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VARIATION OF PERIMETER MEASURE IN SUB-RIEMANNIAN GEOMETRY

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Abstract. We derive a formula for the first variation of horizontal perimeter measure for C^2 hypersurfaces of completely general sub-Riemannian manifolds, allowing for the existence of characteristic points. When the manifold admits dilations, we establish a sub-Riemannian Minkowski formula. For C^2 hypersurfaces in vertically rigid sub-Riemannian manifolds we also produce a second variation formula for variations supported away from the characteristic locus.

1. Introduction

Optimization problems lie at the heart of many pure and applied problems, two of which, the minimal and isoperimetric surface problems, have played a central role in mathematical development over the last century. In the last decade, there has been increasing interest in these problems in the setting of sub-Riemannian spaces. This interest is driven in part by applications of sub-Riemannian geometry to optimal control as well as to more novel applications such as the recent sub-Riemannian model of the primary visual cortex [8, 22, 31, 32]. We are also motivated by a deep conjecture of Pansu [29] concerning the isoperimetric profile in the sub-Riemannian Heisenberg group, which has seen a great deal of recent activity with many partial results [11, 24, 28, 34] (see also [5] for an overview of this problem).

One of the basic approaches to such optimization problems uses the tools of the calculus of variations to determine the geometric and analytical properties of their solutions. Recent investigations of minimal and isoperimetric surfaces have focused on this approach with many authors deriving first and second variation formulae. In [1, 3, 10, 11, 30, 33, 34], the various authors compute first variation formulae for C^2 smooth noncharacteristic surfaces in the Heisenberg group. We note that some of these authors restrict their attention to certain types of graphs (Euclidean graphs: [30, 34], intrinsic graphs [1]), [10, 11] deals with a level set formulation and [3] provides a completely general nonparametric first variation formula. In [6]

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the authors compute a first variation formula for C^2 noncharacteristic graphs in any three dimensional pseudo-hermitian space (including, of course, the Heisenberg group). In [27, 35] the authors independently provide a first variation formula for C^2 noncharacteristic surfaces in general Carnot groups. In [9, 36], the authors provide a first variation formula for C^2 surfaces in Martinet-type space and (2,3) contact manifolds (respectively).

In [20], we compute the first variation formula for C^2 noncharacteristic surfaces in all so-called *vertically rigid* spaces. To formalize this, we recall some of the basic definitions.

Definition 1.1. A **sub-Riemannian (or Carnot-Carathéodory) manifold** is a triple $(M, V_0, \langle\cdot,\cdot\rangle)$ consisting of a smooth manifold M of dimension $n+1 = k+l+1$, a smooth $k+1$-dimensional distribution $V_0 \subset TM$ and a smooth inner product $\langle \cdot\,,\,\cdot \rangle$ on V_0. This structure is endowed with a metric structure given by

$$d_{cc}(x,y) = \inf\left\{ \int \langle \dot\gamma, \dot\gamma\rangle^{\frac{1}{2}} | \gamma(0) = x, \gamma(1) = y, \gamma \in \mathscr{A} \right\}$$

where $\mathscr{A}$ is the space of all absolutely continuous paths whose derivatives, when they are defined, lie in V_0.

Definition 1.2. A **vertical complement** to a sub-Riemannian structure is a smooth complement V to V_0 in TM, i.e. a smooth bundle V such that

$$(V_0)_p \oplus V_p = T_pM$$

at every point p. A sub-Riemannian manifold with a vertical complement will be referred to as a sRC-manifold.

A metric extension for a sRC-manifold is a Riemannian metric g such that

- $g(V_0, V) = 0$ at every point,
- restricted to V_0, $g = \langle \cdot\,,\,\cdot \rangle$.

For a metric extension g of an sRC-manifold, the vertical rigidity 1-form of g is the vertical trace of the Lie derivative of the metric

$$\mathfrak{R}_g(Y) = \frac{1}{2}\sum_\alpha \left(\mathcal{L}_{Y_0} g\right)(T_\alpha, T_\alpha)$$

where T_α is any (local) orthonormal frame for V. While this is defined for vector fields Y, the projection to V_0 actually ensures that $\mathfrak{R}_g$ is tensorial and so is a well-defined 1-form. The extension is **vertically rigid** if $\mathfrak{R}_g \equiv 0$.

This definition of vertical rigidity is more general than the definition introduced in [20], and includes all cases where the previously mentioned first variation formulas have been computed. In fact (see Theorem 5.5), a slight modification of the argument from [20] allows us to compute a first variation formula in *any* sub-Riemannian manifold with vertical complement:

Theorem A. *Let Σ be a C^2 noncharacteristic hypersurface in a sRC-manifold M with metric extension g. Suppose F is a $C^{1;2}$ variation of Σ with horizontal variation function ρ_0. Then*

$$\left.\frac{d}{dt}\right|_{t=0} P_0(\Sigma_t) = \int_\Sigma \rho_0 \left(H - \mathfrak{R}_g(\nu)\right)\Lambda.$$

Here, P_0 denotes the sub-Riemannian perimeter measure, H denotes the horizontal mean curvature and ν is the horizontal unit normal (see Section 3 for precise definitions). As a consequence, we have:

Theorem B. *A necessary and sufficient condition for a C^2 hypersurface Σ to be a noncharacteristic critical point for the horizontal perimeter measure in the category of C^1 hypersurfaces with fixed boundary in a sub-Riemannian manifold is*

$$div\ \nu = H - \mathfrak{R}_g(\nu) = 0.$$

If the vertical structure is rigid, the second term drops out and the equation becomes

$$div\ \nu = H = 0.$$

Recent work of Cheng-Hwang-Yang [7] and Ritoré-Rosales [33] have shown how to extend the first variation formula to allow for variations over the characteristic locus in the Heisenberg group. Our first main result of this paper is to prove a similar extension of the previous Theorems to include variations over the characteristic locus.

Theorem C. *Let Σ be a C^2 hypersurface in a sRC-manifold M with metric extension g and characteristic locus $\Sigma(M)$. Further suppose that the Riemannian curvature tensor of Σ is bounded and that the horizontal mean curvature of Σ, H, is in $L^1(\Sigma)$. Suppose F is a compactly supported $C^{1;2}$ variation of Σ with F_0 C^2 and variation function ρ. Then*

$$\left.\frac{d}{dt}\right|_{t=0^+} P_0(\Sigma_t) = \int_{\Sigma\setminus C(\Sigma)} \rho(div\ \nu) - div_\Sigma(\rho\nu^\top)\, dV_\Sigma.$$

Here, div and div_Σ are the Riemannian divergence operators on M and Σ respectively, N is the Riemannian normal to Σ, ν is the unit horizontal normal and $\nu^\top$ is the (Riemannian) component of ν tangential to Σ.

In keeping with historical terminology, we call critical points of the perimeter variation *minimal surfaces.*

As an application of the general first variation formula, we prove a version of the Minkowski formula in this setting (see Theorem 6.4 and Corollary 6.5):

Theorem D. *Suppose Ω is a compact C^2 domain with $\Sigma = \partial\Omega$ that is a critical point for perimeter measure with volume constraint. Then*

$$(Q-1)P_0(\Sigma) = QH\, Vol(\Omega).$$

We note that this formula was shown in groups of Heisenberg type in [11] and in the Heisenberg group in [33].

A number of authors [10, 27, 35] have also computed second variation formulae in the setting of Carnot groups as a tool in the investigation of stable minimal surfaces. As has been shown recently [1, 12, 13, 14], stability plays a crucial role in the study of minimal surfaces in the Heisenberg group. Specifically, these papers study analogues of the sub-Riemannian Bernstein problem and show that without the imposition of stability on critical points of the first variation of perimeter, there is no Bernstein-type rigidity. On the other hand, there is rigidity in the presence of the stability condition. The most general of these results is an analogue of Riemannian results of Fischer-Colberie/Schoen [17] and Do Carmo/Peng [16]:

Theorem 1.1 ([14])**.** *The only stable C^2 complete embedded noncharacteristic minimal surfaces without boundary in the first Heisenberg group are the vertical planes.*

To facilitate further study of stable minimal surfaces, we derive a second variation formula for C^2 noncharacteristic surfaces in vertically rigid spaces (Theorem 7.4):

Theorem E. *Suppose M is a vertically rigid sRC-manifold and F is a noncharacteristic $C^{\infty;3}$ variation of $\Sigma\backslash C(\Sigma)$ with compactly supported horizontal variation function. Then*

$$\begin{aligned}
\frac{d^2}{dt^2}\bigg|_{t=0} P_0(\Sigma_t) = &\int_{\Xi} [(\partial_t\widehat{\rho}_0)H\Lambda]_{|t=0} + \int_{\Sigma} \left|\nabla^{0,\Sigma}\rho_0\right|^2 \Lambda \\
&+ \int_{\Sigma} \rho_0^2 \Big[- Ric^{\nabla}(\nu,\nu) - tr(I\!I_0^{\top} I\!I_0) + H^2 \\
&\qquad + \langle\, tr_0\,(\, TOR_2 - \nabla\, Tor)\,(\nu)\,,\, \mathcal{Y}\,\rangle \\
&\qquad - 2\langle\, Tor(\nu, e_j)\,,\, \nabla_j\mathcal{Y}\,\rangle - \langle\, Tor(\nu, e_j)\,,\, \mathcal{Y}\,\rangle^2 \Big]\Lambda.
\end{aligned} \tag{1.1}$$

Here $\mathcal{Y}$ is the Riemannian normal with unit normalized horizontal component, $\nu = \mathcal{Y}_0$ is the unit horizontal normal and the $\{e_j\}$ form an orthonormal basis for the horizontal tangent space to Σ. The matrix $I\!I_0$ denotes the horizontal second fundamental form and the ∇ is the canonical connection for M (see Section 3).

Furthermore, if either the surface is minimal or the surface is CMC and variation preserves volume, then the first integral cancels out with the H^2 term of the third.

We note that this second variation formula, when restricted to the special case of the first Heisenberg group matches with others in the literature [7, 10, 11, 33].

2. Notation and conventions

To improve economy with the intensive computations throughout this paper, we shall following the following notations and conventions:

(A) Unless explicitly stated otherwise, roman indices will run from $1\ldots k$, overlined roman indices from $0\ldots k$ and greek indices from $1\ldots l = n-k$.

(B) $\omega^{\bullet} = \omega^1 \wedge \cdots \wedge \omega^k$.

(C) $\eta^{\bullet} = \eta^1 \wedge \cdots \wedge \eta^l$.

(D) $\omega^{(j)}$ denotes the ordered wedge product of all possible (by index conventions) 1-forms ω^i with the jth form omitted, e.g

$$\omega^{(2)} = \omega^1 \wedge \omega^3 \wedge \cdots \wedge \omega^k$$
$$\omega^{(\bar{2})} = \omega^0 \wedge \omega^1 \wedge \omega^3 \wedge \cdots \wedge \omega^k.$$

We shall also use $\omega^{(i,j)}$ with $i<j$ to denote the ordered wedge product with both ith and jth terms missing and extend to all indices by setting $\omega^{(i,j)} = -\omega^{(j,i)}$.

(E) (Summation Convention) Whenever the same index appears twice in a term obeying the above conventions, we shall assume that there is an implicit sum over all possible values.

(F) If a metric extension g is understood, we shall extend the notation $\langle\cdot,\cdot\rangle$ to include the g-inner product of non-horizontal vectors.

3. SubRiemannian manifolds and vertical structures

In [21], the first author introduced a new connection for sRC-manifolds with metric extensions. The main result shown in [21] is the following:

Theorem 3.1. *If M is an sRC-manifold with metric extension g then there is a unique* **canonical** *connection ∇ such that*

- *$\nabla g = 0$, i.e. ∇ is compatible with g*
- *V_0 and V are parallel*
- *$Tor(V_0, V_0) \subset V$, $Tor(V, V) \subset V_0$*
- *If X, Y are horizontal vectors and T, U are vertical vectors then*

$$\langle\, Tor(X,T)\,,\, U\,\rangle = \langle\, T\,,\, Tor(X,U)\,\rangle, \qquad \langle\, Tor(X,T)\,,\, Y\,\rangle = \langle\, X\,,\, Tor(Y,T)\,\rangle.$$

Furthermore, if X, Y are horizontal vector fields, then ∇X and $Tor(X,Y)$ depend only the sRC-manifold and not the choice of metric extension.

These connections simultaneously generalize the Levi-Civita connection for Riemannian manifolds and the Tanaka-Webster connection on strictly pseudoconvex pseudohermitian manifolds. The rigidity tensor can also easily be described in terms of the canonical connection. See [21] for details.

Lemma 3.1. *If X is a horizontal vector field and $\{T_\beta\}$ is any (local) orthonormal frame for V then*

$$\mathfrak{R}_g(X) = \sum_\beta \langle\, Tor(X, T_\beta)\,,\, T_\beta\,\rangle.$$

Remark 3.1. It is the vanishing of this torsion trace that makes vertically rigid extensions much easier to work.

Remark 3.2. As the torsion doesn't vanish, frequently terms involving torsion occur where they do not in the Riemannian setting. The following tensors often arise in computations with second derivatives.

$$\begin{aligned}(\nabla\mathrm{Tor})(X,Y,Z) &= \nabla_Z \mathrm{Tor}(X,Y) - \mathrm{Tor}(\nabla_Z X, Y) - \mathrm{Tor}(X, \nabla_Z Y),\\ \mathrm{TOR}_2(X,Y,Z) &= \mathrm{Tor}(X, \mathrm{Tor}(Y,Z)).\end{aligned} \tag{3.1}$$

We now begin the study of hypersurface geometry using this canonical connection. First for a C^1 hypersurface Σ in a subRiemannian manifold, we define the characteristic set of Σ to be

$$C(\Sigma) = \{p \in \Sigma \colon (V_0)\big|_p \subset T_p\Sigma\}.$$

If Σ is oriented, we define N to be a choice unit Riemannian (with respect to g) normal to Σ and N_0 as the orthogonal projection of N to V_0. The characteristic set $C(\Sigma)$ can then be thought of as the points at which $N_0 = 0$. Away from the characteristic set, we define the **horizontally normalized normal** $\mathcal{Y}$ and the **unit horizontal normal** ν by

$$\mathcal{Y} = \frac{N}{|N_0|}, \qquad \nu = \frac{N_0}{|N_0|} = \mathcal{Y}_0.$$

Definition 3.1. The horizontal perimeter measure of Σ is defined to be

$$P_0(\Sigma) = \int_\Sigma |N_0|\, dV_\Sigma$$

where $dV_\Sigma = N \lrcorner dV_g$.

