International Electronic Journal of Geometry
Volume 6 No.1 pp. 41-45 (2013) ©IEJG

DUALISTIC STRUCTURES ON DOUBLY WARPED PRODUCT MANIFOLDS

ABDOUL SALAM DIALLO

(Communicated by Yusuf YAYLI)

Abstract. In this note we prove that the projection of a dualistic structure defined on a doubly warped product spaces induces dualistic structures on the base and the fiber manifolds. Conversely dualistic structures on the base and the fiber induces a dualistic structures on the doubly warped product space.

1. Introduction

Geometry of conjugate connections is a natural generalization of geometry of Levi-Civita connections from Riemannian manifolds theory. Since conjugate connections arise from affine differential geometry and from geometric theory of statistical inferences [1].

Let (M, g) be a Riemannian manifold and ∇ an affine connection on M. A connection ∇^* is called *conjugate connection* of ∇ with respect to the metric g if

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z), \tag{1.1}$$

for arbitrary X, Y and$Z \in \mathfrak{X}(M)$ where $\mathfrak{X}(M)$ is the set of all tangent vectors fields on M. The triple of a Riemannian metric and a pair of conjugates connection (g, ∇, ∇^*) satisfying (1.1) is called a *dualistic structure* on M.

Dualistic structures are a fundamental mathematics concept of information geometry, specially in the investigation of the natural differential geometric structure possessed by families of probability distributions. The information geometry is nowdays applied in a broad variety of different fields and contexts which include, for instance, information theory, stochastic processes, dynamical systems and times series, statistical physics, quantum systems and the mathematical theory of neural networks [2].

Date: December 03, 2011 and Accepted: September 13, 2012.
1991 *Mathematics Subject Classification.* 53B05, 53B15, 53C50.
Key words and phrases. conjugate connections; doubly warped product.
Work begun during the author's stay at the Institut de Mathématiques et de Sciences Physiques of Porto-Novo, Benin, supported by the ICTP's office of External Activities through the ICAC-3 Program.

The manifold M endowed with a dualistic structure (g, ∇, ∇^*) is called a *dually flat space* if both dual connections ∇ and ∇^* are torsion free and flat; that is the curvature tensors with respect to ∇ and ∇^* respectively vanishe identically. This does not imply that the manifold is Euclidean, because the Riemannian curvature due to the Levi-Civita connections does not necessarily vanish. Moreover the existence of a dually flat structure on a manifold points out some topological and geometrical properties of the manifold. For example if a manifold M admits a dually flat structure (g, ∇, ∇^*) and if one of the dual connection, say ∇, is complete, then only the first homotopy group of M is non trivial, and any two points in M can be joined by a $\nabla-$geodesic [2].

In [1] the author proved that a dualistic structure on a manifold M induces, through the canonical projection on a submanifold, a dualistic structure on any submanifold of M. Recently, Todjihounde proved that the projection of a dualistic structure defined on a warped product space induces dualistic structures on the base and the fiber manifold (see [4] for more information about the dualistic structure on the warped product manifold). Our aim in this note, is to study the dualistic structure on the doubly warped product manifold.

2. Dualistic structures on doubly warped product spaces

Let (B, g_B) and (F, g_F) be Riemannian manifolds of dimensions r and s, respectively, and let $\pi : B \times F \longrightarrow B$ and $\sigma : B \times F \longrightarrow F$ be the canonical projections. Also let $b : B \longrightarrow \mathbb{R}^+$ and $f : F \longrightarrow \mathbb{R}^+$ be positive smooth functions. Then the *doubly warped product* of Riemannian manifolds (B, g_B) and (F, g_F) with warping functions b and f is the product manifold $B \times F$ with metric tensor $g = f^2 g_B \oplus b^2 g_F$ given by

$$g = (f \circ \sigma)^2 \pi^* g_B + (b \circ \pi)^2 \sigma^* g_F. \tag{2.1}$$

We denote this Riemannian manifold (M, g) by $B_f \times_b F$. In particular, if either $b = 1$ or $f = 1$, but not both, then we obtain a warped product. If both $b = 1$ and $f = 1$, then we obtain a direct product. If neither b nor f is constant, then we have a *proper doubly warped product.*

Let $\bar{\phi} : B \to \mathbb{R} \in \mathcal{C}(B)$ then the lift of $\bar{\phi}$ to $B \times F$ is $\phi = \bar{\phi} \circ \pi \in \mathcal{C}(B \times F)$, where $\mathcal{C}(B)$ is the set of all smooth real-valued functions on B.

Moreover, one can define lifts of tangent vectors as follows: let $\bar{X}_p \in T_pB$ and $q \in F$ then the lift $X_{(p,q)}$ of $\bar{X}_p$ is the unique tangent vector in $T_{(p,q)}(B \times \{q\})$ such that $d\pi_{(p,q)}(X_{(p,q)}) = \bar{X}_p$ and $d\sigma_{(p,q)}(X_{(p,q)}) = 0$. We will denote the set of all lifts of all tangents vectors of B to $B \times F$ by $L_{(p,q)}(B)$.

