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CHARACTERIZATION OF GENERAL HELIX IN THE 3 - DIMENSIONAL LORENTZ-HEISENBERG SPACE

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Abstract. In this paper, making use of method in [2], [6], [9], we obtained characterizations of a curve with respect to the Frenet frame of the three-dimensional Lorentz-Heisenberg space H_3. We prove that all of the non-geodesic non-null timelike biharmonic curves in H_3 are helices.

1. Introduction

In the theory of curves in Lorentzian space, one of the important and interesting problems is the characterizations of a regular curve. In 1845, de Saint Venant first proved that a space curve is a general helix if and only if the ratio of curvature to torsion be constant (see [14] for details). In [9] T. Ikawa obtained the following equation

$$D_X^3 X - (k^2 - \tau^2) D_X X = 0,$$

for the circular helices which corresponds to the case that the curvatures k and τ of a timelike curve γ on the Lorentzian manifold M are constant.

In [6] N. Ekmekçi and H. H. Hacisalihoğlu generalized T. Ikawa's result to the case of general helices and gave the following characterization

$$D_X^3 X - \left(\frac{3k'}{k}\right) D_X^2 X - \left(\frac{k''}{k} - \frac{3k'}{k^2} + k^2 - \tau^2\right) D_X X = 0$$

for timelike curve with its tangent vector fields on any point.

In [7] N. Ekmekçi and K. İlarslan obtained characterizations of timelike null helices in terms of principal normal or binormal vector fields. H. Balgetir, M. Bektas and M. Ergüt in [2] obtained a geometric characterization of null Frenet curve with constant ratio of curvature and torsion. In [12] A. O. Ogrenmis, M. Ergut and M. Bektas obtained characterizations of helix for a curve with respect to the Frenet frame in $3-$dimensional Galilean space G_3.

Recently, in [11] Y. Nakanishi prove the following lemma.

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Lemma 1.1. *A unit speed curve c in M_c is a helix if and only if there exist a constant λ such that*

$$D_X^3 X = \lambda D_X X.$$

This paper generalizes the lemma stated above to the case of a general helix.

2. Preliminaries

2.1. **Lorentz-Heisenberg group.** Let H_3 be the Lorentz-Heisenberg group endowed with a left-invariant lorentzian metric

$$g_\lambda = dx^2 + dy^2 - \big(dz + \lambda(ydx - xdy)\big)^2, \quad \lambda \in \mathbb{R}. \tag{2.1}$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\lambda y \\ 0 & 1 & \lambda x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}, \tag{2.2}$$

where $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$, $\partial_z = \frac{\partial}{\partial z}$.

The characterising properties of this algebra are the following commutation relations:

$$[e_1, e_2] = 2\lambda e_3, \quad [e_1, e_3] = [e_2, e_3] = 0,$$

with

$$g_\lambda(e_1, e_1) = g_\lambda(e_2, e_2) = 1, \quad g_\lambda(e_3, e_3) = -1.$$

The Levi-Civita connection ∇ of the left-invariant metric g_λ is explicitly given as follows

$$\begin{aligned} \begin{pmatrix} \nabla_{e_1} e_1 \\ \nabla_{e_1} e_2 \\ \nabla_{e_1} e_3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \lambda & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \\ \begin{pmatrix} \nabla_{e_2} e_1 \\ \nabla_{e_2} e_2 \\ \nabla_{e_2} e_3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & -\lambda \\ 0 & 0 & 0 \\ -\lambda & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \\ \begin{pmatrix} \nabla_{e_3} e_1 \\ \nabla_{e_3} e_2 \\ \nabla_{e_3} e_3 \end{pmatrix} &= \begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \end{aligned} \tag{2.3}$$

The dual coframe field $\omega = (\omega_1, \omega_2, \omega_3)$ associated to $B = (e_1, e_2, e_3)$ is a triplet of 1-*forms* which $\omega_i(e_j) = \delta_{ij}$. This coframe field is given by

$$(\omega_1, \omega_2, \omega_3) = \big(dx, dy, dz + \lambda(ydx - xdy)\big).$$

Note that the $1-$form η_3 is a *contact form* on H_3.

The Riemannian curvature tensor is given by

$$R(X, Y, Z, T) = g_\lambda\big(R(X, Y)Z, T\big).$$

If we put

$$R_{abc} = R\big(e_a, e_b\big)e_c, \qquad R_{abcd} = R\big(e_a, e_b, e_c, e_d\big),$$

where the indices a, b, c and d take the values $1, 2, 3$.

