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HOMOTHETIC MOTIONS AND BICOMPLEX NUMBERS

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Abstract. In this study, one of the concepts of conjugate which is defined [1] for bicomplex numbers is investigated. In this case, the metric, in four dimensional semi-Euclidean space E_2^4, has been defined by the help of the concept of the conjugate. We define a motion in E_2^4 with the help of the metric in bicomplex numbers. We show that the motions defined by a curve lying on a hypersurface M of E_2^4 are homothetic motions . Furthermore, it is shown that the motion defined by a regular curve of order r and derivations of the curve on the hypersurface M has only one acceleration centre of order (r-1) at every t- instant.

1. Introduction

In 2006, Dominic Rochon and S.Tremblay , presented a paper based on bicomplex quantum mechanics : II. The Hilbert Space [1, 2]. Bicomplex (hyperbolic) numbers are given in this paper from a number of different points of view of Hilbert Space for quantum mechanics.

In this study , a new operator similar to Hamilton operator [3] has been given for bicomplex numbers [4] , homothetic motion has been defined by the help of the components of the hyper surface and different theorems have been given . It is shown that this study can be repeated for bicomplex numbers,which is a homothetic motion in four-dimensional semi-Euclidean spaces and this homothetic motion satisfies all of the properties [5].

2. Bicomplex Numbers

Bicomplex numbers are defined by [1, 2, 6]

$$T = \{z_1 + z_2\ i_2 : z_1, z_2 \in C(i_1)\}$$

where the imaginary units i_1, i_2 and j are governed by the rules:

$$i_1^2 = i_2^2 = -1\ ,\quad j^2 = 1$$

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$$i_1.i_2 = i_2.i_1 = j \ : \ i_1.j = j.i_1 = -i_2 \ : \ i_2.j = j.i_2 = -i_1$$

where we define $C(i_k)=\{x + y\ i_k \ : i_k^2 = -1 \text{ and } x, y \in R\}$ for $k = 1, 2$. Hence it is easy to see that the multiplication of two bicomplex numbers is commutative . It is also convenient to write the set of bicomplex numbers as

$$T = \{w \mid w = w_1 + w_2\ i_1 + w_3\ i_2 + w_4\ j \quad \mid (w_1, w_2, w_3, w_4) \in R.\}$$

Complex conjugation plays an important role both for algebraic and geometric properties of complex numbers [1, 2].

$$\begin{aligned} \overline{w} &= \overline{z_1 + z_2 i_2} = z_1 - z_2 i_2 \\ &= w_1 + w_2\ i_1 - w_3\ i_2 - w_4\ j \end{aligned}$$

where $\overline{w}.w \ = w_1^2 - w_2^2 + w_3^2 - w_4^2 \ + \ 2i_2\ (w_1w_2 + w_3w_4)$ and $w_1w_2 + w_3w_4 = 0$ i.e

$$\overline{w}.w \ = w_1^2 - w_2^2 + w_3^2 - w_4^2 \ \in \ R$$

The system $\{T, \oplus, R, +, ., \circ, \otimes\}$ is a commutative algebra. It is referred as the bicomplex number algebra and shown with T, briefly one of the bases of this algebra is $\{1, i_1, i_2,\ j\}$ and the dimension is 4.

2.1. **Multiplication Operation.** The operation

$$\begin{aligned} \otimes : \quad T \times T &\to \quad T \\ (u, w) &\to \quad u \otimes w \quad = w \otimes u \end{aligned}$$

is defined with the following multiplication

$$\begin{aligned} u \otimes w &= (u_1 + i_1u_2 + i_2u_3 + ju_4) \otimes (w_1 + i_1w_2 + i_2w_3 + jw_4) \qquad (2) \\ &= (u_1w_1 - u_2w_2 - u_3w_3 + u_4w_4) + i_1(u_1w_2 + u_2w_1 - u_3w_4 - u_4w_3) \\ &\quad + i_2(u_1w_3 - u_2w_4 + u_3w_1 - u_4w_2) + j(u_1w_4 + u_2w_3 + u_3w_2 + u_4w_1) \end{aligned}$$

It is possible to give the production in T similar to the Hamilton operators which has been given in [3]. Because it is not a quaternion commutative matrix, there are two different matrixes for each of the right and left-multiplications. However, here only one matrix is obtained. Because it is similar to Hamilton operators. (for Hamilton operators see [3, 5]). If $w = w_1 + w_2\ i_1 + w_3\ i_2 + w_4\ j$ is a bicomplex number ,then $N^+ = N^- = N$ is defined as

$$N(w) = \begin{bmatrix} w_1 & -w_2 & -w_3 & w_4 \\ w_2 & w_1 & -w_4 & -w_3 \\ w_3 & -w_4 & w_1 & -w_2 \\ w_4 & w_3 & w_2 & w_1 \end{bmatrix}$$