For noncharacteristic surfaces, P_0 has the alternative descriptions

$$P_0(\Sigma) = \int_\Sigma \nu \lrcorner dV_\Sigma = \sup \left\{ \int_\Sigma X \lrcorner dV_\Sigma : X \in \Gamma(V_0), |X| = 1 \right\}.$$

There are several natural questions associated to this perimeter measure.

Question. *Among hypersurfaces with the same boundary, which minimizes the horizontal perimeter measure? Can such surfaces be characterized as solutions to a PDE?*

Question. *Among domains of the same volume, which has boundary minimizing horizontal perimeter measure?*

These problems are studied using variational techniques which may yield critical points rather than true minima. Thus there is another natural question:

Question. *Of the critical points of horizontal perimeter measure, which are* **stable***, i.e.* $\frac{d^2}{dt^2}\big|_{t=0} P_0(\Sigma_t) \geq 0$ *for any variation of* Σ*?*

Under a slightly more restrictive definition of rigidity, and the assumption of characteristic points and C^2 regularity, Questions 3 and 3 were answered in [20] in terms of the horizontal mean curvature.

Suppose $e_0 = \nu, e_1, \dots e_k$ forms a (local) orthonormal frame for V_0 such that on $\Sigma \backslash C(\Sigma)$, e_0 is the unit horizontal normal to Σ. Then away from $C(\Sigma)$, the horizontal second fundamental form for Σ is defined by

$$I\!I_0 = \begin{pmatrix} \langle \nabla_{e_1} e_0, e_1 \rangle & \dots & \langle \nabla_{e_1} e_0, e_k \rangle \\ \vdots & \vdots & \vdots \\ \langle \nabla_{e_k} e_0, e_1 \rangle & \dots & \langle \nabla_{e_k} e_0, e_k \rangle \end{pmatrix}.$$

The horizontal mean curvature is defined by

$$H = \operatorname{trace}(I\!I_0) = \operatorname{trace}(\nabla \nu). \tag{3.2}$$

We remark that the connection used in these definitions can be either the Levi-Cevita connection for the Riemannian metric or any connection adapted to the vertical structure.

In a vertically rigid sRC-manifold, C^2 minimizers of P_0 with fixed boundary constraint were shown in [20] to satisfy $H = 0$ away from characteristic points. Likewise C^2 minimizers subject to the volume constraint satisfied the condition that the horizontal mean curvature was locally constant away from the characteristic set.

We finish this section by making some remarks on the nature of the equation $H = c$ off the characteristic set. Since the canonical connection is metric compatible, we obtain the following result about the ambient divergence on M from standard results in Riemannian geometry (see [23], appendix 6):

$$\operatorname{div} Z = \operatorname{trace}(\nabla Z + \operatorname{Tor}(Z, \cdot))$$

Thus if $\{X_{\bar{i}}\}$ denotes a local horizontal orthonormal frame, $\{T_\beta\}$ a local vertical orthonormal frame and Z is a horizontal vector field

$$\begin{aligned} \operatorname{div} Z &= \langle \nabla_{X_{\bar{i}}} Z , X_{\bar{i}} \rangle + \langle \operatorname{Tor}(Z, T_\beta) , T_\beta \rangle \\ &= \langle \nabla_{X_{\bar{i}}} Z , X_{\bar{i}} \rangle - \langle [Z, T_\beta]) , T_\beta \rangle \\ &= \operatorname{trace}(\nabla Z) - \mathfrak{R}_g(Z). \end{aligned}$$

Applying this to ν yields

$$\operatorname{div} \nu = H - \mathfrak{R}_g(\nu).$$

In the rigid case, the second term drops out and H naturally takes the form

$$H = \operatorname{div} \nu.$$

4. Bundles and Variations

Throughout this section M is a sRC-manifold with a fixed metric extension g. To describe the variational properties of the horizontal perimeter measure, we shall define a variety of bundles over M.

First, we shall denote by $\mathcal{S}(M)$ the **contact manifold of normalized hypersurface elements**, i.e $\pi\colon \mathcal{S}(M) \to M$ is the unit tangent bundle over M viewed as a bundle of Riemannian unit normals. We define the 1-form Θ on $\mathcal{S}(M)$ by

$$\Theta_{|(p,E)}(X) = \langle \pi_* X, E\rangle_p.$$

An immersion ι of an n-dimensional manifold into $\mathcal{S}(M)$ is said to be **transverse** if $\pi \circ \iota$ is an immersion and $\iota^*\Theta = 0$.

Definition 4.1. The function $N_0\colon \mathcal{S}(M) \to V_0(M)$ is defined by

$$N_0(p, E) = ((\pi_* E)_0)\Big|_p.$$

Here, we use the convention that if W is a vector field on M then $(W)_0$ is its projection to a vector field in V_0.

The characteristic slice, $C_{\mathcal{S}}$ of $\mathcal{S}(M)$ is the zero level set of N_0.

There is a natural projection $\pi_{\mathcal{F}}$ from the Riemannian frame bundle $\mathcal{F}(M)$ to $\mathcal{S}(M)$ given by

$$\pi_{\mathcal{F}}\colon (p, E_0, \dots E_n) \mapsto (p, E_0).$$

We note that if $E^0, \dots E^n$ denote the tautological forms on $\mathcal{F}(M)$, (i.e. at the point $(p, E_0, \dots E_n)$, $E^j(X) = \langle \pi_* X, E_j\rangle$, $j = 0, \dots, n$) then

$$\Theta = \sigma^* E^0$$

for any section σ of $\pi_{\mathcal{F}}$.

Definition 4.2. A differential form ψ on $\mathcal{S}(M)$ is semibasic if

$$X \lrcorner \psi = 0$$

wherever $\pi_* X = 0$. Thus ψ depends only the the projection to M and the choice of E_0.

For example, it is clear that Θ is a semibasic 1-form.

The **bundle of graded orthonormal frames** is the subbundle $\mathcal{GF}(M) \subset \mathcal{F}(M)$ such that

$$(E_0 \dots E_n) = (e_0 \dots e_k, t_1, \dots, t_l)$$

where the e_j's are all horizontal and $t_1, \dots, t_l$ span V. The reduced structure group of the bundle is then $O(k+1) \times O(l)$.

Unfortunately, these graded bundles do not encode enough information to describe the geometry of hypersurfaces of M. To compensate for this we also introduce the augmented bundles

$$\mathcal{GF}_0 = \mathcal{GF} \times \mathbb{R}^l.$$

The additional elements will be used to keep track of the dependence of the hypersurface normal directions on the vertical vector fields.

There is an alternative presentation of $\mathcal{S}(M)\backslash C_{\mathcal{S}}$ that will prove computationally simpler to work with for noncharacteristic variations. We define the **contact manifold of horizontally normalized hypersurface elements** to be

$$\mathcal{S}_0(M) = \{(p, Z) \in TM : |Z| = 1, Z \in (V_0)_p\} \times \mathbb{R}^l.$$

There is then a natural projection map $\pi_{\mathcal{G}}\colon \mathcal{GF} \to \mathcal{S}_0(M)$ and a bundle isomorphism $\mathcal{S}(M)\backslash C_{\mathcal{S}} \cong \mathcal{S}_0(M)$ given by

$$(p, E) \mapsto \big(p, |N_0|^{-1}(E)_0, |N_0|^{-1}\langle E, T_\beta\rangle\big)$$

with inverse

$$(p, e_0, a_\beta) \mapsto \big(p, \frac{1}{\sqrt{1+|a|^2}}(e_0 + a_\beta T_\beta)\big).$$

We shall identify $\mathcal{S}_0(M)$ with $\mathcal{S}(M)\backslash C_{\mathcal{S}}$ using this bundle isomorphism.

On the bundle $\pi\colon \mathcal{GF}(M) \to M$ we can define tautological 1-forms ω^j and η^j by

$$\omega^j(X) = \langle \pi_* X, e_j\rangle, \qquad j = 0 \dots k$$
$$\eta^j(X) = \langle \pi_* X, t_j\rangle, \qquad j = 1 \dots n-k.$$

We define a 1-form θ on $\mathcal{S}_0(M)$ by

$$\theta(X) = \langle \pi_* X, e_0 + a_\beta T_\beta\rangle$$

and note that on $\mathcal{S}_0(M)$ we have

$$|N_0| = \frac{1}{\sqrt{1+|a|^2}} \text{ and } \Theta = |N_0|\theta.$$

With respect to any section of the natural projection map $\pi_{\mathcal{G}}$,

$$\theta = \sigma^*(\omega^0 + a_\beta \eta^\beta)$$

Since we shall frequently be computing on the frame bundles, we shall often implictly identify $\omega^0 + a_\beta\eta^\beta$ with θ.

For the remainder of this section, we shall suppose that Σ is an oriented, embedded C^2 hypersurface of M realized as the image of the C^2 embedding

$$\iota\colon \Xi\widehat{\omega}okrightarrowM$$

for some smooth compact, oriented manifold (possibly with boundary) Ξ.

Definition 4.3. A variation of Σ is a map

$$F\colon \Xi \times (-\epsilon, \epsilon) \to M$$

such that

- each $F_t = F(\cdot, t)$ is an immersion of Ξ into M,
- $F_0 = \iota$.

The lifted variation $\widehat{F}\colon \Xi \times (-\epsilon, \epsilon) \to \mathcal{S}(M)$ is the map defined by

$$\widehat{F}(\xi, t) = (F(\xi, t), N\big|_{F(\xi,t)})$$

where N is the (local) Riemannian unit normal vector to the immersed surface $F_t(\Xi)$ such that the pullback of $N \lrcorner dV$ matches the fixed orientation of Ξ.

The variation function of F is $\rho = \widehat{\rho}(\cdot, 0)$ where $\widehat{F}^*\Theta = \widehat{\rho}dt$. The variation is said to be compactly supported if $\widehat{\rho}(\cdot, t)$ has compact support for all t.

The characteristic sets of the variation are the preimages of the characteristic slice,

$$C^t_\Xi = \widehat{F}_t^{-1}(C_{\mathcal{S}}).$$

Definition 4.4. When the lifted variation $\widehat{F}$ maps into the complement of the characteristic slice $C_{\mathcal{S}}$, i.e. $C^t_\Xi = \emptyset$ for all t, we shall refer to the variation as noncharacteristic. The horizontal variation function for F is then defined by $\rho_0 = \widehat{\rho}_0(\cdot,0)$, $\widehat{F}^*\theta = \widehat{\rho}_0 dt$.

Remark 4.1. Since $C_{\mathcal{S}}$ is closed, if Σ has no characteristic points then, shrinking ϵ if necessary, any variation F will map into $\mathcal{S}_0(M)$. The relationship between the variational functions is then just

$$\rho = |N_0 \circ \widehat{F}|\rho_0.$$

So far, we have not put any regularity conditions on our variations. However, we shall need precise descriptions of regularity to make our theory optimal.

Definition 4.5. The classes of $C^{i;j}$ maps from $\mathbb{R}^n_x \times \mathbb{R}_t$ to $\mathbb{R}$ for $i, j \geq 0$ are defined inductively by

- $C^{0;0} = C^0$, i.e. continuous maps,
- $F \in C^{i+1;0}$ if and only if $F, \frac{\partial F}{\partial x^m} \in C^{i;0}$ for all $m = 1 \dots n$,
- $F \in C^{0;j+1}$ if and only if $F, \frac{\partial F}{\partial t} \in C^{0;j}$,
- $F \in C^{i+1;j+1}$ if and only if $F \in C^{i;j+1} \cap C^{i+1;j}, \frac{\partial F}{\partial t} \in C^{i+1;j}$ and $\frac{\partial F}{\partial x^m} \in C^{i;j+1}$ for all $m = 1 \dots n$.

Thus a map is $C^{i;j}$ if up to i continuous spatial (x) derivatives and j temporal (t) continuous derivatives can be taken in any order.

Using coordinate charts, this definition extends naturally to define $C^{i;j}$ maps

$$\Xi \times \mathbb{R} \to M$$

for smooth manifolds Ξ and M.

Remark 4.2. We note in passing that

$$C^m = \bigcap_{i+j=m} C^{i;j} \subset C^{m;m}.$$

The following approximation result will be useful later

Lemma 4.1. *Given a $C^{i;j}$ map $F\colon \Xi \times \mathbb{R} \to M$ that is constant outside $K \times \mathbb{R}$ for some compact set $K \subset \Xi$, there exists a sequence F_m of $C^{\infty;j}$ maps such that*

(1) *F_m converges to F in $C^{i;j}$,*
(2) *if $F(\cdot,t_0) \in C^p$ then $F_m(\cdot,t_0)$ converges to $F(\cdot,t_0)$ in C^p.*

This lemma is a version of standard approximation theorems adapted to allow parameters. The reader is referred to [19] pp.41-55. for a proof that C^r maps between smooth manifolds can be approximated by smooth maps. Theorem 2.3 in [19] can easily be adapted to give an approximation of $C^{i;j}$ maps from $\Xi \times \mathbb{R} \to \mathbb{R}^m$ by $C^{\infty;j}$ maps, with the observation that the mollification process should only be in the Ξ coordinates. The details are standard and are left to the reader.

Lemma 4.2. *If F is a $C^{i;j}$ variation $i, j \geq 1$, then $\widehat{F}$ is a $C^{i-1,j}$ map.*

Proof. The tangent space to the immersed surface $F_t(\Xi)$ is locally spanned by the vector fields $F_*\partial_{\xi^m}$. Therefore the Gram-Schmidt algorithm followed by a horizontal projection and rescaling, expresses the unit horizontal normal to $F_t(\Xi)$ ν as a smooth combination of these spanning vector fields. Thus ν can be viewed as a $C^{i-1;j}$ function.

$\square$

We now list a few basic regularity properties

Lemma 4.3. *If F is a $C^{i;j}$ variation and ψ is a smooth semibasic differential form on $\mathcal{S}(M)$ then $\widehat{F}^*\psi$ is a $C^{i-1;j-1}$ form on $\Xi \times (-\epsilon, \epsilon)$.*

Proof. The real issue here is that as a map into $\mathcal{S}(M)$, $\widehat{F}$ only has $C^{i-1;j}$ regularity. However, $\widehat{F}^*\psi$ depends tensorially on the projected input $F_*\frac{d}{dt}$, $F_*\frac{d}{d\xi^m}$, which are $C^{i;j-1}$, $C^{i-1;j}$ vector fields respectively, and its position $\widehat{F}(x,t)$ which is also at least $C^{i-1;j}$.