Then the components of the curvature tensor field are

$$\begin{aligned} R_{121} &= 3\lambda^2\ e_2, \quad R_{131} = R_{232} = -\lambda^2\ e_3, \\ R_{1212} &= 3\lambda^2, \qquad R_{1313} = R_{2323} = \lambda^2. \end{aligned} \tag{2.4}$$

For tangent vectors

$$X = x_1e_1 + x_2e_2 + x_3e_3, \quad Y = y_1e_1 + y_2e_2 + y_3e_3$$

in H_3, the Lorentzian exterior product $X \wedge_L Y$ is computed as

$$X \wedge_L Y = (x_2y_3 - x_3y_2)e_1 + (x_3y_1 - x_1y_3)e_2 + (x_2y_1 - x_1y_2)e_3. \tag{2.5}$$

The product

$$g_\lambda (X \wedge_L Y, Z) = [X, Y, Z] \tag{2.6}$$

is called the mixed product.

2.2. **Biharmonic maps.** First we should recall some notions and results related to the harmonic and the biharmonic maps between Riemannian manifolds.

Harmonic maps $\psi : (M, g) \to (N, \tilde{g})$ between Riemannian manifolds are the critical points of the energy functional

$$E_1 : C^\infty(M, N) \to \mathbb{R}, \quad E_1(\psi) = \frac{1}{2}\int_M |d\psi|^2\, v_g,$$

and is characterized by the vanishing of the first tension field

$$\tau_1(\psi) = trace\nabla d\psi.$$

We remind that the bienergy of ψ is given by

$$E_2 : C^\infty(M, N) \to \mathbb{R}, \quad E_2(\psi) = \frac{1}{2}\int_M |\tau(\psi)|^2\, v_g,$$

and the bitension field $\tau_2(\psi)$ hase the expression

$$\tau_2(\psi) = -\Delta^\psi \tau(\psi) - trace_g R^N(d\psi, \tau(\psi))d\psi,$$

where $\Delta^\psi = -trace(\nabla^\psi)^2 = -trace(\nabla^\psi\nabla^\psi - \nabla^\psi_\nabla)$.

A smooth map ψ is *biharmonic* if it satisfies the following biharmonic equation

$$\tau_2(\psi) = 0.$$

Biharmonic maps are the critical points of the bienergy functional E_2. We call proper biharmonic the non-harmonic biharmonic maps. Biharmonic curves ψ of a Riemannian manifold are the solutions of the fourth order differential equation

$$\nabla^3_{\phi'}\phi' - R(\phi', \nabla_{\phi'}\phi')\phi' = 0. \tag{2.7}$$

3. Timelike biharmonic curves in H_3

In this section we study the non-geodesic non-null timelike biharmonic curves in H_3. We show that every timelike biharmonic curve in the Lorentzian Heisenberg group H_3 is a helix.

Eells and Sampson in [5] introduced the notion of biharmonic maps as a natural generalization of the well-known harmonic maps. Chen and Ishikawa in [4] classified biharmonic curves in semi-Euclidean 3-space. The biharmonic curves in the Heisenberg group $\mathbb{H}_3$ are investigated in [3].

Let $\phi : I \to H_3$ be a non geodesic timelike biharmonic curve in H_3 parametrized by arclength and let $\{T, N, B\}$ be the orthonormal moving Frenet frame along the curve ϕ in H_3 such that $T = \phi'$ is the unit vector field tangent to ϕ, N is the unit vector field in the direction $\nabla_T T$ normal to ϕ (principal normal) and $B = T \wedge_L N$ (binormal vector). Then we have the following Frenet equations

$$\begin{pmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{3.1}$$

where

$$k^2 = g_\lambda(\tau_1(\phi), \tau_1(\phi)) = g_\lambda(\nabla_T T, \nabla_T T),$$

is the curvature of ϕ and τ is its torsion.

From (3.1) we have

$$\nabla_T^3 T = (3kk')T + (k'' + k^3 - k\tau^2)N + (2k'\tau + k\tau')B, \tag{3.2}$$

where $k' = \frac{dk}{ds}$, $k'' = \frac{d^2k}{ds^2}$, $\tau' = \frac{d\tau}{ds}$.

