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WEIERSTRASS REPRESENTATION FOR MINIMAL IMMERSIONS INTO DAMEK-RICCI SPACES

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Abstract. We give a setting for constructing Weierstrass representation formulas for simply connected minimal surfaces into four-dimensional Damek-Ricci spaces $\mathbb{S}_4$. We derive Weierstrass representations, and establish the generating equations for minmal surfaces into $\mathbb{S}_4$.

1. Introduction

The Weierstrass representation of minimal surfaces is a very important notion in mathematics and physics for its applications. They are for instance, surface waves, propagation of flame fronts, growth of crystals, deformation of menbranes, dynamics of vortex sheets, many problems of hydrodynamics connected with motion of boundaries between region of differing densities and viscosities (see[6]). The classical Weierstrass representation for minimal surfaces in $\mathbb{R}^3$ with its generalizations to $\mathbb{R}^n$ has been also proved to be an extremely useful tool for the study of the minimal surfaces in those spaces (see for exemple [1, 2]). Weierstrass representation for minimal surfaces into Hyperbolic space have been obtained by Kokubu [4].
In [5], Mercuri, Montaldo and Piu described a method to obtain Weierstrass representation type formulas for simply connected immersed minimal surfaces into three-dimensional Heisenberg groups $\mathbb{H}_3$ and into the product space $\mathbb{H}^2 \times \mathbb{R}$ of the hyperbolic plane with the real line. Later, Turhan and Köpinar gave in [8] Weierstrass-type representation formulas for minimal surfaces into $\mathbb{H}_3 \times \mathbb{S}^1$.
Damek-Ricci spaces are Lie groups (connected and simply connected) whose Lie algebras are semi-direct sums of Heisenberg algebras with one dimensional vectors spaces, endowed with left invariant metrics defined by scalar products [3]. In fact, Damek-Ricci spaces are semi-direct products of Heisenberg groups and one-dimensional Lie groups, and can be considered as certain solvable Lie groups equipped with left-invariant metrics.
In this paper, we applied the general setting on Damek-Ricci spaces and described a method to derive Weierstrass-type representation formulas for simply connected minimal surfaces into four-dimensional Damek-Ricci spaces.

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2. Preliminaries

Let (M^n, g) be an n-dimensional Riemannian manifold, $\Sigma \subset M$ be a Riemannian surface and $f : \sum \longrightarrow M$ be a minimal conformal immersion. The pull-back bundle $f^\star(TM)$ is endowed with a metric and compatible connection ∇, the pull-back connection induced by the Riemannian metric, and with the Levi-Civita connection of M.
Let us consider the complexified bundle $\mathbb{E} = f^\star(TM) \otimes \mathbb{C}$.
Let (u, v) be a local coordinates on Σ, $z = u + iv$ the local conformal complex parameter and $(x_1, ..., x_n)$ be a system of local coordinates in a neighborhood U of M such that $U \cap f(\Sigma) \neq \emptyset$. The pull-back connection extends to a complex connection on $\mathbb{E}$, Hermitian with respect to $\langle ., . \rangle$ and it is well known that $\mathbb{E}$ has a unique holomorphic structure such that a section $\Upsilon : \Sigma \longrightarrow \mathbb{E}$ is holomorphic if and only if

$$\nabla_{\frac{\partial}{\partial \bar{z}}} \Upsilon = 0. \tag{2.1}$$

In the sequel we will consider the section

$$\Upsilon = f_z = \frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial u} - i\frac{\partial f}{\partial v}).$$

The induced metric on Σ is

$$ds^2 = \lambda^2(du^2 + dv^2) = \lambda^2|dz|^2.$$

The Beltrami-Laplace operator on M, with respect to the induced metric is given by

$$\Delta = \lambda^2(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}).$$

We recall that $f : \Sigma \longrightarrow M$ is harmonic if and only if

$$\tau(f) = trace \nabla df = 0,$$

where $\tau(f)$ is the tension field of f.
Let us consider the local decomposition of Υ :

$$\Upsilon = \sum_{j=1}^{n} \Upsilon_j \frac{\partial}{\partial x_j}$$

for complex-valued function Υ_j defined on Ω. Then tension field of f can be written as:

$$\tau(f) \quad = \quad 4\lambda^{-2} \sum_i \{\frac{\partial \Upsilon_i}{\partial \bar{z}} + \Gamma^i_{jk} \bar{\Upsilon}_j \Upsilon_k\} \frac{\partial}{\partial x_i}$$

where Γ^i_{jk} are the Christoffel symbol of M. We have

$$\tau(f) = 4\lambda^{-2}(\nabla_{\frac{\partial}{\partial \bar{z}}} \Upsilon).$$