$\square$

In particular, the lemma implies that we can make sense of the pullback of semibasic forms by $C^{1;2}$ variations despite the fact that the lifted variations are only continuous maps into $\mathcal{S}(M)$.

Corollary 4.1. *For a $C^{i;j}$ variation F and smooth semibasic form ψ, the form*

$$(\xi, t) \mapsto F_t^*\psi$$

has $C^{i-1;j}$ regularity.

Proof. The proof is identical to the previous lemma except that we no longer need dependence on $F_*\frac{d}{dt}$.

$\square$

Corollary 4.2. *For a $C^{i;2}$ variation F, $i \geq 1$, the variation function is C^{i-1}.*

Furthermore, it will be of interest to note that, locally at least, every function on Σ can be realized as a variation function.

Lemma 4.4. *For every point $p = \iota(\xi) \in \Sigma$ there exists a neighborhood $\xi \in U \subset \Xi$ such that every C^j function ρ, $j = 1, 2$, on U is the restriction of the variation function for a $C^{j;\infty}$ variation of Σ.*

This is shown using standard arguments with Pfaff coordinates (see [4], p.16). The restriction $j = 1, 2$ is due to the fact that Σ is only assumed to be C^2.

Definition 4.6. On $\mathcal{GF}$ we define the smooth n-form Λ by

$$\Lambda = \omega^\bullet \wedge \eta^\bullet = \omega^1 \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \wedge \eta^{n-k}.$$

Note that

$$\pi^* dV = \omega^0 \wedge \omega^\bullet \wedge \eta^\bullet$$

thus for any local section σ of $\pi_{\mathcal{G}}$,

$$\sigma^* \Lambda_{|(p,e_0)}(E_1, \ldots, E_n) = dV(e_0, \pi_* E_1, \ldots, \pi_* E_n)$$

which is independent of σ. Thus Λ descends to a well-defined form on $\mathcal{S}_0(M)$ which we shall also denote by Λ. In fact, the same argument shows that $\omega^\bullet$ and $\eta^\bullet$ descend to $\mathcal{S}_0(M)$ also.

For a variation F of Σ, we define

$$\Lambda_t = \begin{cases} \widehat{F}_t^* \Lambda, & \text{on } \Xi \backslash C_\Xi^t \\ 0, & \text{on } C_\Xi^t. \end{cases}$$

Lemma 4.5. *Each Λ_t is continuous on Ξ.*

Proof. Let $\xi_1, \ldots, \xi_n$ be a frame for Ξ that pushes forward under F_t to the oriented orthonormal frame

$$E_0 = \cos\phi e_0 + \sin\phi T_{n-k}, e_1, \ldots e_k, T_1, \ldots T_{n-k-1}, \sin\phi e_0 - \cos\phi T_{n-k},$$

where E_0 is the vector field determined by $\widehat{F}_t$ and $0 \leq \phi < \pi$. Locally the characteristic set C_Ξ^t is then given by $\phi = \pi/2$. Then away from C_Ξ^t,

$$\begin{aligned} \Lambda_t(\xi_1, \ldots \xi_n) &= \Lambda(\widehat{F}_{t*}\xi_1, \ldots, \widehat{F}_{t*}\xi_n) \\ &= dV(e_0, E_1, \ldots, E_k, T_1, \ldots T_{n-k-1}, \sin\phi e_0 - \cos\phi T_{n-k}) \\ &= \cos\phi \end{aligned}$$

which continuously extends to zero across C_Ξ^t.

□

Lemma 4.6. *Suppose $F \colon \Xi \times (-\epsilon, \epsilon)$ is a $C^{0;2}$ variation of Σ. Then*

$$P_0(\Sigma_t) = \int_\Xi \Lambda_t.$$

Proof. Let $\xi_1, \ldots, \xi_n$ be a frame for Ξ as in the previous lemma. Then away from C_Ξ^t,

$$\begin{aligned} \Lambda_t(\xi_1, \ldots, \xi_n) &= \widehat{F}_t^* \Lambda(\xi_1, \ldots, \xi_n) \\ &= dV(e_0, (\pi \circ \widehat{F}_t)_* \xi_1, \ldots, (\pi \circ \widehat{F}_t)_* \xi_n) \\ &= dV(e_0, F_{t*}\xi_1, \ldots F_{t*}\xi_n) \\ &= F_t^*(e_0 \lrcorner dV)(\xi_1, \ldots \xi_n). \end{aligned}$$

Therefore since Λ_t continuously extends to zero over C_Ξ^t,

$$P_0(\Sigma_t) = \int_{\Sigma_t} e_0 \lrcorner dV = \int_\Xi \Lambda_t.$$

□

Using the embedding map $\iota \colon \Xi \to \Sigma \subset M$, can identify Λ_0 with the n-form $\iota^{-1*}\Lambda_0$. Abusing notation slightly, we will also write this form as Λ and so write

$$P_0(\Sigma) = \int_\Sigma \Lambda.$$

5. Horizontal Perimeter Measure and the First Variation

Throughout this section we suppose M is a subRiemannian manifold with a chosen vertical structure.

Most computations will be undertaken on the frame bundles. In addition to the tautological forms, we also have the bundle structural equations for the connection (see [20])

$$\begin{aligned} d\omega^{\bar{j}} &= \omega^{\bar{m}} \wedge \omega^{\bar{j}}_{\bar{m}} + \tau^{\bar{j}} \\ d\eta^{\beta} &= \eta^{\alpha} \wedge \eta^{\beta}_{\alpha} + \tilde{\tau}^{\beta}. \end{aligned} \tag{5.1}$$

Here $\tau^{\bar{j}}$ and $\tilde{\tau^{\beta}}$ are the standard torsion The parallel property of V_0 and V is what implies that there are no cross terms of the form $\eta^{\alpha} \wedge \omega^{\bar{m}}_{\alpha}$ or $\omega^{\bar{m}} \wedge \eta^{\alpha}_{\bar{m}}$ in the above equations. As the connection is metric compatible, it follows immediately that $\omega^{\bar{j}}_{\bar{j}}$ and η^{α}_{α} vanish for all $\bar{j}$ and α. The torsion properties of the connection also ensure that

$$\tau^{\bar{i}} \wedge \eta^{\bullet} = 0 = \tilde{\tau}^{\beta} \wedge \omega^{\bullet}.$$

The effect of vertical rigidity is given in the following elementary equation.

$$(-1)^{\beta} \omega^{\bullet} \wedge \tilde{\tau}^{\beta} \wedge \eta^{(\beta)} = (-1)^{k} \mathfrak{R}_g(e_0) \omega^0 \wedge \Lambda. \tag{5.2}$$

Next we define a form for our computation

Definition 5.1. We define the 1-form Ψ on $\mathcal{GF}_0$ by

$$\Psi = (-1)^{j-1} \omega^j_0 \wedge \omega^{(j)} \wedge \eta^{\bullet}.$$

Lemma 5.1. *If σ is any section of $\pi_{\mathcal{G}}$ then on $\mathcal{S}_0(M)$,*

$$d\Lambda = \theta \wedge \left(\sigma^* \Psi - \mathfrak{R}_g(e_0) \Lambda\right).$$

If the vertical structure is rigid, then

$$d\Lambda = \theta \wedge \sigma^* \Psi.$$

Proof. The rigid case was proved in [20] section 4. For the general case, we compute on $\mathcal{GF}_0$ noting that $\omega^j_j = 0 = \eta^{\beta}_{\beta}$,

$$\begin{aligned} d\Lambda &= d\omega^{\bullet} \wedge \eta^{\bullet} + (-1)^k \omega^{\bullet} \wedge d\eta^{\bullet} \\ &= (-1)^{j-1} d\omega^j \wedge \omega^{(j)} \wedge \eta^{\bullet} + (-1)^{k+\beta-1} \omega^{\bullet} \wedge d\eta^{\beta} \wedge \eta^{(\beta)} \\ &= (-1)^{j-1} (\omega^{\bar{m}} \wedge \omega^j_{\bar{m}} + \tau^j) \wedge \omega^{(j)} \wedge \eta^{\bullet} + (-1)^{k+\beta-1} \omega^{\bullet} \wedge (\eta^{\alpha} \wedge \eta^{\beta}_{\alpha} + \tilde{\tau}^{\beta}) \wedge \eta^{(\beta)} \\ &= (-1)^{j-1} \omega^0 \wedge \omega^j_0 \wedge \omega^{(j)} \wedge \eta^{\bullet} + (-1)^{j-1} \tau^j \wedge \omega^{(j)} \wedge \eta^{\bullet} \\ &\quad + (-1)^{k+\beta-1} \omega^{\bullet} \wedge \tilde{\tau}^{\beta} \wedge \eta^{(\beta)} \\ &= (-1)^{j-1} \omega^0 \wedge \Psi + (-1)^{j-1} \tau^j \wedge \omega^{(j)} \wedge \eta^{\bullet} + (-1)^{k+\beta-1} \omega^{\bullet} \wedge \tilde{\tau}^{\beta} \wedge \eta^{(\beta)}. \end{aligned}$$

From the defining properties of the connection, when we pull back we see that $\tau^j \wedge \eta^{\bullet} = 0$. Furthermore $(\theta - \omega^0) \wedge \eta^{\bullet} = 0$. Thus from (5.2) we see that on $\mathcal{S}_0(M)$

$$d\Lambda = \theta \wedge \sigma^* \Psi - \sigma^* \left(\mathfrak{R}_g(e_0) \omega^0 \wedge \Lambda\right) = \theta \wedge \left(\sigma^* \Psi - \mathfrak{R}_g(e_0) \Lambda\right).$$

∎

The importance of Ψ lies in Lemma 5.1 and the immediate consequence that for a noncharacteristic variation of Σ,

$$(\sigma \circ \widehat{F}_0)^* \Psi = \omega_0^j(e_j)\Lambda_0 = H\Lambda_0 \tag{5.3}$$

where H is the horizontal mean curvature of Σ.

We are now in a position to compute the first variation for noncharacteristic surfaces.

Theorem 5.1. *Suppose Σ is a C^2 noncharacteristic hypersurface in M and F is a $C^{1;2}$ variation of Σ with horizontal variation function ρ_0. Then*

$$\left.\frac{d}{dt}\right|_{t=0} P_0(\Sigma_t) = \int_\Sigma \rho_0 \left(H - \mathfrak{R}_g(\nu)\right)\Lambda.$$

Proof. First note

$$\left.\frac{d}{dt}\right|_{t=0} P_0(\Sigma_t) = \left.\left[\int_\Xi \mathcal{L}_{\partial_t}\widehat{F}^*\Lambda\right]\right|_{t=0}.$$

Now Λ is a spatial form on $\mathcal{S}_0(M)$, so we can only guarantee $\widehat{F}^*\Lambda$ is $C^{0;1}$ on $\Xi \times (-\epsilon, \epsilon)$. However, since the variation has $C^{1;2}$ regularity and Σ itself is a C^2 hypersurface, at $t = 0$ we can differentiate on $\mathcal{GF}_0$ to see

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} P_0(\Sigma_t) &= \left[\int_\Xi d(\partial_t \lrcorner \widehat{F}^*\Lambda)\right]_{\big|t=0} + \left[\int_\Xi \partial_t \lrcorner \widehat{F}^* d\Lambda\right]_{\big|t=0} \\
&= \int_{\partial\Xi} \left.\left[\partial_t \lrcorner \widehat{F}^*\Lambda\right]\right|_{t=0} + \left.\int_\Xi \partial_t \lrcorner \widehat{F}^*(\theta \wedge (\sigma^*\Psi - \mathfrak{R}_g(e_0)\Lambda))\right|_{t=0} \\
&= 0 + \int_\Xi \rho_0 \left[(\sigma \circ \widehat{F}_0)^*\Psi - \mathfrak{R}_g(\nu)\Lambda_0\right] \\
&= \int_\Sigma \rho_0 \left(H - \mathfrak{R}_g(\nu)\right)\Lambda.
\end{aligned}$$

□

Corollary 5.1. *A necessary and sufficient condition for a noncharacteristic C^2 hypersurface Σ to be a critical point for the horizontal perimeter measure in the category of C^1 hypersurfaces with fixed boundary is*

$$div\ \nu = H - \mathfrak{R}_g(\nu) = 0.$$

If the extension is rigid, the second term drops out and the equation becomes

$$div\ \nu = H = 0.$$

Remark 5.1. This is the first result of this nature for completely general subRiemannian manifolds. The rigid case was shown in [20]. Prior results include numerous important cases: level sets in Carnot groups [11, 27], for graphs in three dimensional strictly pseudoconvex pseudohermitian manifolds [6], Martinet-type spaces [9], for graphs in the Heisenberg group [18, 30], for parametrized surfaces in the Heisenberg group [3], for intrinsic graphs in the Heisenberg groups [1], and for surfaces in (2,3) contact manifolds [36].

In the presence of characteristic points, the situation becomes more complicated. It is to this case that we now direct our attention. To avoid needless repetition, we shall make the following assumptions throughout this section.

(A) Σ is an oriented C^2 hypersurface with piecewise C^1 boundary in some $n+1$-dimensional VR manifold M.
(B) The Riemannian unit normal to Σ will be denoted N. Off the characteristic set $C(\Sigma)$, the unit horizontal normal $\nu = \frac{1}{|N_0|} N_0$.
(C) Σ is the image of the immersion $\iota\colon \Xi \to M$ with $\Xi \subset \mathbb{R}^n$.
(D) $F\colon \Xi \times (-\epsilon, \epsilon) \to M$ is a $C^{1;2}$ variation of Σ with F_0 a C^2 mapping. In particular this implies that $\widehat{F}$ is a $C^{0;2}$ map.
(E) The horizontal mean curvature of Σ, $H \in L^1(\Sigma)$.
(F) The Riemannian curvature tensor of Σ is bounded.

Remark 5.2. For the natural embedding of Σ into $\mathcal{S}_0(M)$, the N_0 referred to above is equivalent to the pullback of the function N_0. Likewise we can pull N_0 back to $\Xi \times (-\epsilon, \epsilon)$ and we shall not make any notational distinction between them.

