$ in α-indirgenmiş olması için gerek ve yeter şartın; $M[x, x^{-1}]$ in α-indirgenmiş olması gerektiği ispat edilmiştir.

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