Thus f is harmonic if and if only if Υ is a holomorphic section on $\mathbb{E}$. Or equivalently, Υ is a holomorphic section on $\mathbb{E}$ if and only if

$$\frac{\partial \Upsilon_i}{\partial \bar{z}} + \sum_{k,j} \Gamma^i_{jk} \bar{\Upsilon}_j \Upsilon_k = 0, \quad i = 1, 2, ..., n. \tag{2.2}$$

By considering (2.2) as a system of integral differential equations in the Υ_i, where Γ's are computed in $f_i = 2\Re e \int_{z_0}^{z} \Upsilon_i dz$, it can be written as:

$$\frac{\partial \Upsilon_i}{\partial \bar{z}} + 2\sum_{j>k} \Gamma^i_{jk} \Re e(\bar{\Upsilon}_j \Upsilon_k) + \sum_j \Gamma^i_{jj} |\Upsilon_j|^2 = 0, \quad i = 1, ..., n.$$

This implies that $\frac{\partial \Upsilon_i}{\partial \bar{z}} \in \mathbb{R}$, and ensures that (locally) the 1-forms $\Upsilon_i dz$ don't have real periods as it has been mentioned in [5]. Therefore we have the following:

Proposition 2.1. [7] *Let (M, g) be a riemannian manifold and $(x_1, ..., x_n)$ local coordinates. Let $\Upsilon_j, \quad j = 1, ..., n$, be complex-valued functions in an open simply connected domain $\Omega \subset \mathbb{C}$ which are solutions of* (2.2). *Then the map*

$$f_j(u, v) = 2\Re e\left(\int_{z_0}^{z} \Upsilon_j dz\right) \tag{2.3}$$

is well defined and defines a minimal conformal immersion if and only if the following conditions are satisfied:

(1) $\sum_{j,k=1}^{n} g_{jk} \Upsilon_j \bar{\Upsilon}_k \neq 0$
(2) $\sum_{j,k=1}^{n} g_{jk} \Upsilon_j \Upsilon_k = 0$.

For more details see [8] or [5].
It has been proved in [5] that if M is a Lie group then the system of partial differential equations (2.2) is reduced to a system of partial differential equations with constant coefficients.

3. Minimal surfaces into 4-dimensional Damek-Ricci spaces

Let $\mathfrak{v}_m$ and $\mathfrak{z}_n$ be real inner product vector spaces of dimensions m and n, respectively, and $\beta : \mathfrak{v}_m \times \mathfrak{v}_m \longrightarrow \mathfrak{z}_n$ a skew-symetric bilinear map. We endow the direct sum $\mathfrak{h}_{m+n} = \mathfrak{v}_m \oplus \mathfrak{z}_n$ with an inner product $\langle ., . \rangle_{\mathfrak{h}_{m+n}}$ such that $\mathfrak{v}_m$ and $\mathfrak{z}_n$ are perpendicular and define an $\mathbb{R}$-algebra homomorphism

$$J : \mathfrak{z}_n \longrightarrow End(\mathfrak{v}_n), \quad Z \mapsto J_Z$$

by

$$\langle J_Z U, V \rangle_{\mathfrak{h}_{m+n}} = \langle \beta(U, V), Z \rangle_{\mathfrak{h}_{m+n}}, \ \forall U, V \in \mathfrak{v}_m \ , \ Z \in \mathfrak{z}_n.$$

We define a lie algebra structure on $\mathfrak{h}_{m+n}$ by

$$[U + X, V + Y]_{\mathfrak{h}_{m+n}} := \beta(U, V) \ , \ \forall U, V \in \mathfrak{v}_m \ , \ X, Y \in \mathfrak{z}_n.$$