The necessary observation for studying variations for hypersurfaces with characteristic points is the following:

Suppose F is a variation of Σ. Note that C^0_Ξ is a closed set in Ξ. Furthermore by the results of the appendix, $C(\Xi)$ must have Hausdorff dimension $\leq n-1$. Let U be any open subset of Ξ containing C^0_Ξ. By shrinking ϵ if necessary, F induces a noncharacteristic variation F_H of $\Sigma \backslash \iota(U)$ as discussed in Section 4. Furthermore, if ρ is the variation function for F, then $|N_0|^{-1}\rho$ is the variation function for F_H. In particular

$$(\partial_t \lrcorner \widehat{F}^* \Theta)_{|t=0} dV_\Sigma = (\partial_t \lrcorner \widehat{F}_H^* \theta)_{|t=0} \Lambda.$$

Before diving into the general first variation formula, we shall need some technical lemmas.

Lemma 5.2. *With the assumptions listed above,*

- $|N_0(\cdot, 0)|$ *is a Lipschitz function on* $\Xi \times \{t\}$,
- $|N_0(\cdot, t)|$ *is* $C^{0;1}$ *off* $C(\Sigma_t)$ *and has bounded distributional temporal derivative on all of* Σ,
- *the one-sided derivative* $\left.\frac{d}{dt}\right|_{t=0^+} |N_0(t)|$ *exists everywhere, is continuous off* $C(\Sigma)$ *and is bounded on* Σ .

Proof. The first part follows from the fact that $N_0(\cdot, 0)$ is a C^1 map as F_0 is C^2. The remaining parts of the lemma are obvious properties of the absolute value of a C^1 function from $\mathbb{R}$ to $\mathbb{R}$.

□

The next lemma addresses an important technical subtlety with these variational calculations. The horizontal perimeter measure on Σ is defined using the weighted volume $|N_0| dV_\Sigma$, whereas all of our computations have been in terms of the form $\Lambda = \nu \lrcorner dV$. Up until now, we have been able to use these forms interchangeably as they are equivalent *when restricted to* Σ. However, when dealing with hypersurfaces with non-characteristic locus we shall need to contract these forms by vector fields transverse to the surface and this equivalence fails. Importantly, for a transverse vector field X we must be careful about how to restrict $X \lrcorner \Lambda$ to submanifolds S within Σ itself. In particular, for a codimension one submanifold $S \subset \Sigma$, $(X \lrcorner \Lambda)_{|S} \neq |N_0| \langle X\, , \, N_S \rangle dV_S$ as might be expected. The necessary correction term described in the following lemma is the cause of additional constraints on the characteristic locus required for a hypersurface Σ to be minimal. Failure to observe this detail leads to some incorrect over-simplifications of the characteristic case present in the literature.

Lemma 5.3. *Suppose U is an open set in Σ such that ∂U does not intersect $C(\Sigma)$. Then for any vector field X on M,*

$$X \lrcorner \Lambda_{|\partial U} = |N_0| \langle X, N_{\partial U} \rangle dV_{\partial U} - \langle \nu, N_{\partial U} \rangle \langle X, N \rangle dV_{\partial U}$$

where $N_{\partial U}$ is the Riemannian unit normal inside Σ to ∂U.

Proof. Note that off $C(\Sigma)$, $N = |N_0|\nu + |N_0| a_\beta T_\beta$ for constants a_β. The last piece can be rewritten as $\tilde{a}\tilde{T}$ for some unit vector $\tilde{T}$ orthogonal to V_0. Thus we can construct a vector field $\tilde{e}$ along $\Sigma \backslash C(\Sigma)$ by $\tilde{e} = \tilde{a}\nu - |N_0|\tilde{T}$. Then since we must have $|N_0|^2 + \tilde{a}^2 = 1$, clearly $\nu = |N_0|N + \tilde{a}\tilde{e}$.

Now $X \lrcorner \Lambda = X \lrcorner \nu \lrcorner dV$, thus splitting ν into pieces orthogonal and tangent to Σ we have

$$X \lrcorner \Lambda = |N_0| X \lrcorner dV_\Sigma + \tilde{a} X \lrcorner \tilde{e} \lrcorner dV.$$

Pulling back to ∂U immediately yields

$$X \lrcorner \Lambda_{|\partial U} = |N_0| \langle X, N_{\partial U} \rangle dV_{\partial U} - \tilde{a} \langle X, N \rangle \langle \tilde{e}, N_{\partial U} \rangle dV_{\partial U}.$$

Noting that $\tilde{a}\tilde{e}$ is the tangential component of ν then completes the proof.

□

Lemma 5.4. *There exists a family $\Omega_\delta \subset \Sigma$, $\delta > 0$ such that*

- *the portion of the boundary $\partial\Omega_\delta$ in the interior of Σ is piecewise C^2,*
- *$C(\Sigma) \subset \Omega_\delta$ for all $\delta > 0$,*
- *$\mu^n(\Omega_\delta) \to 0$ as $\delta \to 0$,*
- *$\int_{\partial\Omega_\delta} |N_0| dV_{\partial\Omega_\delta} \to 0$ as $\delta \to 0$,*

where μ^n is the n-dimensional Riemannian spherical Hausdorff measure.

Proof. By Theorem A.1, the characteristic set $C(\Sigma)$ is compact and has Hausdorff dimension $\leq n-1$. Thus for any $\delta, p > 0$ we can construct a finite collection of Riemannian balls $\mathcal{U}$ of radius ϵ covering $C(\Sigma)$ such that

$$\sum_{\mathcal{U}} \epsilon^{n-1+p} < \delta.$$

For each $\delta > 0$ take $\Omega_\delta = \Sigma \cap \bigcup_{\mathcal{U}} B$ for some $0 < p << 1$. This family clearly satisfies the first three properties.

The standing assumption on the Riemannian curvature tensor (F) implies that for some constant C

$$\int_{\partial\Omega_\delta} |N_0| dV_{\Omega_\delta} \leq \epsilon \int_{\partial\Sigma} dV_{\Omega_\delta} + C \sum_{\mathcal{U}} \epsilon^{n-1+1}$$

which tends to zero as $\delta \to 0$.

□

We are now in a position to state and proof the main result of this section, the first variation formula for perimeter measure of C^2 surfaces.

Theorem 5.2. *Suppose F is a compactly supported $C^{1;2}$ variation of Σ with F_0 C^2 and variation function ρ. Then*

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0^+} P_0(\Sigma_t) &= \int_{\Sigma\setminus C(\Sigma)} \rho(div\ \nu) dV_\Sigma - \lim_{\delta\to 0}\int_{\partial\Omega_\delta} \rho\langle\, \nu\,,\, N_{\Omega_\delta}\,\rangle dV_{\Omega_\delta} \\ &= \int_{\Sigma\setminus C(\Sigma)} \rho(div\ \nu) - div_\Sigma(\rho\nu^\top)\, dV_\Sigma\end{aligned}$$

where Ω_δ is any family satisfying the conditions of Lemma 5.4 and $\nu^\top$ is the Riemannian orthogonal projection of ν onto $T\Sigma$.

Proof. Using the existence of a family of neighborhoods of $C(\Sigma)$ as in Lemma 5.4 and pulling back to Ξ, we can immediately decompose

$$\frac{d}{dt}\bigg|_{t=0^+} P_0(\Sigma_t) = \left[\int_{\Xi\setminus\Xi_\delta} \mathcal{L}_{\partial_t}(\widehat{F}^*\Lambda)\right]\bigg|_{t=0} + \frac{d}{dt}\bigg|_{t=0^+}\int_{\Omega_\delta} (|N_0(t)| dV_{\Sigma_t})\,. \tag{5.4}$$

Using the results of Theorem 5.1 we can reduce the first term to

$$\int_{\Sigma\setminus\Omega_\delta} \rho(\mathrm{div}\ \nu) dV_\Sigma + \left[\int_{\partial\Xi_\delta} \partial_t \lrcorner \widehat{F}^*\Lambda\right]\bigg|_{t=0}.$$

This equals

$$\begin{aligned}\int_{\Sigma\setminus\Omega_\delta} \rho(\mathrm{div}\ \nu) dV_\Sigma + \int_{\partial\Omega_\delta} |N_0|\langle\, \widehat{F}_*\frac{d}{dt}\bigg|_{t=0}\,,\, N_{\partial\Omega_\delta}\,\rangle dV_{\partial\Omega_\delta} \\ - \int_{\partial\Omega_\delta} \rho\langle\, \nu^\top\,,\, N_{\partial\Omega_\delta}\,\rangle dV_{\partial\Omega_\delta}\end{aligned}$$

by Lemma 5.3. However the middle term can be neglected as $|N_0|$ is Lipschitz on Σ, with the other terms bounded, and so the integral will vanish as $\delta \to 0$ by Lemma 5.4. Thus we need only consider the contribution of

$$\int_{\Sigma\setminus\Omega_\delta} \rho(\mathrm{div}\ \nu) dV_\Sigma - \int_{\partial\Omega_\delta} \rho\langle\, \nu^\top\,,\, N_{\partial\Omega_\delta}\,\rangle dV_{\partial\Omega_\delta}$$

which by the Riemannian divergence theorem can also be expressed as

$$\int_{\Sigma\setminus\Omega_\delta} \rho(\mathrm{div}\ \nu) - \mathrm{div}_\Sigma(\rho\nu^\top)\, dV_\Sigma.$$

Now the second term of (5.4) decomposes as

$$\int_{\Omega_\delta} \left(\frac{d}{dt}\bigg|_{t=0^+} |N_0(t)|\right) dV_\Sigma + \int_{\Xi_\delta} |N_0| \mathcal{L}_{\partial_t}\iota_t^* dV_{\Sigma_t}.$$

By Lemma 5.2, $|N_0|$ is Lipschitz on Σ and vanishes on $C(\Sigma)$. The second integral can therefore be uniformly bounded by a fixed constant times $\delta \int_{\Xi_\delta} \mathcal{L}_{\partial_t}\iota_t^* dV_{\Sigma_t}$. Furthermore by Lemma 5.2 again, $|N_0|$ has bounded distributional derivative, so the first integral is bounded by $\mu^n(\Omega_\delta)$. As $\widehat{F}$ is C^1, $\int_{\Xi_\delta} \mathcal{L}_{\partial_t}\iota_t^* dV_{\Sigma_t}$ is bounded near $t = 0$. Therefore as $\delta \to 0$ the second term of (5.4) tends to zero. Therefore letting $\delta \to 0$ yields the desired result.

$\square$

Corollary 5.2. *A necessary and sufficient condition for a C^2 surface to be a critical point of horizontal perimeter measure in the category of C^2 hypersurfaces with fixed boundary is*

$$div\ \nu = H - \mathfrak{R}_g(\nu) = 0$$

on $\Sigma \backslash C(\Sigma)$ and

$$\lim_{\delta \to 0} \int_{\partial \Omega_\delta} \rho \langle \nu\, ,\, N_{\Omega_\delta} \rangle dV_{\Omega_\delta} = - \int_{\Sigma \backslash C(\Sigma)} div_\Sigma(\rho \nu^\top) dV_\Sigma = 0 \tag{5.5}$$

for all compactly supported C^1 functions ρ.

Proof. This follows from the fact that every compactly supported C^2 function can be realized as the variation function of a C^2 variation of Σ and our assumption (E) that the horizontal mean curvature is in L^1. □

Corollary 5.3. *A C^2 perimeter critical domain in the category of C^2 domains with volume constraint must have boundary Σ satisfying both $div\ \nu = c$ off $C(\Sigma)$ for some constant c and* (5.5)*.*

Proof. Take any open $\tilde{U} \subset M$ small enough so that $dV = d\mu$ is exact on $\tilde{U}$ and then set $U = \tilde{U} \cap \Sigma$. Then any variation with variation function ρ supported inside U must satisfy

$$\left.\frac{d}{dt}\right|_{t=0} \left(P_0(\Sigma_t) - \int_{\Sigma_t} c\mu \right) = 0$$

for some constant c. But

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \int_{\Sigma_t} c\mu &= \int_\Sigma c\mathcal{L}_{\partial_t}\mu = \int_\Sigma \partial_t \lrcorner dV + d(\partial_t \lrcorner \mu) \\ &= \int_\Sigma \partial_t \lrcorner \Theta \wedge dV_\Sigma = \int_\Sigma c\rho dV_\Sigma. \end{aligned}$$

Thus by using a partition of unity we have that

$$\int_{\Sigma \backslash C(\Sigma)} (\mathrm{div}\ \nu - c)\, dV_\Sigma - \lim_{\delta \to 0} \int_{\partial \Omega_\delta} \rho \langle \nu\, ,\, N \rangle dV_{\partial \Omega_\delta}$$

must vanish for all C^2 functions ρ on Σ. Here we can use the same constant c on each supporting patch of the partition as the constants must agree on overlaps.

□

Remark 5.3. For any C^2 hypersurface Σ such that the characteristic set has Hausdorff dimension $< n - 1$, the family Ω_δ can be chosen so that condition (5.5) is automatically satisfied. This follows easily from the observation that if we follow the construction of Lemma 5.4 then $\int_{\partial \Omega_\delta} dV_{\partial \Omega_\delta} \to 0$. For example, as seen in [2, 7, 26], in the Heisenberg groups $\mathbb{H}^m$ the characteristic set of any C^2 hypersurface has dimension $\leq m$. Thus for $m > 1$ there is no constraint on the characteristic set of minimal surfaces.

Remark 5.4. Let Σ be a critical point for perimeter variation (with or without volume constraint). Suppose that $p \in C(\Sigma)$ and in a small neighborhood U of p, $C(\Sigma)$ is an embedded submanifold of dimension $n - 1$ dividing U into two regions U^+ and U^-. If we further suppose that ν extends continuously to ν^+, ν^- on the boundaries of U^+ and U^- respectively, then

$$\langle \nu^+\, ,\, N_c \rangle - \langle \nu^-\, ,\, N_c \rangle = 0$$

where N_c is the normal to $C(\Sigma)$ in U pointing into U^-.

This follows immediately from the divergence integral form of (5.5). Note that, when restricted to the Heisenberg group, this was observed in [6, 33] where the structure of the characteristic locus is known to be either lower dimensional or of this form.

6. Application: a Minkowski formula for CMC surfaces

Throughout this section we shall suppose M is a sRC-manifold with vertically rigid extension g.

Definition 6.1. *M admits a dilating flow if there is a global orthonormal frame $\{T_\beta\}$ for V and a smooth map $D\colon M \times \mathbb{R} \longrightarrow M$ together with constants γ_β such that*

- $(D_\lambda)_*$ maps V_0 to V_0 for $j = 0, \ldots, L$,
- $\langle (D_\lambda)_* Y , (D_\lambda)_* Z \rangle_{D_\lambda p} = e^{2\lambda} \langle Y , Z \rangle_p$ for all sections Y, Z of V_0,
- $(D_\lambda)_* T_\beta = e^{\gamma_\beta \lambda} T_\beta$ for each β.