Using (2.4) one obtains

$$R(T, N, T, N) = \lambda^2(1 + 4B_3^2), \quad R(T, N, T, B) = 4\lambda^2 B_3 N_3, \tag{3.3}$$

where

$$\begin{cases} T = T_1 e_1 + T_2 e_2 + T_3 e_3 \\ N = N_1 e_1 + N_2 e_2 + N_3 e_3 \\ B = T \wedge_L N = B_1 e_1 + B_2 e_2 + B_3 e_3. \end{cases}$$

Theorem 3.1. *Let $\phi : I \to H_3$ be a timelike curve parametrized by arclength. ϕ is a non geodesic biharmonic curve if and only if*

$$\begin{cases} k = constant \neq 0 \\ k^2 - \tau^2 = \lambda^2(1 + 4B_3^2) \\ \tau' = 4\lambda^2 B_3 N_3. \end{cases} \tag{3.4}$$

Proof. The biharmonic equation of ϕ is:

$$\begin{aligned} \tau_2(\phi) &= \nabla_T^3 T - R(T, \nabla_T T)T \\ &= (3kk')T + (k'' + k^3 - k\tau^2)N + (2k'\tau + k\tau')B - kR(T, N)T \\ &= 0. \end{aligned} \tag{3.5}$$

From (3.5) it follows that ϕ is biharmonic curve if and only if

$$\begin{cases} kk' = 0 \\ k'' + k^3 - k\tau^2 = kR(T, N, T, N) \\ 2k'\tau + k\tau' = kR(T, N, T, B). \end{cases}$$

Since $k \neq 0$ (ϕ is a non geodesic)

$$\left\{ \begin{array}{l} k = \text{constant} \neq 0 \\ k^3 - k\tau^2 = kR(T,N,T,N) = k\lambda^2(1+4B_3^2) \\ k\tau' = kR(T,N,T,B) = k4\lambda^2 B_3 N_3. \end{array} \right. \tag{3.6}$$

These, together with (3.6), complete the proof of the theorem. □

Corollary 3.1. *If $\tau = 0$ and $k =$ constant $\neq 0$ for a timelike curve ϕ. ϕ is a non geodesic biharmonic curve if and only if*

$$\left\{ \begin{array}{l} k^2 = \lambda^2(1+4B_3^2) \\ B_3 N_3 = 0. \end{array} \right.$$

Theorem 3.2. *Let $\phi : I \to H_3$ be a timelike curve with $k =$constant. If*

$$B_3 N_3 \neq 0,$$

then ϕ is not biharmonic.

Proof. From (3.1) and (2.3) we have

$$\left\{ \begin{array}{l} \nabla_T T = (T_1' - 2\lambda T_2 T_3)e_1 + (T_2' + 2\lambda T_1 T_3)e_2 + (T_3')e_3 \\ \nabla_T N = (N_1' - \lambda(T_2 N_3 + T_3 N_2))e_1 + (N_2' + \lambda(T_1 N_3 + T_3 N_1))e_2 + (N_3' + \lambda B_3)e_3 \\ \nabla_T B = (B_1' - \lambda(T_2 B_3 + T_3 B_2))e_1 + (B_2' + \lambda(T_1 B_3 + T_3 B_1))e_2 + (B_3' + \lambda N_3)e_3. \end{array} \right.$$

It follows that the third component of these vectors are given by

$$\left\{ \begin{array}{l} g_\lambda(\nabla_T T, e_3) = -T_3' \\ g_\lambda(\nabla_T N, e_3) = -(N_3' + \lambda B_3) \\ g_\lambda(\nabla_T B, e_3) = -(B_3' + \lambda N_3). \end{array} \right. \tag{3.7}$$

From (3.1) we have

$$\left\{ \begin{array}{l} g_\lambda(\nabla_T T, e_3) = -kN_3 \\ g_\lambda(\nabla_T N, e_3) = -kT_3 - \tau B_3 \\ g_\lambda(\nabla_T B, e_3) = \tau N_3. \end{array} \right. \tag{3.8}$$

By comparing (3.7) and (3.8) we obtain

$$\left\{ \begin{array}{l} T_3' = kN_3 \\ N_3' = kT_3 + \tau B_3 - \lambda B_3 \\ B_3' = -\tau N_3 - \lambda N_3. \end{array} \right. \tag{3.9}$$

Assume now that ϕ is biharmonic, then using (3.4) and

$$\tau' = 4\lambda^2 B_3 N_3 \neq 0,$$

we obtain

$$\tau = -\frac{B_3'}{N_3}. \tag{3.10}$$

From (3.9), we obtain $\lambda N_3 = 0$, which is a contradiction. □

Corollary 3.2. *$\phi : I \to H_3$ is timelike non geodesic biharmonic curve if and only if*

$$\left\{ \begin{array}{l} k = \textit{constant} \neq 0 \\ \tau = \textit{constant} \\ k^2 - \tau^2 = \lambda^2(1+4B_3^2) \\ B_3 N_3 = 0. \end{array} \right.$$

4. Timelike general helix in H_3

Definition 4.1. Let ϕ be a curve in 3- dimensional Lorentz-Heisenberg H_3 and $\{T, N, B\}$ be the Frenet frame on H_3 along ϕ.