The Lie algebra $\mathfrak{h}_{m+n}$ is said to be a *generalized Heisenberg algebra* if

$$J_Z^2 = -\langle Z, Z \rangle_{\mathfrak{h}_{m+n}} id_{\mathfrak{v}_m}, \forall Z \in \mathfrak{z}_n.$$

The associated simply connected Lie group $\mathbb{H}_{m+n}$, endowed with the induced left-invariant Riemannian metric g, is called a *generalized Heisenberg group.* We define a new vector space

$$\mathfrak{s}_{m+n+1} := \mathfrak{h}_{m+n} \oplus \mathfrak{a}$$

as the direct sum of $\mathfrak{h}_{m+n}$ and $\mathfrak{a}$. A vector in $\mathfrak{s}_{m+n+1}$ can be writtten in a unique way as $V + Y + sA$ for some $V \in \mathfrak{v}_m$, $Y \in \mathfrak{z}_n$, $s \in \mathbb{R}$ and A a non-zero vector in $\mathfrak{a}$. We define an inner product $\langle ., . \rangle$ and a Lie bracket $[., .]$ on $\mathfrak{s}_{m+n+1}$ by

$$\langle U + X + rA, V + Y + sA \rangle := \langle U + X, V + Y \rangle_{\mathfrak{h}_{m+n}} + rs$$

and

$$[U+X+rA, V+Y+sA] := [U,V]_{\mathfrak{h}_{m+n}} + \frac{1}{2}rV - \frac{1}{2}sU + rY - sX.$$

In this way $\mathfrak{s}_{m+n+1}$ becomes a Lie algebra with an inner product. The associated simply connected Lie group $\mathbb{S}_{m+n+1}$, equiped with the induced left-invariant metric, is called *Damek-Ricci space*.
The Levi-Civita connection ∇ of the Damek-Ricci space $\mathbb{S}_{m+n+1}$ is given by:

$$\begin{aligned} \nabla_{V+Y+sA}(U+X+rA) &= -\frac{1}{2}J_Y U - \frac{1}{2}J_X V - \frac{1}{2}rV - \frac{1}{2}[U,V] \\ &- rY + \frac{1}{2}\langle U,V\rangle A + \langle X,Y\rangle A. \end{aligned} \tag{3.1}$$

We have:
$\mathfrak{s}_{m+n+1} = \mathfrak{b}_m \oplus \mathfrak{z}_n \oplus \mathfrak{a}$, $\mathfrak{b}_m$ has dimension m , $\mathfrak{z}_n$ has dimension n and $\mathfrak{a}$ has dimension 1.
$[\mathfrak{b}_m;\mathfrak{b}_m] = \mathfrak{z}_m$, $[\mathfrak{b}_m;\mathfrak{a}] = \frac{1}{2}\mathfrak{a}$, $[\mathfrak{z}_n;\mathfrak{a}] = \mathfrak{a}$ and others are zeros.
The Lie group $\mathbb{S}_{m+n+1}$ is a semi-direct product $\mathbb{H}_{m+n} \times_F \mathbb{R}$, with

$$F : \mathbb{R} \longrightarrow Aut(\mathbb{H}_{m+n})$$
$$s \mapsto F_s$$

defined by

$$F_s(exp_{\mathfrak{h}_{m+n}}(V+Y)) = exp_{\mathfrak{h}_{m+n}}(e^{\frac{s}{2}}V + e^s Y), \forall \quad V+Y \in \mathfrak{s}_{m+n+1},$$

where $exp_{\mathfrak{h}_{m+n}}$ is the Lie exponential of $\mathbb{H}_{m+n}$ (see [3]).
Since the Lie exponential map $exp_{\mathfrak{s}_{m+n+1}} : \mathfrak{s}_{m+n+1} \longrightarrow \mathbb{S}_{m+n+1}$ of $\mathbb{S}_{m+n+1}$ is a diffeomorphism, it induces global coordinates on $\mathbb{S}_{m+n+1}$.
Let $\mathbb{S}_4$ be a four-dimensinal Damek-Ricci space and x, y, z, t global coordinates on $\mathbb{S}_4$. The left invariant Riemannian metric g on $\mathbb{S}_4$ is given by