Associated to a dilating flow are the dilation operators defined by

$$\delta_\lambda = D_{\log \lambda}$$

and the generating vector field X defined by

$$X_p = \frac{d}{d\lambda}_{|\lambda=0} D_\lambda(p).$$

The associated homogeneous dimension of M is

$$Q = k + 1 + \sum_\beta \gamma_\beta.$$

For compactness of notation, we shall write λp for $D_\lambda(p)$ and $\lambda_* Y$ for $(D_\lambda)_* Y$.

A dilating flow can be lifted to a global flow $\widehat{D}$ on the contact bundle $\mathcal{S}_0(M)$. For $\widehat{p} = (p, \nu, a_\beta)$ we can define

$$\lambda \widehat{p} = \left(\lambda p, e^{-\lambda} \lambda_* \nu, e^{(1-\gamma_\beta)\lambda} a_\beta \right). \tag{6.1}$$

This lifts ensures that the middle term is still unit length and that if $\nu + a_\beta T_\beta$ is a normal vector for the surface $\Sigma = \{\phi = 0\}$, then $e^{-\lambda} \lambda_* \nu + e^{(1-\gamma_\beta)\lambda} a_\beta T_\beta$ is a normal vector to $\Sigma_\lambda = \{D^*_{-\lambda} \phi = 0\}$. We also note that $\pi \circ \widehat{D}_\lambda = D_\lambda \circ \pi$. The generator of the lifted flow will be denoted $\widehat{X}$.

Lemma 6.1. *The contact form $\theta_{\widehat{p}}(\widehat{Y}) = \langle \pi_* \widehat{Y} , \nu + a_\beta T_\beta \rangle_p$ has the property*

$$\mathcal{L}_{\widehat{X}} \theta = \theta.$$

Proof. We compute

$$\begin{aligned} (\widehat{D}^*_\lambda \theta)_{\widehat{p}}(\widehat{Y}) &= \theta_{\lambda \widehat{p}}(\lambda_* \widehat{Y}) = \langle \pi_* \lambda_* \widehat{Y} , e^{-\lambda} \lambda_* \nu + e^{(1-\gamma_\beta)\lambda} a_\beta T_\beta \rangle_{\lambda p} \\ &= e^{\lambda} \langle \pi_* \widehat{Y} , \nu + a_\beta T_\beta \rangle_p = e^{\lambda} \theta_{\widehat{p}}(\widehat{Y}). \end{aligned}$$

The result is a direct consequence.

□

Lemma 6.2. *The horizontal perimeter measure form Λ on $\mathcal{S}_0(M)$ has the following dilation property:*

$$\mathcal{L}_{\widehat{X}} \Lambda = (Q-1)\Lambda.$$

Proof. We first note that clearly

$$D^*_\lambda dV = e^{\lambda Q} dV.$$

Now

$$\begin{aligned}
(\widehat{D}^*_\lambda \Lambda)_{\widehat{p}} \left(\widehat{Y}_1, \dots, \widehat{Y}_n\right) &= \Lambda_{\lambda \widehat{p}} \left(\lambda_* \widehat{Y}_1, \dots \lambda_* \widehat{Y}_n\right) \\
&= \pi^* \left(e^{-\lambda} \lambda_* \nu \lrcorner dV\right)_{\lambda \widehat{p}} \left(\lambda_* \widehat{Y}_1, \dots \lambda_* \widehat{Y}_n\right) \\
&= e^{-\lambda} dV_{\lambda p} \left(\lambda_* \nu, \lambda_* \pi_* \widehat{Y}_1, \dots, \lambda_* \pi_* \widehat{Y}_n\right) \\
&= e^{-\lambda} (D_\lambda)^* dV_p \left(\nu, \pi_* \widehat{Y}_1, \dots \pi_* \widehat{Y}_n\right) \\
&= e^{(Q-1)\lambda} dV_p \left(\nu, \pi_* \widehat{Y}_1, \dots \pi_* \widehat{Y}_n\right) \\
&= e^{(Q-1)\lambda} \Lambda_{\widehat{p}} \left(\widehat{Y}_1, \dots \widehat{Y}_n\right).
\end{aligned}$$

The result immediately follows

□

In the presence of a dilation, we can define

$$\Upsilon = Q^{-1} X \lrcorner dV$$

so that $d\Upsilon = Q^{-1} \mathcal{L}_X dV = dV$ and $\mathcal{L}_X \Upsilon = Q\Upsilon$. We also define $\widehat{\Upsilon} = \pi^* \Upsilon$, the pullback of Υ to $\mathcal{S}_0(M)$. Since $\widehat{D}^*_\lambda \widehat{\Upsilon} = \pi^* D^*_\lambda \Upsilon$, we immediately see that $\mathcal{L}_{\widehat{X}} \widehat{\Upsilon} = Q\widehat{\Upsilon}$.

Now suppose Σ is a noncharacteristic C^2 hypersurface of M with constant mean curvature H. Then Σ embeds naturally as $\widehat{\Sigma}$ into $\mathcal{S}_0(M)$ and

$$\int_{\widehat{\Sigma}} \mathcal{L}_{\widehat{X}} (\Lambda - H\widehat{\Upsilon}) = \int_{\widehat{\Sigma}} (Q-1)\Lambda - QH\widehat{\Upsilon}.$$

But since $\pi^* dV = \theta \wedge \Lambda$ and $d\Lambda = H\pi^* dV$

$$\begin{aligned}
\int_{\widehat{\Sigma}} \mathcal{L}_{\widehat{X}} (\Lambda - H\widehat{\Upsilon}) &= \int_{\partial\widehat{\Sigma}} \widehat{X} \lrcorner \Lambda + \int_{\widehat{\Sigma}} \widehat{X} \lrcorner (H\pi^* dV) - HQ\widehat{\Upsilon} \\
&= \int_{\partial\widehat{\Sigma}} \widehat{X} \lrcorner \Lambda.
\end{aligned} \tag{6.2}$$

After pulling back to Σ along the natural inclusion into $\mathcal{S}_0(M)$, we have now established a Minkowski type identity for C^2 noncharacteristic patches. Namely

$$(Q-1)P_0(\Sigma) = (Q-1)\int_\Sigma \Lambda = QH \int_\Sigma \Upsilon + \int_{\partial\Sigma} X \lrcorner \Lambda. \tag{6.3}$$

Theorem 6.1. *Suppose M admits a dilating flow and Σ is a C^2 hypersurface with piecewise C^1 boundary such that H is constant off $C(\Sigma)$ and Σ satisfies the constraint* (5.5)*. Then*

$$(Q-1)P_0(\Sigma) = Q\int_\Sigma H\Upsilon + \int_{\partial\Sigma} X \lrcorner \Lambda.$$

Proof. As before we use the family of open sets Ω_δ containing $C(\Sigma)$ constructed in Lemma 5.4 and set $\Sigma_\delta = \Sigma - \Omega_\delta$.

Then by (6.3) we know that

$$(Q-1)P_0(\Sigma_\delta) = Q\int_{\Sigma_\delta} H\Upsilon + \int_{\partial\Sigma_\delta} X \lrcorner \Lambda. \tag{6.4}$$

But by an argument identical to 5.2, the internal portions of boundary integral will tend to zero as $\delta \to 0$, leaving the desired equality.

□

Corollary 6.1. *Suppose Ω is a compact C^2 domain with $\Sigma = \partial\Omega$ that is a critical point for perimeter measure with volume constraint. Then*

$$(Q-1)P_0(\Sigma) = QH\,Vol(\Omega).$$

For the Heisenberg groups, this result was first shown in [33].

7. The Second Variation

We shall now attempt the arduous task of describing a general second variation formula under the assumption of vertical rigidity. This unfortunately is just a long tedious exercise in computing derivatives using the structural equations of the adapted connection in $\mathcal{GF}_0(M)$. The underlying idea is differentiate on both the frame bundle and on $\Xi \times (-\epsilon, \epsilon)$ and compare results.

To aid with the long computations to follow, we shall briefly list the standing assumptions and notational conventions of this section. To save time and space, we shall also adopt the habit of absorbing all unnecessary terms that do not affect the relevant computations into "junk" collections.

(A) M is a vertically rigid subRiemannian manifold of dimension $n+1 = k+1+l$.
(B) Unless otherwise specified $\Sigma = \iota(\Xi)$ is a C^∞ hypersurface in M with no characteristic points.
(C) $F\colon \Xi \times (-\epsilon, \epsilon)$ is a $C^{\infty;3}$ noncharacteristic variation with $\widehat{F}^*\theta = \widehat{\rho}_0$ and $\rho_0 = \widehat{\rho}_0(\cdot, 0)$.
(D) And recall: roman indices run from $1\dots k$, barred roman indices from $0\dots k$ and greek indices from $1\dots l$.

We use the framing $dt, \widehat{F}^*\omega^j, \widehat{F}^*\eta^\alpha$ on $\Xi \times (-\epsilon, \epsilon)$ and will generally omit the $\widehat{F}^*$. We shall use the notation $\omega^\bullet = \omega^1 \wedge \cdots \wedge \omega^k$ and $\eta^\bullet = \eta^1 \wedge \cdots \wedge \eta^l$.

We define a variety of tensors by pulling back the structural equations to $\Xi \times (-\epsilon, \epsilon)$.

$$\begin{aligned}
\widehat{F}^*\omega^{\bar{i}}_{\bar{j}} &= \gamma^{\bar{i}}_{\bar{j}} dt + \Gamma^{\bar{i}}_{\bar{j}m}\omega^m + \Gamma^{\bar{i}}_{\bar{j}\alpha}\eta^\alpha \\
\widehat{F}^*\eta^\alpha_\beta &= \gamma^\alpha_\beta dt + \Gamma^\alpha_{\beta m}\omega^m + \Gamma^\alpha_{\beta\delta}\eta^\delta \\
\tau^{\bar{j}} &= A^{\bar{j}}_{\bar{i}\alpha}\omega^{\bar{i}} \wedge \eta^\alpha + B^{\bar{j}}_{\alpha\beta}\eta^\alpha \wedge \eta^\beta \\
\tilde{\tau}^\beta &= C^\beta_{\bar{j}\bar{i}}\omega^{\bar{j}} \wedge \omega^{\bar{i}} + D^\beta_{\bar{j}\alpha}\omega^{\bar{j}} \wedge \eta^\alpha + E^\beta_{\alpha\gamma}\eta^\alpha \wedge \eta^\gamma
\end{aligned}$$

with the understanding that each C^β, E^β and $B^{\bar{j}}$ are skew-symmetric. Vertical rigidity corresponds to $\sum_\beta D^\beta_{\bar{j}\beta} = 0$ for all $\bar{j}$.

Before diving into the main computation, we shall warm up by using our techniques to derive an integration by parts formula.

Definition 7.1. For a differential operator X on Σ we define the horizontal adjoint $X^{\#}$ by

$$\int_\Sigma fXh\,\Lambda_0 = \int_\Sigma hX^{\#}f\,\Lambda_0$$

for compactly supported functions f, h.

The key step to computing the horizontal adjoint of a vector field is the following computations on $\mathcal{GF}_0$:

First

$$\begin{aligned} d\eta^\bullet &= (-1)^{\beta-1}d\eta^\beta\wedge\eta^{(\beta)} = (-1)^{\beta-1}\left(\eta^\alpha\wedge\eta^\beta_\alpha + \tilde{\tau}^\beta)\right)\wedge\eta^{(\beta)} \\ &= -\eta^\beta_\beta\wedge\eta^\bullet + (-1)^{\beta-1}\tilde{\tau}^\beta\wedge\eta^{(\beta)} \\ &= (-1)^{\beta-1}\tilde{\tau}^\beta\wedge\eta^{(\beta)} \end{aligned} \tag{7.1}$$

and so, recalling that $D^\beta_{\bar{j}\beta}=0$ for all j, we see

$$\begin{aligned} d(\omega^{(j)}\wedge\eta^\bullet) &= d\omega^{(j)}\wedge\eta^\bullet + (-1)^{k-1}\omega^{(j)}\wedge d\eta^\bullet \\ &= (-1)^{m-1}\omega^{\bar{i}}\wedge\omega^m_{\bar{i}}\wedge\omega^{(m,j)}\wedge\eta^\bullet \\ &\quad + (-1)^{k+\beta}\omega^{(j)}\wedge\tilde{\tau}^\beta\wedge\eta^{(\beta)} \\ &= (-1)^{m-1}\omega^0\wedge\omega^m_0\wedge\omega^{(m,j)}\wedge\eta^\bullet + (-1)^{j+m}\omega^m_j\wedge\omega^{(m)}\wedge\eta^\bullet \\ &\quad + (-1)^{k+\beta+j-1}2C^\beta_{0j}\omega^0\wedge\omega^\bullet\wedge\eta^{(\beta)} \\ &\qquad + D^\beta_{0\beta}\omega^0\wedge\omega^{(j)}\wedge\eta^\bullet + (-1)^{j-1}D^\beta_{\bar{j}\beta}\omega^\bullet\wedge\eta^\bullet \\ &= (-1)^{m-1}\theta\wedge\omega^m_0\wedge\omega^{(m,j)}\wedge\eta^\bullet + (-1)^{j+m}\omega^m_j\wedge\omega^{(m)}\wedge\eta^\bullet \\ &\quad + (-1)^{k+\beta+j-1}2C^\beta_{0j}\theta\wedge\omega^\bullet\wedge\eta^{(\beta)} + (-1)^{j-1}2a_\beta C^\beta_{0j}\omega^\bullet\wedge\eta^\bullet. \end{aligned} \tag{7.2}$$

Lemma 7.1. *For each e_j with $j>0$ we have*

$$e_j^{\#} = -e_j - 2a_\beta C^\beta_{0j} - \Gamma^m_{jm}$$

Proof. First note that (7.2) implies

$$d(\omega^{(j)}\wedge\eta^\bullet)_{|\Sigma} = (-1)^{j-1}\left(2a_\beta C^\beta_{0j} + \Gamma^m_{jm}\right)\Lambda_0. \tag{7.3}$$