If $w = z_1 + z_2\ i_2$ then

$$N(w) = \begin{bmatrix} N(z_1) & -N(z_2) \\ N(z_2) & N(z_1) \end{bmatrix}$$

Using the definition of N, the multiplication of two bicomplex numbers x and y is given by

$$w \otimes u = N(w).u \qquad : u, w \in T$$

and

$$\det N(w) = \left[w_1^2 - w_2^2 + w_3^2 - w_4^2 + \ 2(w_1 w_2 + w_3 w_4)\right]^2$$

3. Homothetic Motions at E_2^4

Let,

$$\begin{array}{rcl} M & = & \{w = (w_1, w_2, w_3, w_4) \mid w_1 w_2 + w_3 w_4 = 0\} \text{ be a hyper surface} \\ S_2^3 & = & \left\{w = (w_1, w_2, w_3, w_4) \mid w_1^2 - w_2^2 + w_3^2 - w_4^2 = 1\right\} \text{ be a unit sphere ,} \\ K & = & \left\{w = (w_1, w_2, w_3, w_4) \mid w_1^2 - w_2^2 + w_3^2 - w_4^2 = 0\right\} \text{ be a null cone in } E_2^4. \end{array}$$

Let us consider the following curve:

$$\alpha \quad : \quad I \subset R \ \rightarrow \ M \subset E_2^4 \ \text{ defined by}$$

$$\alpha(t) \quad = \quad [w_1(t), w_2(t), w_3(t), w_4(t)] \quad \text{for every } t \in I.$$

We suppose that $\alpha(t)$ is a differentiable curve of order r. The operator B, corresponding to $\alpha(t)$ is defined by

$$B = N(w) = \begin{bmatrix} w_1 & -w_2 & -w_3 & w_4 \\ w_2 & w_1 & -w_4 & -w_3 \\ w_3 & -w_4 & w_1 & -w_2 \\ w_4 & w_3 & w_2 & w_1 \end{bmatrix} \tag{3}$$

Let $\alpha(t)$ be a unit velocity curve. The matrix can be represent as

$$B = h \begin{bmatrix} \frac{w_1}{h} & -\frac{w_2}{h} & -\frac{w_3}{h} & \frac{w_4}{h} \\ \frac{w_2}{h} & \frac{w_1}{h} & -\frac{w_4}{h} & -\frac{w_3}{h} \\ \frac{w_3}{h} & -\frac{w_4}{h} & \frac{w_1}{h} & -\frac{w_2}{h} \\ \frac{w_4}{h} & \frac{w_3}{h} & \frac{w_2}{h} & \frac{w_1}{h} \end{bmatrix} \tag{4}$$

$$B = h.A$$

where,

$$h : I \subset R \ \rightarrow \ R \ , t \rightarrow h(t) = \sqrt{|w_1^2 - w_2^2 + w_3^2 - w_4^2|} \text{ and } \ \alpha(t) \neq 0, \alpha(t) \notin K.$$

Theorem 3.1. *Let $\alpha(t) \ \in \ S_2^3 \cap M$. In equation $B = hA$, the matrix B is $SO(4,2)$.*

Proof. If $\alpha(t) \in S_2^3$,where $w_1^2 - w_2^2 + w_3^2 - w_4^2 = 1$, using equation (4) , in equation $B = hA$, we find $B^{-1} = \varepsilon B \varepsilon$ and $\det A = 1$. □

Theorem 3.2. *In equation $B = hA$,the matrix A in E_2^4 is semiorthogonal matrix.*

Proof. Let $\alpha(t) \notin K$, and $w_1(t)w_2(t) + w_3(t)w_4(t) = 0$. In equation $B = hA$, the matrix A has been shown by $A^T\varepsilon\ A{=}\varepsilon$. Let the signature matrix , given in [7], be

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

where ,the matrix A is semiorthogonal matrix and $\det A = 1$. □

4. A Motion with One Parameter

Let the fixed space and the motinal space be ,respectively, R_0 and R .In this case, one- parametric motion of R_0 with respect to R will be denoted by R_0 / R. This motion can be expressed by

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} hA & C \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} X_0 \\ 1 \end{bmatrix} \tag{5}$$

where, X and X_0 represent position vectors of any point ,respectively, in R and R_0, and C represent any translation vector.