$$g = e^{-t}dx^2 + e^{-t}dy^2 + e^{-2t}(dz + \frac{c}{2}ydx - \frac{c}{2}xdy)^2 + dt^2$$

where $c \in \mathbb{R}$. The Lie algebra $\mathfrak{s}_4$ of $\mathbb{S}_4$ has an orthonormal basis

$$e_1 = e^{\frac{t}{2}}(\frac{\partial}{\partial x} - \frac{c}{2}y\frac{\partial}{\partial z}), \quad e_2 = e^{\frac{t}{2}}(\frac{\partial}{\partial y} + \frac{c}{2}x\frac{\partial}{\partial z}), \quad e_3 = e^t\frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}. \tag{3.2}$$

The corresponding Lie brackets are

$$[e_1,e_2] = ce_3; \quad [e_1,e_4] = -\frac{1}{2}e_1; \quad [e_2,e_4] = -\frac{1}{2}e_2,$$
$$[e_3,e_4] = -e_3; [e_1,e_3] = [e_2,e_3] = 0.$$

From (3.1) the Levi-Civita connection w,r,t this orthonormal basis is given by:

$$\nabla_{e_1}e_1 = 0; \quad \nabla_{e_1}e_2 = \frac{c}{2}e_3; \quad \nabla_{e_1}e_3 = -\frac{c}{2}e_2; \quad \nabla_{e_1}e_4 = -\frac{1}{2}e_1; \quad \nabla_{e_2}e_1 = -\frac{c}{2}e_3;$$
$$\nabla_{e_2}e_2 = 0; \quad \nabla_{e_2}e_3 = \frac{c}{2}e_1; \quad \nabla_{e_2}e_4 = -\frac{1}{2}e_2; \quad \nabla_{e_3}e_1 = -\frac{c}{2}e_2; \quad \nabla_{e_3}e_2 = \frac{c}{2}e_1;$$
$$\nabla_{e_3}e_3 = 0; \quad \nabla_{e_3}e_4 = -e_3; \quad \nabla_{e_4}e_1 = 0; \quad \nabla_{e_4}e_2 = 0; \quad \nabla_{e_4}e_3 = 0; \quad \nabla_{e_4}e_4 = 0.$$

The non zero Christoffel coefficientsL_{ij}^k are then given by:

$$L_{12}^3 = c, \quad L_{13}^2 = -c, \quad L_{14}^1 = -1, \quad L_{21}^3 = -c$$
$$L_{23}^1 = c, \quad L_{24}^2 = -1, \quad L_{31}^2 = -c, \quad L_{32}^1 = c, \quad and \quad L_{34}^3 = -2. \tag{3.3}$$

Let us put

$$\Upsilon = \sum_{k=1}^{4} \Upsilon_k \frac{\partial}{\partial x_k} = \sum_{k=1}^{4} \chi_k e_k. \tag{3.4}$$

For some complex functions Υ_k and χ_k. There exists an invertible matrix $A = (A_{ij})$, with component functions $A_{ij} : f(\Omega) \bigcap U \longrightarrow \mathbb{R}, i, j = 1, 2, 3, 4$, such that

$$\Upsilon_i = \sum_{i=1}^{4} A_{ij} \chi_j,$$

where

$$A = \begin{pmatrix} e^{\frac{t}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{t}{2}} & 0 & 0 \\ -\frac{c}{2} e^{\frac{t}{2}} y & \frac{c}{2} e^{\frac{t}{2}} x & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It follows that, the section Υ is holomorphic if and only if

$$\frac{\partial \chi_i}{\partial \bar{z}} + \frac{1}{2} \sum_{j,k} L^i_{jk} \bar{\chi_j} \chi_k = 0, \quad i = 1, 2, 3, 4. \tag{3.5}$$

Remark 3.1. Compared to (2.2), equations (3.5) have the advantage to be partial differential equations with constant coefficients, which allows us to have explicite solutions.