Thus

$$d(fh\omega^{(j)}\wedge\eta^\bullet)_{|\Sigma} = (-1)^{j-1}(fe_jh + he_jf)\Lambda_0 + (fh)d(\omega^{(j)}\wedge\eta^\bullet)_{|\Sigma}.$$

Thus

$$\int fe_jh\Lambda_0 = \int(-he_jf - 2fha_\beta C^\beta_{0j} - fh\Gamma^m_{jm})\Lambda_0.$$

□

Now we return to the derivation of a second variation formula. We begin by computing $\widehat{F}^* d\theta = d\widehat{F}^*\theta$ in two different ways and equating the results. Firstly

$$
\begin{aligned}
d\theta &= d(\omega^0 + a_\beta \eta^\beta)\\
&= \omega^j \wedge \omega_j^0 + \tau^0 + da_\beta \wedge \eta^\beta + a_\beta \eta^\alpha \wedge \eta_\alpha^\beta + a_\beta \tilde{\tau}^\beta\\
&= -\omega^j \wedge \omega_0^j + A_{0\alpha}^0 \omega^0 \wedge \eta^\alpha + A_{j\alpha}^0 \omega^j \wedge \eta^\alpha + da_\beta \wedge \eta^\beta\\
&\qquad + a_\beta \Gamma_{\alpha\bar{j}}^\beta \eta^\alpha \wedge \omega^{\bar{j}} + a_\beta 2C_{0j}^\beta \omega^0 \wedge \omega^j\\
&\qquad + a_\beta D_{0\alpha}^\beta \omega^0 \wedge \eta^\alpha + a_\beta D_{j\alpha}^\beta \omega^j \wedge \eta^\alpha\\
&\qquad + \eta^\alpha \wedge \eta^\beta \cdot \mathrm{junk} + \omega^j \wedge \omega^m \cdot \mathrm{junk}\\
&= -\omega^j \wedge \omega_0^j + \omega^j \wedge \theta\big(- 2a_\beta C_{0j}^\beta\big)\\
&\quad + \omega^j \wedge \eta^\alpha\big(- a_\beta \Gamma_{\alpha j}^\beta + A_{j\alpha}^0 + (e_j a_\alpha) + a_\beta D_{j\alpha}^\beta + 2a_\alpha a_\beta C_{0j}^\beta\big)\\
&\qquad + \theta \wedge \eta^\alpha \cdot \mathrm{junk} + \eta^\alpha \wedge \eta^\beta \cdot \mathrm{junk} + \omega^j \wedge \omega^m \cdot \mathrm{junk}.
\end{aligned}
\tag{7.4}
$$

Thus

$$
\begin{aligned}
\widehat{F}^* d\theta &= \omega^j \wedge dt\big(- \gamma_0^j - 2a_\beta C_{0j}^\beta\big)\\
&\quad + \omega^j \wedge \eta^\alpha\big(- \Gamma_{0\alpha}^j - a_\beta \Gamma_{\alpha j}^\beta + A_{j\alpha}^0 + (e_j a_\alpha) + a_\beta D_{j\alpha}^\beta + 2a_\alpha a_\beta C_{0j}^\beta\big)\\
&\qquad + dt \wedge \eta^\alpha \cdot \mathrm{junk} + \eta^\alpha \wedge \eta^\beta \cdot \mathrm{junk} + \omega^j \wedge \omega^m \cdot \mathrm{junk}.
\end{aligned}
\tag{7.5}
$$

But from the definitions we see that $\widehat{F}^*\theta = \widehat{\rho}_0 dt$ so

$$
d(\widehat{F}^*\theta) = d\widehat{\rho}_0 \wedge dt = (e_j \widehat{\rho}_0)\omega^j \wedge dt + \eta^\beta \wedge dt \cdot \mathrm{junk}.
$$

Comparing with (7.5) thus yields

$$
\begin{aligned}
e_j \widehat{\rho}_0 &= -\gamma_0^j - 2\widehat{\rho}_0 a_\beta C_{0j}^\beta\\
0 &= -\Gamma_{0\alpha}^j - a_\beta \Gamma_{\alpha j}^\beta + A_{j\alpha}^0 + (e_j a_\alpha) + a_\beta D_{j\alpha}^\beta + 2a_\alpha a_\beta C_{0j}^\beta.
\end{aligned}
\tag{7.6}
$$

Using metric compatibility of the connection thus yields the following useful identities

$$
\begin{aligned}
\gamma_0^j &= -e_j \widehat{\rho}_0 - 2\widehat{\rho}_0 a_\beta C_{0j}^\beta\\
\Gamma_{0\alpha}^j &= -a_\beta \Gamma_{\alpha j}^\beta + A_{j\alpha}^0 + a_\beta D_{j\alpha}^\beta + (e_j a_\alpha) + 2a_\alpha a_\beta C_{0j}^\beta.
\end{aligned}
\tag{7.7}
$$

Returning to the main computation. Recall that, since we are assuming vertical rigidity,

$$
\Psi = (-1)^{j-1} \omega_0^j \wedge \omega^{(j)} \wedge \eta^\bullet
$$

and so

$$
\partial_t \lrcorner \widehat{F}^* \Psi = (-1)^{j-1} \gamma_0^j \wedge \omega^{(j)} \wedge \eta^\bullet. \tag{7.8}
$$

In particular, this implies that

$$
\begin{aligned}
d(\partial_t \lrcorner \widehat{F}^* \Psi) &= (-1)^{j-1} d\gamma^j \wedge \omega^{(j)} \wedge \eta^\bullet + (-1)^{j-1}\gamma_0^j d(\omega^{(j)} \wedge \eta^\bullet) \\
d(\partial_t \lrcorner \widehat{F}^* \Psi)_{|\Sigma} &= \left(e_j \gamma_0^j + \gamma_0^j \Gamma_{jm}^m + 2\gamma_0^j a_\beta C_{0j}^\beta\right)\Lambda_0 \\
&= \left[e_j(-e_j\widehat{\rho}_0 - 2\widehat{\rho}_0 a_\beta C_{0j}^\beta) + (-e_j\widehat{\rho}_0 - 2\widehat{\rho}_0 a_\beta C_{0j}^\beta)(\Gamma_{jm}^m + 2a_\beta C_{0j}^\beta)\right]\Lambda_0 \\
&= \Big[-e_j^2\widehat{\rho}_0 - 4a_\beta C_{0j}^\beta e_j \widehat{\rho}_0 - \Gamma_{jm}^m e_j\widehat{\rho}_0 - 2\widehat{\rho}_0 e_j(a_\beta C_{0j}^\beta) \\
&\qquad - 4\widehat{\rho}_0 (a_\beta C_{0j}^\beta)^2 - 2\widehat{\rho}_0 \Gamma_{jm}^m a_\beta C_{0j}^\beta\Big]\Lambda_0.
\end{aligned}
\tag{7.9}
$$

Also if we define curvature 2-forms by

$$
d\omega_{\bar{j}}^{\bar{i}} = \omega_{\bar{j}}^{\bar{m}} \wedge \omega_{\bar{m}}^{\bar{i}} + \Omega_{\bar{j}}^{\bar{i}}
\tag{7.10}
$$

then

$$
\begin{aligned}
d\Psi &= (-1)^{j-1} d(\omega_0^j \wedge \omega^{(j)} \wedge \eta^\bullet) \\
&= (-1)^{j-1}\left(\omega_0^m \wedge \omega_m^j + \Omega_0^j\right) \wedge \omega^{(j)} \wedge \eta^\bullet + (-1)^j \omega_0^j \wedge d(\omega^{(j)} \wedge \eta^\bullet) \\
&= (-1)^{j-1}\left(\omega_0^m \wedge \omega_m^j + \Omega_0^j\right) \wedge \omega^{(j)} \wedge \eta^\bullet \\
&\quad + (-1)^{m+j}\theta \wedge \omega_0^j \wedge \omega_0^m \wedge \omega^{(m,j)} \wedge \eta^\bullet + (-1)^m \omega_0^j \wedge \omega_j^m \wedge \omega^{(m)} \wedge \eta^\bullet \\
&\quad + (-1)^{k+\beta} 2C_{0j}^\beta \theta \wedge \omega_0^j \wedge \omega^\bullet \wedge \eta^{(\beta)} - 2a_\beta C_{0j}^\beta \omega_0^j \wedge \omega^\bullet \wedge \eta^\bullet \\
&= (-1)^{j-1}\Omega_0^j \wedge \omega^{(j)} \wedge \eta^\bullet + (-1)^{m+j}\theta \wedge \omega_0^j \wedge \omega_0^m \wedge \omega^{(m,j)} \wedge \eta^\bullet \\
&\quad + (-1)^{k+\beta} 2C_{0j}^\beta \theta \wedge \omega_0^j \wedge \omega^\bullet \wedge \eta^{(\beta)} - 2a_\beta C_{0j}^\beta \omega_0^j \wedge \omega^\bullet \wedge \eta^\bullet.
\end{aligned}
\tag{7.11}
$$

Thus

$$
\begin{aligned}
\partial_t \lrcorner \widehat{F}^* d\Psi = \Big[&\widehat{\rho}_0\big(2\Omega_{00j}^j - \Gamma_{0m}^j \Gamma_{0j}^m + \Gamma_{0j}^j \Gamma_{0m}^m \\
&- 2C_{0j}^\beta \Gamma_{0\beta}^j\big) - 2a_\beta C_{0j}^\beta \gamma_0^j\Big]\omega^\bullet \wedge \eta^\bullet.
\end{aligned}
\tag{7.12}
$$

When we restrict to Σ we can use (7.7) and the facts that $H = -\Gamma_{0j}^j$ and $\rho_0 = \widehat{\rho}_0(\cdot, 0)$ to see

$$
\begin{aligned}
(\partial_t \lrcorner \widehat{F}^* d\Psi)_{|\Sigma} &= \rho_0\Big(\big(2\Omega_{00j}^j - \Gamma_{0m}^j \Gamma_{0j}^m + H^2\big) \\
&\qquad - 2a_\beta C_{0j}^\beta(-e_j\rho_0 - 2\rho_0 a_\beta C_{0j}^\beta) \\
&\qquad - 2\rho_0 C_{0j}^\beta(-a_\delta \Gamma_{\beta j}^\delta + A_{j\beta}^0 + a_\gamma D_{j\beta}^\gamma + (e_j a_\beta) + 2a_\beta a_\alpha C_{0j}^\alpha)\Big)\Lambda_0 \\
&= e_j\rho_0\Big(2a_\beta C_{0j}^\beta\Big)\Lambda_0 \\
&\quad + \rho_0\Big(2\Omega_{00j}^0 - \Gamma_{jm}^0 \Gamma_{mj}^0 + H^2 + 2C_{0j}^\beta a_\delta \Gamma_{\beta j}^\delta \\
&\qquad - 2C_{0j}^\beta A_{j\beta}^0 - 2a_\gamma D_{j\beta}^\gamma C_{0j}^\beta - 2C_{0j}^\beta(e_j a_\beta)\Big)\Lambda_0.
\end{aligned}
\tag{7.13}
$$

We now encode all this computation in the following lemma.

Lemma 7.2. *Suppose M is a vertically rigid sRC-manifold and F is a noncharacteristic $C^{\infty;3}$ variation of $\Sigma\backslash C(\Sigma)$ with compactly supported horizontal variation* *function. Then*

$$
\begin{aligned}
\frac{d^2}{dt^2}_{\big|t=0} P_0(\Sigma_t) &= \int_\Xi [(\partial_t\widehat{\rho}_0)H\Lambda]_{\big|t=0} + \int_\Sigma \left|\nabla^{0,\Sigma}\rho_0\right|^2 \Lambda \\
&\quad + \int_\Sigma \rho_0^2 \big[- Ric^{\nabla}(\nu,\nu) - tr(I\!I_0^\top I\!I_0) + H^2 \\
&\qquad + \langle tr_0\,(TOR_2 - \nabla\, Tor)\,(\nu), \mathcal{Y}\rangle \\
&\qquad - 2\langle Tor(\nu,e_j), \nabla_j \mathcal{Y}\rangle - \langle Tor(\nu,e_j),\mathcal{Y}\rangle^2\Big]\Lambda
\end{aligned} \tag{7.14}
$$

where $\mathcal{Y} = \nu + \alpha_\beta T_\beta$.

Proof. As was shown Theorem 5.1

$$\frac{d}{dt}P_0(\Sigma_t) = \int_\Xi \widehat{\rho}_0 F_t^* \Psi.$$

So

$$
\begin{aligned}
\frac{d^2}{dt^2}_{\big|t=0} P_0(\Sigma_t) &= \left[\int_\Xi \mathcal{L}_{\partial_t}(\widehat{\rho}_0 F_t^*\Psi)\right]_{\big|t=0} \\
&= \int_\Xi \left[(\partial_t\widehat{\rho}_0)H + \rho_0 \mathcal{L}_{\partial_t}(\widehat{F}^*\Psi)\right]_{\big|t=0}. \\
&= \int_\Xi \left[(\partial_t\widehat{\rho}_0)H\Lambda + \rho_0 \mathcal{L}_{\partial_t}(\widehat{F}^*\Psi)\right]_{\big|t=0}.
\end{aligned}
$$

Now as previously shown, locally

$$
\begin{aligned}
\left[\mathcal{L}_{\partial_t}\widehat{F}^*\Psi\right]_{\big|t=0} &= \Big[\sum_{j=1}^k e_j^{\#} e_j \rho_0 + \rho_0\big(2\Omega^j_{00j} - \Gamma^0_{jm}\Gamma^0_{mj} + H^2 \\
&\quad - 2C^\beta_{0j}A^0_{j\beta} - 2a_\gamma D^\gamma_{j\beta}C^\beta_{0j} - 2e_j(a_\beta C^\beta_{0j}) - 2C^\beta_{0j}(e_j a_\beta) \\
&\quad - 4(a_\beta C^\beta_{0j})^2\big) - 2\Gamma^m_{jm} a_\beta C^\beta_{0j} + 2C^\beta_{0j} a_\delta \Gamma^\delta_{\beta j}\Big]\Lambda.
\end{aligned} \tag{7.15}
$$

The Lemma is completed by converting to the invariant form and integrating by parts once.