Similarly, we can define lifts of vectors fields. Let $\bar{X} \in \mathfrak{X}(B)$ then the lift of $\bar{X}$ to $B \times F$ is the vector field $X \in \mathfrak{X}(B \times F)$ whose value at each (p, q) is the lift of $\bar{X}_p$ to (p, q). We will denote the set of lifts of all vector fields of B by $\mathcal{L}(B)$.

For any vector field $X \in \mathcal{L}_H(B)$, we denote $\pi_*(X)$ by $\bar{X}$, and for any vector field $U \in \mathcal{L}_V(F)$, we denote $\sigma_*(U)$ by $\bar{U}$.

Lemma 2.1. [3] *Let* $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(B)$ *and* $X, Y, Z \in \mathcal{L}_H(B)$ *be their corresponding horizontal lifts respectively. Let* $\bar{U}, \bar{V}, \bar{W} \in \mathfrak{X}(F)$ *and* $U, V, W \in \mathcal{L}_V(F)$ *be their corresponding vertical lifts respectively. Then*

$$\bar{X} \cdot g(\bar{Y}, \bar{Z}) \circ \pi = X \cdot g(Y, Z), \tag{2.2}$$

$$\bar{U} \cdot g(\bar{V}, \bar{W}) \circ \sigma = U \cdot g(V, W). \tag{2.3}$$

Let (g, D, D^*) be a dualistic structure on $B \times F$. For $X, Y \in \mathcal{L}_H(B)$ and $U, V \in \mathcal{L}_V(F)$ we put:

$$\pi_*(D_X Y) = {}^B\nabla_{\bar{X}} \bar{Y} \quad \text{and} \quad \pi_*(D^*_X Y) = {}^B\nabla^*_{\bar{X}} \bar{Y}, \tag{2.4}$$

and

$$\sigma_*(D_U V) = {}^F\nabla_{\bar{U}} \bar{V} \quad \text{and} \quad \sigma_*(D^*_U V) = {}^F\nabla^*_{\bar{U}} \bar{V}. \tag{2.5}$$

Since D are D^* are affine connections on $B \times F$ and π and σ are the projections of $B \times F$ on B and F respectively, ${}^B\nabla$ and ${}^B\nabla^*$ are affine connections on B and ${}^F\nabla$ and ${}^F\nabla^*$ are affine connections on F. We have the following result:

Proposition 2.1. *The triple* $(g_B, {}^B\nabla, {}^B\nabla^*)$ *is a dualistic structure on* B *and the triple* $(g_F, {}^F\nabla, {}^F\nabla^*)$ *is a dualistic structure on* F.

Proof. Let $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(B)$ and $X, Y, Z \in \mathcal{L}_H(B)$ be their corresponding horizontal lifts respectively. Denoting the inner product g by $<,>$. We have:

$$\begin{aligned}
\bar{X} \cdot g_B(\bar{Y}, \bar{Z}) \circ \pi &= (f \circ \sigma)^{-2} X \cdot < Y, Z > \\
&= (f \circ \sigma)^{-2} \Big[< D_X Y, Z > + < Y, D^*_X Z > \Big] \\
&= (f \circ \sigma)^{-2} \Big[(f \circ \sigma)^2 g_B(\pi_*(D_X Y), \pi_*(Z)) \circ \pi \\
&+ (f \circ \sigma)^2 g_B(\pi_*(Y), \pi_*(D^*_X Z)) \circ \pi \Big] \\
&= g_B({}^B\nabla_{\bar{X}} \bar{Y}, \bar{Z}) \circ \pi + g_B(\bar{Y}, {}^B\nabla^*_{\bar{X}} \bar{Z}) \circ \pi \\
&= \Big[g_B({}^B\nabla_{\bar{X}} \bar{Y}, \bar{Z}) + g_B(\bar{Y}, {}^B\nabla^*_{\bar{X}} \bar{Z}) \Big] \circ \pi.
\end{aligned}$$

Thus

$$\bar{X} \cdot g_B(\bar{Y}, \bar{Z}) = g_B({}^B\nabla_{\bar{X}} \bar{Y}, \bar{Z}) + g_B(\bar{Y}, {}^B\nabla^*_{\bar{X}} \bar{Z}).$$

Hence ${}^B\nabla$ and ${}^B\nabla^*$ are dual w.r.t g_B.