We have:

Lemma 3.1. *χ satisfies the equation (3.5) if and only if*

$$\frac{\partial \chi_1}{\partial \bar{z}} - \frac{1}{2} \bar{\chi_1} \chi_4 + c\Re e(\bar{\chi_2} \chi_3) = 0 \quad ; \tag{3.6}$$

$$\frac{\partial \chi_2}{\partial \bar{z}} - \frac{1}{2} \bar{\chi_2} \chi_4 - c\Re e(\bar{\chi_1} \chi_3) = 0 \quad ; \tag{3.7}$$

$$\frac{\partial \chi_3}{\partial \bar{z}} - \bar{\chi_3} \chi_4 + ic\Im m(\bar{\chi_1} \chi_2) = 0 \quad ; \tag{3.8}$$

$$\frac{\partial \chi_4}{\partial \bar{z}} = 0 \quad . \tag{3.9}$$

Proof. substitute (3.3) into (3.5) gives the result. □

Therefore:

Theorem 3.1. *Let $\chi_i, i = 1, 2, 3, 4,$ be complex-valued functions defined in a simply connected domain $\Omega \subset \mathbb{C}$ such that:*
•$|\chi_1|^2 + |\chi_2|^2 + |\chi_3|^2 + |\chi_4|^2 \neq 0$;
•$\chi_1^2 + \chi_2^2 + \chi_3^2 + \chi_4^2 = 0$;
•χ_i *are solutions of* $(3.6) - (3.9)$.
Then the map $f : \Omega \longrightarrow \mathbb{S}_4$, defined by

$$\begin{aligned} f_1(u, v) &= 2\Re e \int_{z_0}^{z} e^{\frac{f_4}{2}} \chi_1 dz, \\ f_2(u, v) &= 2\Re e \int_{z_0}^{z} e^{\frac{f_4}{2}} \chi_2 dz, \end{aligned}$$

$$\begin{aligned} f_3(u,v) &= 2\Re e(\int_{z_0}^{z} e^{\frac{f_4}{2}}\frac{c}{2}(-\chi_1 f_2+\chi_2 f_1)+e^{f_4}\chi_3 dz), \\ f_4(u,v) &= 2\Re e\int_{z_O}^{z}\chi_4 dz \end{aligned}$$

is a conformal minimal immersion.

Proof. By substituting (3.3) in (3.4), we obtain

$$\Upsilon_1=\chi_1,\quad \Upsilon_2=\chi_2,\quad \Upsilon_3=-\frac{c}{2}e^{\frac{t}{2}}y\chi_1+\frac{c}{2}e^{\frac{t}{2}}x\chi_2+e^t\chi_3,\quad \Upsilon_4=\chi_4.$$

From 2.3, we have the result. Using proposition 2.1, $f:\Omega\longrightarrow\mathbb{S}_4$ is a conformal minimal immersion. □

Example 3.1. If $\chi_4=0$ then f_4 is constant and this implies that $\chi_1^2+\chi_2^2+\chi_3^2=0$. We can give a simple geometric description of almost solutions of the equation $\chi_1^2+\chi_2^2+\chi_3^2=0$, which suggests of two complex functions

$$G=\sqrt{\frac{1}{2}(\chi_1-i\chi_2)}, H=\sqrt{-\frac{1}{2}(\chi_1+i\chi_2)} \tag{3.10}$$

By using the same arguments as in [5], we get the following Weierstrass Representations:
•For the vertical plane:

$$\begin{aligned} f_1(u,v) &= 2e^{\frac{f_4}{2}}\Re e((1-e^{2i\theta})\tilde{G}), \\ f_2(u,v) &= 2e^{\frac{f_4}{2}}\Re e((1+e^{2i\theta})\tilde{G}), \\ f_3(u,v) &= 2\Re e(\int_{z_0}^{z} e^{\frac{f_4}{2}}\frac{c}{2}(-(1-e^{2i\theta})G^2 f_2+i(1+e^{2i\theta})G^2 f_1)+2e^{f_4}e^{i\theta}G^2 dz), \\ f_4(u,v) &= constant \end{aligned}$$