First we note that $[e_i, e_j]$ is always tangent to the hypersurface so, modulo terms in $H \cap T\Sigma$, is in the span of the vector fields $\alpha_\beta\nu - T_\beta$. Thus

$$\Gamma^0_{jm} - \Gamma^0_{mj} = \langle [e_m, e_j], \nu\rangle = \alpha_\beta \langle \mathrm{Tor}(e_m, e_j), T_\beta\rangle = 2\alpha_\beta C^\beta_{mj}$$

and so

$$\Gamma^0_{jm}\Gamma^0_{mj} = \mathrm{tr}(I\!I_0^\top I\!I_0) + (\Gamma^0_{jm} - \Gamma^0_{mj})\Gamma^0_{mj} = \mathrm{tr}(I\!I_0^\top I\!I_0) + 2\alpha_\beta \Gamma^0_{mj} C^\beta_{mj}.$$

Next we can compute directly, that

$$
\begin{aligned}
\langle \mathrm{tr}_0 \mathrm{TOR}_2(\nu), \mathcal{Y}\rangle &= \langle \mathrm{Tor}(e_i, \mathrm{Tor}(e_i,\nu)), \mathcal{Y}\rangle \\
&= \langle \mathrm{Tor}(e_i, 2C^\beta_{i0}T_\beta), \mathcal{Y}\rangle \\
&= \langle 2A^{\bar{j}}_{i\beta} C^\beta_{i0} e_{\bar{j}} + 2D^\gamma_{i\beta} C^\beta_{i0} T_\gamma, e_0 + \alpha_\beta T_\beta\rangle \\
&= -2A^0_{i\beta} C^\beta_{0i} - 2\alpha_\gamma D^\gamma_{i\beta} C^\beta_{0i},
\end{aligned}
$$

$$\begin{aligned}\langle\big(\mathrm{tr}_0\nabla\mathrm{Tor}\big)(\nu),\mathcal{Y}\rangle &= \langle\nabla\mathrm{Tor}(\nu,e_i,e_i),\mathcal{Y}\rangle\\ &= \langle\nabla_i\mathrm{Tor}(\nu,e_i)-\mathrm{Tor}(\nabla_i\nu,e_i)-\mathrm{Tor}(\nu,\nabla_i e_i),\mathcal{Y}\rangle\\ &= e_j\langle\mathrm{Tor}(\nu,e_j),\mathcal{Y}\rangle-\langle\mathrm{Tor}(\nu,e_i),\nabla_j\mathcal{Y}\rangle\\ &\quad+\langle-2\Gamma^m_{0i}C^\beta_{mi}T_\beta-2\Gamma^m_{ii}C^\beta_{0m}T_\beta,\mathcal{Y}\rangle\\ &= e_j\left(2\alpha_\beta C^\beta_{0j}\right)-\langle\mathrm{Tor}(\nu,e_i),\nabla_j\mathcal{Y}\rangle\\ &\quad-2\alpha_\beta\Gamma^m_{0i}C^\beta_{mi}-2\alpha_\beta\Gamma^m_{ii}C^\beta_{0m}\\ &= 2e_j\left(\alpha_\beta C^\beta_{0j}\right)-\langle\mathrm{Tor}(\nu,e_i),\nabla_j\mathcal{Y}\rangle\\ &\quad+2\alpha_\beta\Gamma^0_{mi}C^\beta_{mi}+2\alpha_\beta\Gamma^i_{mi}C^\beta_{0m}\end{aligned}$$

and

$$\begin{aligned}\langle\mathrm{Tor}(\nu,e_i),\nabla_i\mathcal{Y}\rangle &= \langle 2C^\beta_{0i}T_\beta, e_j(\alpha_\gamma)T_\gamma+\alpha_\gamma\nabla_j T_\gamma\rangle\\ &= 2C^\beta_{0i}e_i(\alpha_\beta)-2C^\beta_{0i}\Gamma^\gamma_{\beta i}\alpha_\gamma.\end{aligned}$$

Putting all of this together completes the proof.

□

Theorem 7.1. *Suppose M is a vertically rigid subRiemannian manifold, Σ is a C^2 hypersurface and F is a noncharacteristic $C^{1;3}$ variation of $\Sigma\backslash C(\Sigma)$. Then whenever either of the following holds*

- *$H=0$ on $\Sigma\backslash C(\Sigma)$*
- *H is constant on $C(\Sigma)$ and F preserves $\int_\Xi F_t^*\mu$ for any smooth form with $\mu=dV$.*

we have

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0}P_0(\Sigma_t)=\int_\Sigma\Big(&\left|\nabla^{0,\Sigma}\rho_0\right|^2-\rho_0^2\big[Ric^\nabla(\nu,\nu)+tr(\mathit{II}_0^\top\mathit{II}_0)\\ &+\langle tr_0\left(\nabla Tor-TOR_2\right)(\nu),\mathcal{Y}\rangle\\ &+2\langle Tor(\nu,e_j),\nabla_j\mathcal{Y}\rangle+\langle Tor(\nu,e_j),\mathcal{Y}\rangle^2\big]\Big)\Lambda.\end{aligned}$$

Proof. Since F is supported away from the characteristic set, we have that

$$\frac{d}{dt}\int_\Xi F_t^*\mu=\int_\Xi\mathcal{L}_{\partial_t}F_t^*\mu=\int_\Xi\widehat{\rho}_0(\xi,t)\Lambda_t.$$

Thus if F preserves volume then $\int_\Xi\widehat{\rho}_0(\xi,t)\Lambda_t=0$. If H is constant then differentiating yields

$$\int_\Xi(\partial_t\widehat{\rho}_0)H\Lambda_0+\int_\Xi\rho_0^2H^2\Lambda_0=0.$$

Therefore the effect of either condition is that the first term of (7.14) cancels the $+\rho_0^2H^2$ term within the second integral. Thus the theorem is proved for $C^{\infty;3}$ variations.

All that remains is to show that the result still holds with the restricted regularity conditions. The difficulty is that for the computations to hold, we must have ρ_0 being C^2 on Ξ, whereas for a $C^{1;3}$ variation we can only guarantee that $\widehat{\rho}_0$ is continuous. However since F_0 itself is C^2 we see ρ_0 is C^1. Fortunately, the right hand side of (7.14) requires only C^1 regularity in ρ_0. All the other terms are in fact tensorial, so the restricted regularity will not cause problems.

Now note that

$$\widehat{F}_t^* \Lambda = \lambda(\xi, t) d\xi^1 \wedge \dots d\xi^n.$$

Furthermore since Λ is semibasic, we see by Corollary 4.1 that λ is $C^{0;2}$. The second variation functional

$$F \mapsto \frac{d^2}{dt^2}\Big|_{t=0} P_0(F_t(\Xi))$$

is therefore continuous from $C^{1;3}$ variations to $\mathbb{R}$. By Lemma 4.1 we see that we can approximate F by $C^{\infty;3}$ variations such that the restrictions to $t=0$ converge in C^2 to F_0. The second variation formula of Lemma 7.2 holds for these approximations and the formula itself is continuous as a functional on C^2 embeddings.

□

8. Examples

This second variation formula is hideously complicated in general so we shall attempt to illuminate it with some remarks and examples.

Firstly, recall that the horizontal second fundamental form is asymmetric but does have real valued entries. Thus its eigenvalues $\lambda_1, \dots \lambda_k$ are either real or come in conjugate pairs. From elementary linear algebra we can then deduce

$$\begin{aligned} H = \operatorname{Trace}(I\!I_0) &= \sum_{j=1}^k \lambda_j \\ \operatorname{Trace}(I\!I_0^\top I\!I_0) &= \sum_{j=1}^k |\lambda_j|^2 . \end{aligned} \tag{8.1}$$

8.1. **Strictly pseudoconvex pseudohermitian manifolds.** Recall a pseudohermitian manifold (M, J, η) consists of:

- a $2n+1$-dimensional smooth manifold M
- a non-vanishing 1-form η defining the horizontal distribution $V_0 = \ker(\eta)$
- a bundle map $J : V_0 \to V_0$ such that $J^2 = -1$

with the integrability condition that the Nijenhuis tensor (see [37]) vanishes. The manifold is strictly pseudoconvex if the Levi metric

$$g(X,Y) = d\eta(X, JY) + \eta(X)\eta(Y)$$

is positive definite. In this instance, a rigid vertical structure can be imposed by taking T to be the Reeb vector field of η, i.e. $\eta(T) = 1$ and $T \lrcorner d\eta = 0$. The canonical connection for this extension is the Tanaka-Webster connection ([21], [37], [38]). This connection has the properties that $\operatorname{Tor}(X,Y) = \langle JX, Y\rangle T$, $\nabla T = 0$ and $\nabla J = 0$.

Lemma 8.1. *For the Tanaka-Webster connection and horizontal vector fields X, Y and Z*

$$\nabla\, Tor(X,Y,Z) = 0.$$

Proof. Using the properties of the Tanaka-Webster connection, we see

$$\begin{aligned} \nabla \operatorname{Tor}(X,Y,Z) &= (Z\langle JX, Y\rangle)T - \langle J\nabla_Z X, Y\rangle T - \langle JX, \nabla_Z Y\rangle T \\ &= (Z\langle JX, Y\rangle - \langle \nabla_Z JX, Y\rangle - \langle JX, \nabla_Z Y\rangle)\, T \\ &= 0. \end{aligned}$$

□

Now,

$$\mathrm{TOR}_2(X,X,Y) = \mathrm{Tor}(X, \langle JX, Y\rangle T) = \langle JX, Y\rangle \mathrm{Tor}(X,T).$$

So

$$\mathrm{tr}_0 \mathrm{TOR}_2(\nu) = -\mathrm{Tor}(J\nu, T).$$

Since in this setting $\mathcal{Y} = \nu + aT$ and $\mathrm{Tor}(\nu, e_i) = \langle J\nu, e_i\rangle T$, the second variation formula reduces to

$$\begin{aligned}\frac{d^2}{dt^2}_{|t=0} P_0(\Sigma_t) = \int_\Sigma \Big[&|\nabla^{0,\Sigma}\rho_0|^2 - \rho_0^2\big(\mathrm{Ric}^\nabla(\nu,\nu) + \mathrm{Tr}(I\!I_0^\top I\!I_0) \\ &+ \langle \mathrm{Tor}(J\nu, T), \nu\rangle + 2(J\nu)a + a^2\big)\Big]\Lambda.\end{aligned} \tag{8.2}$$

If the pseudohermitian structure is normal (i.e. $\nabla_T X = [T, X]$, see [37] for equivalent definitions and consequences) then the torsion term vanishes. For the case $n = 1$ this example first appeared in [6], although it should be noted that their presentation of pseudohermitian manifolds causes $C_{01} = 1$ instead.

A few particular examples are especially important in the literature:

8.2. **The Heisenberg Group.** $\mathbb{H}^n = \mathbb{R}^{2n}_{x,y} \times \mathbb{R}_t$ with the horizontal distribution spanned by

$$X_j = \partial_{x^j} - \frac{1}{2} y^j \partial_t, \quad Y_j = \partial_{y^j} + \frac{1}{2} x^j \partial_t$$

is an example of both a (normal) strictly pseudoconvex pseudohermitian manifold (with $\eta = dt + \frac{1}{2} y^j dx^j - \frac{1}{2} x^j dy^j$, $JX_j = -Y_j$ and Reeb field $T = \partial_t$) and a Carnot Group. However, the curvature and the horizontal torsion both vanish identically so the second variation becomes

$$\frac{d^2}{dt^2}_{|t=0} P_0(\Sigma_t) = \int_\Sigma \Big[|\nabla^{0,\Sigma}\rho_0|^2 - \rho_0^2\big(\mathrm{tr}(I\!I_0^\top I\!I_0) + 2(J\nu)a + a^2\big)\Big]\Lambda.$$

For $n = 1$, this example was first shown by Danielli, Garofalo and Nhieu in [11].

8.3. **Carnot Groups.** Suppose that for the sRC-manifold M there are global orthonormal frames $\{X_i\}$ and $\{T_\alpha\}$ for V_0 and V respectively and function $L\colon \{\alpha\} \to \mathbb{Z}_{>0}$ such that there are constants c^α_{ij}, $c^\beta_{i\alpha}$ with

$$\begin{aligned}[X_i, X_j] &= c^\alpha_{ij} T_\alpha \\ [X_i, T_\alpha] &= c^\beta_{i\alpha} T_\beta \\ L(\alpha) > 1 &\Rightarrow c^\alpha_{ij} = 0 \\ L(\beta) \neq L(\alpha) + 1 &\Rightarrow c^\beta_{i\alpha} = 0.\end{aligned}$$

All Carnot groups have the above property. It is easy to see that M is vertically rigid and that the canonical connection can be computed explicity (see [21]) as

- $\nabla X_j = 0$
- $\mathrm{Tor}(X_i, X_j) = -c^\alpha_{ij} T_\alpha$
- $\nabla_{X_j} T_\beta = \frac{1}{2}\left(c^\alpha_{j\beta} - c^\beta_{j\alpha}\right) T_\alpha$
- $\mathrm{Tor}(X_j, T_\beta) = -\frac{1}{2}\left(c^\alpha_{j\beta} + c^\beta_{j\alpha}\right) T_\alpha$.