1) If both k and τ are positive constant along ϕ, then is called circular helix with respect to Frenet frame.

2) A curve ϕ such that

$$\frac{k}{\tau} = c, \; c \in \mathbb{R},$$

is called a general helix with respect to Frenet frame.

If k =constant $\neq 0$ and $\tau = 0$, then the curve ϕ is a circle.

Theorem 4.1. *Let ϕ be a curve in 3- dimensional Lorentz-Heisenberg H_3. ϕ is a general helix with respect to the Frenet frame $\{T, N, B\}$ if and only if*

1)

$$\nabla_T^3 T - \sigma_1(s)\nabla_T T = 3k'\nabla_T N, \tag{4.1}$$

where $\sigma_1(s) = \frac{1}{k}\left(k'' + k^3 - k\tau^2\right)$.

2)

$$\nabla_T^3 B - \sigma_2(s)\nabla_T B + 3\tau'\nabla_T N = 0, \tag{4.2}$$

where $\sigma_2(s) = \tau^2 - k^2 - \frac{\tau''}{\tau}$.

Proof. Suppose that ϕ is general helix with respect to the Frenet frame $\{T, N, B\}$.

Then from (3.1), we have

$$\nabla_T^3 T = (k'' + k^3 - k\tau^2)N + (3kk')T + (2k'\tau + k\tau')B \tag{4.3}$$

$$N = \left(\frac{1}{k}\right)\nabla_T T, \quad B = \left(\frac{1}{\tau}\right)\nabla_T N - \left(\frac{k}{\tau}\right)T. \tag{4.4}$$

Since ϕ is general helix, we have

$$k'\tau = k\tau'. \tag{4.5}$$

If we substitute the equation (4.4) and (4.5) in (4.3), we obtain (4.1).

Conversly let us assume that the equation (4.1) holds. Differentiating covariantly (4.4) we obtain

$$\nabla_T N = -\left(\frac{k'}{k^2}\right)\nabla_T T + \left(\frac{1}{k}\right)\nabla_T^2 T$$

and so

$$\begin{aligned}\nabla_T^2 N &= \left(-\frac{k'}{k^2}\right)'\nabla_T T - 2\left(\frac{k'}{k^2}\right)\nabla_T^2 T + \left(\frac{1}{k}\right)\nabla_T^3 T \\ &= k'T + (k^2 - \tau^2)N + \left(\frac{k'\tau}{k}\right)B. \end{aligned} \tag{4.6}$$

Also we obtain

$$\nabla_T^2 N = k'T + (k^2 - \tau^2)N + \tau' B. \tag{4.7}$$

Since (4.6) and (4.7) are equal, then

$$\tau' = \frac{k'\tau}{k}. \tag{4.8}$$

From (4.8), we obtain

$$\frac{k}{\tau} = \text{constant}.$$

This means that ϕ is a general helix.

2) Suppose that ϕ is general helix with respect to the Frenet frame. Then, from (3.1), we have

$$\nabla_T^3 B = -(k'\tau + 2\tau' k)T + (\tau^3 - k^2\tau - \tau'')N - 3\tau\tau' B \tag{4.9}$$

$$N = -\left(\frac{1}{\tau}\right)\nabla_T B, \; B = \frac{1}{\tau}\nabla_T N - \frac{k}{\tau}T. \tag{4.10}$$

Now we replace (4.10) in the above expression of $\nabla_T^3 B$, and we obtain (4.2).

Conversely (4.2) holds.