•For the saddle-type surface:

$$\begin{aligned} f_1(u,v) &= -4au \\ f_2(u,v) &= 4Q(v), \\ f_3(u,v) &= -8acuQ(v), \\ f_4(u,v) &= constant, \end{aligned}$$

•For the Helicoids:

$$\begin{aligned} f_1(u,v) &= \rho(u)cos(v) \\ f_2(u,v) &= \rho(u)sin(v) \\ f_3(u,v) &= (av+b), a,b\in\mathbb{R} \\ f_4(u,v) &= constant, \end{aligned}$$

Example 3.2. Let $f:\mathbb{C}\longrightarrow\mathbb{S}_4$ be a minimal immersion into $\mathbb{S}_4$for which χ_1 and χ_2 are purely imaginary functions, and χ_3 is a real function. Then equations (3.6), (3.7), (3.8) and (3.9) become

$$\frac{\partial\chi_1}{\partial\bar{z}}+\frac{1}{2}\chi_1\chi_4 = 0 \quad ; \tag{3.11}$$

$$\frac{\partial\chi_2}{\partial\bar{z}}+\frac{1}{2}\chi_2\chi_4 = 0 \quad ; \tag{3.12}$$

$$\frac{\partial\chi_3}{\partial\bar{z}}-\chi_3\chi_4 = 0 \quad ; \tag{3.13}$$

$$\frac{\partial\chi_4}{\partial\bar{z}} = 0 \quad . \tag{3.14}$$

The solutions of these equations are:

$$\chi_1 = i\alpha e^{-\frac{1}{2}Q(z)\bar{z}}\ ;\ \chi_2 = i\beta e^{-\frac{1}{2}Q(z)\bar{z}}\ ;\ \chi_3 = \gamma e^{Q(z)\bar{z}}\ ;\ \chi_4 = Q(z),$$

with $\alpha, \beta, \gamma \in \mathbb{R}$ and $Q(z)$ a holomorphic function. Since χ_1 and χ_2 are purely imaginary functions, and χ_3 is a real function, and $\chi_1^2+\chi_2^2+\chi_3^2+\chi_4^2 = 0$ imply that $Q(z) = 0$

Then a Weierstrass representation of $f : \Omega \longrightarrow \mathbb{S}_4$ is given by

$$\begin{aligned} f_1(u,v) &= -2\alpha\kappa v, \\ f_2(u,v) &= -2\beta\kappa v, \\ f_3(u,v) &= 2\gamma\kappa u, \\ f_4(u,v) &= \kappa',\ \kappa = e^{\frac{\kappa'}{2}},\ \kappa' \in \mathbb{R}. \end{aligned}$$

It's a particular case of the saddle-type surface.

References

[1] Barbosa, J.L.M, Colares, A.G. Minimal surfaces in $\mathbb{R}^3$, Lecture notes in Mathematics,1195, Springer-Verlag, berlin, 1986.

[2] Berdinski, D. A. and Taimanov, I. A., Surfaces in three dimensinal Lie groups, Sibirsk. Mat. Zh., 46(2005), No 6, , 1005-1019.

[3] Berndt J., Tricerri, F. and Vanhecke , L., Generalized Heisenberg Groups and Damek-Ricci Harmonic Spaces, Springer-Verlag Berlin Heidelberg 1995.

[4] Kokubu, M., Weierstrass representation for minimal surfaces in hyperbolic spaces, Tohoku Math. J. 49(1997), 367-377.

[5] Mercuri, F., Montaldo, S. and Piu, P., A Weierstrass representation formula for minimal surfaces $\mathbb{H}_3$ and $\mathbb{H}^2 \times \mathbb{R}$, Acta Mathematica Sinica, English series Nov. 2006, vol. 22, No 6, pp. 1603-1612.

[6] Nelson, D., Piran, T. and Weinberg, S., Statiscal mechanics of membranes and surfaces, World Scientific, Singapore, 1989.

[7] Ossermann, R., A survey of minimal surfaces, Van Nostrand, 1969.

[8] Turhan, E. and Körpinar, T., The Weierstrass representation for minimal immersions in the lie group $\mathbb{H}_3 \times \mathbb{S}^1$, J. Adv. Maths. Studies Vol. 3 (2010), No. 1, 117-124.

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