Set

$$\tau_{ijk}^{\beta} = c_{ij}^{\alpha} c_{k\alpha}^{\beta}$$

Note that this the sum over α restricts to a sum of ua with $L(\alpha) = 1$ and that the outcome will be zero unless $L(\beta) = 2$. Then

$$\nabla \mathrm{Tor}(X_i, X_j, X_k) = -c_{ij}^{\alpha} \nabla_{X_k} T_\alpha = -\frac{1}{2} \tau_{ijk}^{\beta} T_\beta,$$

$$\mathrm{TOR}_2(X_i, X_j, X_k) = \frac{1}{2} \tau_{jki}^{\beta} T_\beta.$$

Thus in particular

$$\mathrm{TOR}_2(X_k, X_i, X_j) - \nabla \mathrm{Tor}(X_i, X_j, X_k) = \tau_{ijk}^{\beta} T_\beta.$$

The curvature term also clearly vanishes. Now set $\nu = \bar{p}^i X_i$, then

$$\begin{aligned} \mathrm{tr}_0(\mathrm{TOR}_2 - \nabla \mathrm{Tor})(\nu) &= \sum_j \mathrm{TOR}_2(e_j, e_j, \nu) - \nabla \mathrm{Tor}(\nu, e_j, e_j) \\ &= \bar{p}^i \tau_{ijj}^{\beta} T_\beta \end{aligned}$$

and

$$\begin{aligned} \sum_j \langle \mathrm{Tor}(\nu, e_j) \, , \, \nabla_{e_j} \mathcal{Y} \rangle &= \langle \mathrm{Tor}(\bar{p}^i X_i, X_j) \, , \, X_j(\alpha_\gamma) T_\gamma + \alpha_\gamma \nabla_{X_j} T_\gamma \rangle \\ &= \langle -\bar{p}^i c_{ij}^{\beta} T_\beta \, , \, X_j(\alpha_\gamma) T_\gamma + \frac{1}{2} \alpha_\gamma (c_{j\gamma}^{\delta} - c_{j\delta}^{\gamma}) T_\delta \rangle \\ &= -\bar{p}^i c_{ij}^{\beta} (X_j \alpha_\beta) + \frac{1}{2} \bar{p}^i \alpha_\gamma \tau_{ijj}^{\gamma}. \end{aligned}$$

Finally

$$\begin{aligned} \sum_j \langle \mathrm{Tor}(\nu, e_j) \, , \, \mathcal{Y} \rangle^2 &= \sum_j \left(\langle e_j \, , \, X_k \rangle \langle \mathrm{Tor}(\nu, X_k) \, , \, \mathcal{Y} \rangle \right)^2 \\ &= \sum_{j,k,m} \langle e_j \, , \, X_k \rangle \langle e_j \, , \, X_m \rangle \left(\bar{p}^i c_{ik}^{\beta} \alpha_\beta \right) \left(\bar{p}^l c_{lm}^{\gamma} \alpha_\gamma \right) \\ &= \sum_{k,m} \left(\langle X_k \, , \, X_m \rangle - \langle X_k \, , \, \nu \rangle \langle X_m \, , \, \nu \rangle \right) \left(\bar{p}^i c_{ik}^{\beta} \alpha_\beta \right) \left(\bar{p}^l c_{lm}^{\gamma} \alpha_\gamma \right) \\ &= \sum_{k,m} \left(\delta_{k,m} - \bar{p}^k \bar{p}^m \right) \left(\bar{p}^i c_{ik}^{\beta} \alpha_\beta \right) \left(\bar{p}^l c_{lm}^{\gamma} \alpha_\gamma \right) \\ &= \bar{p}^i \bar{p}^l c_{ik}^{\beta} c_{lk}^{\gamma} \alpha_\beta \alpha_\gamma \end{aligned}$$

as the other term vanishes due to skew-symmetry of the c_{ik}^{β} terms. Putting all of this together second variation formula becomes

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} P_0(\Sigma_t) = \int_\Sigma \Big(\left| \nabla^{0,\Sigma} \rho_0 \right|^2 - \rho_0^2 \Big[&\mathrm{tr}(I\!I_0^\top I\!I_0) \\ &+ 2\bar{p}^i (X_j \alpha_\gamma) c_{ij}^{\gamma} + \bar{p}^i \bar{p}^l c_{ik}^{\beta} c_{lk}^{\gamma} \alpha_\beta \alpha_\gamma \Big] \Big) \Lambda. \end{aligned}$$

Appendix A. Size of the characteristic set

In this section we prove

Theorem A.1. *Suppose $X_1, X_2, \dots X_k$ are smooth globally defined vector fields on $\mathbb{R}^n$ that bracket generate at every point. Then for any C^2 hypersurface Σ, the characteristic set*

$$C(\Sigma) = \{p \in \Sigma \colon (X_j)\Big|_p \in T_p\Sigma \text{ for all } 1 \leq j \leq k\}$$

has Hausdorff dimension $\leq n-2$.

The size and nature of the characteristic locus has been studied widely [2, 15, 25] in various contexts. We include a discussion here for completeness and because, to the best of our knowledge, a complete argument for general sub-Riemannian spaces does not appear in the current literature.

The proof is based on a series of technical lemmas.

Lemma A.1. *Suppose f is a C^1 function on $\mathbb{R}^m$. Then the set*

$$V = \{p \colon f(p) = 0,\ df_{|p} \neq 0\}$$

has Hausdorff dimension $\leq m-1$.

Proof. Fix $p \in V$. Then since the set $K = \{p \colon f(p) = 0,\ df_{|p} = 0\}$ is closed, we can find an open set U containing p such that $f_{|U}$ is a C^1 submersion from U into $\mathbb{R}$. The constant rank theorem implies that $U \cap V$ is a closed embedded submanifold of U. The set $U \cap V$ thus has Hausdorff dimension $m-1$ as a subset of U (and hence as a subset of $\mathbb{R}^m$.)

Therefore we can cover V by open sets U_α such that each $V \cap U_\alpha$ has dimension $\leq m-1$. Since every subset of $\mathbb{R}^m$ is second countable we can find a countable subcover by the Lindelöf theorem. Thus we can express V as a countable union of sets of dimension $\leq m-1$, which is sufficient to prove the result.

□

Our next lemma is a refinement of a result due to Derridj, Lemma 1 in [15].

Lemma A.2. *Suppose Σ is a C^2 hypersurface in $\mathbb{R}^n$ and X and Y are smooth vector fields. Then the set*

$$V = \{p \in \Sigma \colon X_p, Y_p \in T_p\Sigma, [X,Y]_p \notin T_p\Sigma\}$$

has Hausdorff dimension $\leq n-2$.

Proof. Locally we can introduce C^2 slice coordinates $(y, x^1, \dots, x^{n-1})$ so that $\Sigma = \{y = 0\}$. Rewrite X and Y in these coordinates as

$$X = a\partial_y + a^i \partial_{x^i},$$
$$Y = b\partial_y + b^i \partial_{x^i}.$$

Then

$$[X,Y] = \left(a\frac{\partial b}{\partial y} - b\frac{\partial a}{\partial y} + a^i \frac{\partial b}{\partial x^i} - b^i \frac{\partial a}{\partial x^i} \right) \partial_y \mod \partial_{x^1}, \dots \partial_{x^{n-1}}.$$

The condition that $p = (0, p') \in V$ is therefore equivalent to

$$\begin{cases} a(p) = 0 \\ b(p) = 0 \\ \left(a^i \frac{\partial b}{\partial x^i} - b^i \frac{\partial a}{\partial x^i}\right)(p) \neq 0 \end{cases}.$$

Set $a' = a_{|\Sigma}$, $b' = b_{|\Sigma}$. The portion of V lying inside the slice coordinate chart must be contained in $\{a' = 0, da' \neq 0\} \cup \{b' = 0, db' \neq 0\}$. Since a' and b' are (at least) C^1 functions on $\mathbb{R}^{n-1}$, the result now follows from Lemma A.1.

□

Proof. (of Theorem A.1) Generate the countable collection of all vector fields that can be bracket generated by $X_1, \ldots X_k$ and enumerate them as

$$X_1, \ldots X_k, \ldots, X_\alpha, \ldots$$

with the first k matching the original vector fields. Define

$$E_{\alpha\beta} = \{p \in \Sigma\colon (X_\alpha)_{|p}, (X_\beta)_{|p} \in T_p\Sigma, [X_\alpha, X_\beta]_p \notin T_p\Sigma\}.$$

Thus $\{E_{\alpha\beta}\}$ is a countable collection of sets of Hausdorff dimension $\leq n-2$. But since the original vector fields bracket generate at every point, for every $p \in C(\Sigma)$ we must be able to find X_α and X_β such that $p \in E_{\alpha\beta}$. Therefore $C(\Sigma)$ is contained in the countable union of sets of Hausdorff dimension $\leq n-2$ and so must have dimension $\leq n-2$ also.

□

Remark A.1. Without further restrictions on the vector field X_j this result is sharp for all $n > k \geq 2$. To see this set

$$X_j = \partial_{x^j} \quad \text{for } 1 \leq j \leq k-1,$$
$$X_k = \partial_{x^k} + (x^1)\partial_{x^{k+1}} + (x^1)^2\partial_{x^{k+2}} + \cdots + (x^1)^{n-k}\partial_{x^n}.$$

These vector fields bracket generate at step $n-k+1$ at all points of $\mathbb{R}^n$. The smooth surface $\Sigma = \{x^n = (x^1)^2\}$ then has the property

$$\{x^n = x^1 = 0\} \subset C(\Sigma).$$

Thus the Hausdorff dimension of $C(\Sigma)$ must be $\geq n-2$.

In the special case of the higher Heisenberg groups this theorem is decidedly non-sharp. It was shown by Balogh, [2], that for the Heisenberg group of dimension $2n+1$ the characteristic set dimension is bounded by n rather than $2n-1$. Balogh also showed that if the condition C^2 is relaxed to $C^{1,1}$ then the bound $< 2n$ is actually sharp.

The improved bounds for the higher Heisenberg groups was independently shown by Cheng-Hwang [7] for graphs over the horizontal variables. Their technique had the advantage that it used only elementary linear algebra and generalized to graphs in pseudohermitian manifolds in natural coordinates. Here we present a new coordinate free version of this approach which can be used as a tool to study characteristic dimension in general equiregular subRiemannian structures.

Definition A.1. Given a collection of vector fields $\mathcal{X} = \{X_1, \ldots X_n\}$ and a C^2 function ϕ, we define the Hessian of ϕ at p with respect to $\mathcal{X}$ by

$$\mathcal{X}^2(\phi, p) = \left(X_j X_k \phi_{|p}\right).$$

Additionally we define the symmetric Hessian and skew-symmetric Hessian by

$$\mathcal{X}^2_+ = \mathcal{X}^2 + \left(\mathcal{X}^2\right)^\top, \quad \mathcal{X}^2_- = \mathcal{X}^2 - \left(\mathcal{X}^2\right)^\top.$$

Thus we note

$$\mathcal{X}^2_-(\phi,p) = \left([X_j, X_k]_{|p}\phi\right). \tag{A.1}$$

Now at any point p, we can find a non-degenerate constant matrix P such that the skew-symmetric Hessian can be written

$$\mathcal{X}^2_- = P^{-1}\begin{pmatrix} J_\lambda & 0 \\ 0 & 0 \end{pmatrix} P$$

where J_λ is the $2\lambda \times 2\lambda$ matrix with λ copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ along the leading diagonal and zeros everywhere else. Thus we immediately obtain that

$$\operatorname{rank}\left(\mathcal{X}^2_-\right) = 2\lambda, \quad \operatorname{rank}\left(\mathcal{X}^2\right) \geq \lambda.$$

As a basic illustration of the use of these Hessian we present the following lemma, which essentially first appeared in [7].

Lemma A.3. *Suppose (M,η,J) is a $2m+1$ dimensional pseudohermitian manifold such that the Levi form*

$$(X,Y) \mapsto d\eta(X,JY) = -\eta[X,JY], \qquad X,\ Y \text{ horizontal}$$

has signature (p,n) with $p+n \geq 2k$ everywhere. Then is Σ is any C^2 hypersurface, the characteristic set $C(\Sigma)$ has Hausdorff dimension $\leq 2m-k$.

Proof. Choose $p \in C(\Sigma)$ and let ϕ be a C^2 defining function for Σ in a neighbourhood of p. Next choose $\mathcal{X} = \{X_1, \dots X_{2m}\}$ with $X_{m+j} = JX_m$ a frame for the horizontal distribution near p. Now $T\phi$ cannot vanish at p as otherwise $d\phi_{|p} = 0$. Since the Levi form has total signature bounded below by $2k$, from (A.1) have that $\operatorname{rank}\left(\mathcal{X}^2_-(\phi,p)\right) \geq 2k$. Thus $\operatorname{rank}\left(\mathcal{X}^2(\phi,p)\right) \geq k$.

Define a function $F\colon M \to \mathbb{R}^{2m+1}$ by

$$F = \begin{pmatrix} \phi \\ X_j\phi \end{pmatrix}.$$

If we extend $\mathcal{X}$ to by a vector field T to a frame for M near p we see

$$DF_p = \begin{pmatrix} 0 & T\phi \\ \mathcal{X}^2(\phi,p) & * \end{pmatrix}$$

thus DF_p has rank $\geq k+1$. But near p, $C(\Sigma) = F^{-1}(0)$ so the intersection of $C(\Sigma)$ with a neighbourhood of p is contained in an embedded submanifold of dimension $\leq 2m-k$.

□

This technique can be extended to equiregular subRiemannian manifolds of higher step or otherwise more complicated vertical structures, but the generically the derived bounds on characteristic dimension are no better than the general result of Theorem A.1.

Suppose M is an n-dimensional equiregular subRiemannian manifold M with $\mathcal{X}$ a smooth local frame for the horizontal distribution. Then we can produce smooth frames $\mathcal{X}_{(1)}, \mathcal{X}_{(2)}, \dots$ consisting of vector fields produced from $\mathcal{X}$ by $1, 2, \dots$ or less commutations respectibely. Then for any C^2 surface Σ and any point $p \in C(\Sigma)$ we again study the skew-symmetric Hessian $\mathcal{X}^2_-(\phi, p)$. If this Hessian does not vanish at p we can deduce that $C(\Sigma)$ is locally contained in an embedded submanifold of dimension $\leq n-2$. If the $\mathcal{X}^2_-$ does vanish we can immediately deduce that p is actually a characteristic point for the distribution $\mathcal{X}_{(1)}$. We then iterate this argument. If $\mathcal{X}$ is bracket-generating, this must terminate and we have rederived the result of Theorem A.1.

If the step size of the subRiemannian structure is greater than 1, then in particular this argument suggests that generically we cannot expect any improvement on the bound $n-2$. This would not be surprising as the condition that the dimension of the the hypersurface equaling the dimension of the horizontal distribution might be expected to yield a richer theory than the general case. That said, there are examples where this technique can produce improved bounds.

Example A.1. Consider $M = \mathbb{H}^{\mathbb{H}^2}$ the 21-dimensional manifold constructed as follows:

$$X_{j,k} = \partial_{x_{j,k}} - \frac{1}{2} y_{j,k} U_j, \quad Y_{j,k} = \partial_{y_{j,k}} + \frac{1}{2} x_{j,k} U_j, \quad j,k = 1..4$$

with the U_j the horizontal generators of an independent copy of $\mathbb{H}^2$. Thus M consists of 4 copies of $\mathbb{H}^2$ each yielding an element of another $\mathbb{H}^2$ as its characteristic field. M is then a step 2 Carnot group with codimension 5 horizontal distribution.

If Σ is a C^2 hypersurface with defining function ϕ then for any $p \in C(\Sigma)$, either $\mathcal{X}^2_-(\phi, p)$ vanishes identically or has rank ≥ 4. However if $\mathcal{X}^2_-$ vanishes identically then p is a characteristic point for $\mathcal{X}_{(1)}$. But this higher level characteristic set is contained in a submanifold of dimension $\leq 21 - 3$ by an identical argument.

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