Let $\bar{U}, \bar{V}, \bar{W} \in \mathfrak{X}(F)$ and $U, V, W \in \mathcal{L}_V(F)$ their corresponding horizontal lifts respectively. We have:

$$\begin{aligned}
\bar{U} \cdot g_F(\bar{V}, \bar{W}) \circ \sigma &= (b \circ \pi)^{-2} U \cdot < V, W > \\
&= (b \circ \pi)^{-2} \Big[< D_U V, W > + < V, D_U^* W > \Big] \\
&= (b \circ \pi)^{-2} \Big[(b \circ \pi)^2 g_F(\sigma_*(D_U V), \sigma_*(W)) \circ \sigma \\
&+ (b \circ \pi)^2 g_F(\sigma_*(V), \sigma_*(D_U^* W)) \circ \sigma \Big] \\
&= g_F({}^F\nabla_{\bar{U}} \bar{V}, \bar{W}) \circ \sigma + g_F(\bar{V}, {}^F\nabla^*_{\bar{U}} \bar{W}) \circ \sigma \\
&= \Big[g_F({}^F\nabla_{\bar{U}} \bar{V}, \bar{W}) + g_F(\bar{V}, {}^F\nabla^*_{\bar{U}} \bar{W}) \Big] \circ \sigma.
\end{aligned}$$

It follows then that

$$\bar{U} \cdot g_F(\bar{V}, \bar{W}) = g_F({}^F\nabla_{\bar{U}} \bar{V}, \bar{W}) + g_F(\bar{V}, {}^F\nabla^*_{\bar{U}} \bar{W}).$$

Hence ${}^F\nabla$ and ${}^F\nabla^*$ are dual w.r.t g_F. □

Now, we construct a dualistic structure on the doubly warped product space from those on its base and fiber manifolds.

Let $(g_B, {}^B\nabla, {}^B\nabla^*)$ and $(g_F, {}^F\nabla, {}^F\nabla^*)$ be dualistic structures on B and F. For $X, Y \in \mathcal{L}_H(B)$ and $U, V \in \mathcal{L}_V(F)$ we have the following result:

Proposition 2.2. *The triple (g, D, D^*) is a dualistic structure on $B \times F$.*

Proof. Let $X, Y, Z \in \mathcal{L}_H(B)$. We have:

$$\begin{aligned}
X \cdot < Y, Z > &= f^2 \bar{X} \cdot g_B(\bar{Y}, \bar{Z}) \circ \pi \\
&= f^2 \Big[g_B({}^B\nabla_{\bar{X}} \bar{Y}, \bar{Z}) + g_B(\bar{Y}, {}^B\nabla^*_{\bar{X}} \bar{Z}) \Big] \circ \pi \\
&= f^2 \Big[g_B({}^B\nabla_{\bar{X}} \bar{Y}, \bar{Z}) \circ \pi + g_B(\bar{Y}, {}^B\nabla^*_{\bar{X}} \bar{Z}) \circ \pi \Big] \\
&= f^2 \Big[g_B(\pi_*(D_X Y), \pi_*(Z)) \circ \pi + g_B(\pi_*(Y), \pi_*(D_X^* Z)) \circ \pi \Big] \\
&= f^2 g_B(\pi_*(D_X Y), \pi_*(Z)) \circ \pi + f^2 g_B(\pi_*(Y), \pi_*(D_X^* Z)) \circ \pi \\
&= < D_X Y, Z > + < Y, D_X^* Z > .
\end{aligned}$$

Let $U, V, W \in \mathcal{L}_V(F)$. We have:

$$\begin{aligned}
U \cdot < V, W > &= b^2 g_F(\bar{V}, \bar{W}) \circ \sigma \\
&= b^2 \Big[g_F({}^F\nabla_{\bar{U}} \bar{V}, \bar{W}) + g_F(\bar{V}, {}^F\nabla^*_{\bar{U}} \bar{W}) \Big] \circ \sigma \\
&= b^2 \Big[g_F({}^F\nabla_{\bar{U}} \bar{V}, \bar{W}) \circ \sigma + g_F(\bar{V}, {}^F\nabla^*_{\bar{U}} \bar{W}) \circ \sigma \Big] \\
&= b^2 \Big[g_F(\sigma_*(D_U V), \sigma_*(W)) \circ \sigma + g_F(\sigma_*(V), \sigma_*(D_U^* W)) \circ \sigma \Big] \\
&= b^2 g_F(\sigma_*(D_U V), \sigma_*(W)) \circ \sigma + b^2 g_F(\sigma_*(V), \sigma_*(D_U^* W)) \circ \sigma \\
&= < D_U V, W > + < V, D_U^* W > .
\end{aligned}$$

□

Acknowledgments The author would like to acknowledge the support and excellent research facilities provided by the African Institute for Mathematical Sciences, AIMS-Sénégal. Special thanks to the referee for valuable suggestions to improve the paper.

References

[1] Amari, S. and Nagaoka, H., Methods of Information geometry, AMS, Oxford University Press, vol. 191, 2000.

[2] Ay, N. and Tuschmann, W., Dually flat manifolds and global information geometry, Open Syst. and Information Dyn. 9 (2002), 195-200.

[3] O'Neill, B.,Semi-Riemannian geometry, Academic Press, New-York, 1983.

[4] Todjihounde, L.,Dualistic structures on warped product manifolds, Diff. Geom.-Dyn. Syst. 8, (2006), 278-284.

Institut Africain des Sciences Mathématiques, AIMS-Sénégal, Km2 Route de Joal (Centre IRD de Mbour), B.P. 64 566 Dakar Fann

E-mail address: abdoul@aims-senegal.org