Theorem 4.1. *The motion defined by the equation in (5) in semi-Euclidean space E_2^4 is a homothetic motion.*

Proof. The matrix determined by the equation in (5), can be written as B=hA,where, due to $A \in SO(4,2)$, this matrix determined is a motion with one parameter. □

Theorem 4.2. *Let $\alpha(t)$ be a unit velocity curve and $\dot{\alpha}(t) \in M$ then the derivation operator $\dot{B}$ of B=hA is semiorthogonal matrix in E_2^4.*

Proof. Since $\alpha(t)$ is unit velocity curve,

$$\dot{w}_1^2 - \dot{w}_2^2 + \dot{w}_3^2 - \dot{w}_4^2 = 1$$

and $\dot{\alpha}(t) \in M$, then $\dot{w}_1\dot{w}_2 + \dot{w}_3\dot{w}_4 = 0$. Thus $\dot{B}\varepsilon \dot{B}^T = \dot{B}^T \varepsilon \dot{B}$ and $\det \dot{B} = 1$. □

5. Pole Points and Pole Curves of the Motion

To find the pole point ,we have to solve the equation

$$\dot{B}X + \dot{C} = 0. \tag{6}$$

Any solution of equation (6) is a pole point of the motion at that instant in R_0. Because, by Therom 4, we have $\det \dot{B} = 1$. Hence the equation (6) has only one solution , i.e. $X = (-\dot{B}^{-1})(\dot{C})$ at every t-instant. In this case the following theorem can be given.

Theorem 5.1. *If $\alpha(t)$ is a unit velocity curve and $\dot{\alpha}(t) \in M$,then the pole point corresponding to each t-instant in R_0 is the rotation by $\dot{B}$ of the speed vector $\dot{C}$ of the translation vector at that moment.*

Proof. Since the matrix $\dot{B}$ is semiorthogonal, then the matrix $\dot{B}^T$ is semiorthogonal , too. Thus it makes a rotation. □

Theorem 5.2. *Only the pole point corresponding to each t-instant has at the homothetic motion which is defined by the equation in (6) through the space curve in E_2^4.*

Proof. Since equation (6) has only one solution at every t − instant,the proof is obvious. □

6. Accelaration Centres of Order $(r-1)$ of a Motion

Definition 6.1. The set of the zeros of sliding acceleration of order r is called the acceleration centre of order (r-1).

By the above definition ,we have to find the solutions of the equation

$$B^{(r)}X + C^{(r)} = 0 \tag{7}$$

where

$$B^{(r)} = \frac{d^r B}{dt^r} \quad \texttt{and} \quad C^{(r)} = \frac{d^r C}{dt^r}.$$

Let $\alpha(t)$ be a regular curve of order r and $\alpha^{(r)}(t) \in M$.Then we have

$$w_1^{(r)} w_2^{(r)} + w_3^{(r)} w_4^{(r)} = 0.$$

Thus,

$$(w_1^{(r)})^2 - (w_2^{(r)})^2 + (w_3^{(r)})^2 - (w_4^{(r)})^2 \neq 0 \quad , \; w_i^{(r)} = \frac{d^r w_i}{dt^r}.$$

Also,we have

$$\det \ B^{(r)} = \left[(w_1^{(r)})^2 - (w_2^{(r)})^2 + (w_3^{(r)})^2 - (w_4^{(r)})^2 \right]^2 .$$

Then $\det B^{(r)} \neq 0$. Therefore the matrix $B^{(r)}$ has an inverse and by the equation in (7), the acceleration centre of order $(r-1)$ at every t-instant, is

$$X = \left[B^{(r)} \right]^{-1} \left[-C^{(r)} \right].$$

Example 6.2. Let $\alpha{:}I \subset R \ \rightarrow M \subset E_2^4$ be a curve given by

$$t \rightarrow \alpha(t) = \frac{1}{\sqrt{2}}(cht, sht, cht, -sht).$$

Note that $\alpha(t) \in S_2^3$ and since $\left\| \dot{\alpha}(t) \right\| = 1$,then $\alpha(t)$ is a unit velocity curve.Moreover, $\dot{\alpha}(t) \in M, \ddot{\alpha}(t) \in M$,...,$\alpha^{(r)}(t) \in M$. Thus $\alpha(t)$ satisfies all conditions of the above theorems.

ÖZET: Bu çalışmada, bikompleks sayılar için [1] de tanımlanan eşlenik tanımlarından bir tanesini ele aldık. Bu eşlenik yardımıyla , E_2^4 semi - Öklidiyen uzayda bir metrik tanımlandı. Bu metriği kullanarak E_2^4 de bir hareket tanımladık. E_2^4 de bir M hiperyüzeyi üzerindeki bir eğri yardımıyla tanımlanan hareketin homotetik hareket olduğunu gösterdik. Ayrıca, M hiperyüzeyi üzerinde r. mertebeden regüler olan bir eğri yardımıyla tanımlanan hareketin her t anında (r-1) inci mertebeden bir tek ivme merkezinin olduğu gösterildi.

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