Differentiating covariantly of

$$N = -\left(\frac{1}{\tau}\right)\nabla_T B,$$

we obtain

$$\nabla_T^2 N = -\left(\frac{1}{\tau}\right)''\nabla_T B - 2\left(\frac{1}{\tau}\right)'\nabla_T^2 B - \left(\frac{1}{\tau}\right)\nabla_T^3 B. \tag{4.11}$$

If we use (3.1) and (4.2) we get,

$$\nabla_T^2 N = \left(\frac{\tau' k}{\tau}\right)T + (k^2 - \tau^2)N + \tau' B. \tag{4.12}$$

Also we obtain

$$\nabla_T^2 N = k' T + (k^2 - \tau^2)N + \tau' B. \tag{4.13}$$

By comparing (4.12) and (4.13), we obtainwe obtain

$$\left(\frac{k}{\tau}\right)' = 0.$$

From this, we have

$$\frac{k}{\tau} = \text{constant}.$$

Then ϕ is a general helix. □

Corollary 4.1. *Let ϕ be a curve in H_3. ϕ is a circular helix with respect to the Frenet frame $\{T, N, B\}$ if and only if*

1)
$$\nabla_T^3 B - (\tau^2 - k^2)\tau N = 0$$

2)
$$\nabla_T^3 T + k(\tau^2 - k^2)N = 0.$$

Theorem 4.2. *Let ϕ be a curve in H_3. ϕ is a general helix with respect to the Frenet frame $\{T, N, B\}$, then*

$$\nabla_T^3 T + \sigma_1(s)\nabla_T B = 3k'\nabla_T N, \tag{4.14}$$

where $\sigma_1(s) = \frac{1}{\tau}\left(k'' + k^3 - k\tau^2\right)$.

Proof. Suppose that ϕ is general helix with respect to the Frenet frame $\{T, N, B\}$. If we substitute (4.10) in (3.2), we obtain (4.14). □

Corollary 4.2. *Let ϕ be a general helix in H_3. ϕ is a non geodesic timelike biharmonic curve if*

$$R(T, \nabla_T N)T = 3k'\nabla_T N - \sigma_1(s)\nabla_T B.$$

Theorem 4.3. *ϕ is a general helix in H_3 if and only if*

1)

$$[\nabla_T T, \nabla_T^2 T, \nabla_T^3 T] = 0,$$

2)

$$[\nabla_T B, \nabla_T^2 B, \nabla_T^3 B] = 0,$$

3)

$$[\nabla_T N, \nabla_T^2 N, \nabla_T^3 N] = 0.$$

Proof. 1) From (2.6) and

$$\begin{pmatrix} \nabla_T T \\ \nabla_T^2 T \\ \nabla_T^3 T \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ k^2 & k' & \tau k \\ 3kk' & k'' + k^3 - k\tau^2 & 2\tau k' + k\tau' \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

we have

$$[\nabla_T T, \nabla_T^2 T, \nabla_T^3 T] = g(\nabla_T T \wedge_L \nabla_T^2 T, \nabla_T^3 T) = k^5 \left(\frac{\tau}{k}\right)'.$$

Since ϕ is general helix, we have

$$[\nabla_T T, \nabla_T^2 T, \nabla_T^3 T] = 0.$$

The proof is completed.

2) From (2.6) and

$$\begin{pmatrix} \nabla_T B \\ \nabla_T^2 B \\ \nabla_T^3 B \end{pmatrix} = \begin{pmatrix} 0 & -\tau & 0 \\ -k\tau & -\tau' & -\tau^2 \\ -k'\tau - 2k\tau' & -\tau'' + \tau^3 - \tau k^2 & -3\tau\tau' \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

we have

$$\begin{aligned} [\nabla_T B, \nabla_T^2 B, \nabla_T^3 B] &= \tau^5 \left(\frac{\tau}{k}\right)' \\ &= 0. \end{aligned}$$

The proof is completed.

3) From (2.6) and

$$\begin{pmatrix} \nabla_T N \\ \nabla_T^2 N \\ \nabla_T^3 N \end{pmatrix} = \begin{pmatrix} k & 0 & \tau \\ k' & k^2 - \tau^2 & \tau' \\ k'' + k^3 - k\tau^2 & 3kk' - 3\tau\tau' & \tau'' - \tau^3 + \tau k^2 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

we have

$$[\nabla_T N, \nabla_T^2 N, \nabla_T^3 N] = (\tau^2 - k^2)(k\tau'' - \tau k'') + 3(kk' - \tau\tau')(k'\tau - \tau' k).$$

Since ϕ is general helix, we have

$$\begin{aligned} k\tau'' - \tau k'' &= 0 \\ k'\tau - \tau' k &= 0. \end{aligned}$$

The proof is completed. □

5. Spacelike general helix in H_3

Let $\phi : I \rightarrow H_3$ be a non geodesic spacelike curve parametrized by arclength and let $\{T, N, B\}$ be the orthonormal moving Frenet frame along the curve ϕ in H_3 such that $T = \phi^{'}$ is the unit vector field tangent to ϕ. Then we have the following Frenet equations

$$\begin{pmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{5.1}$$

Theorem 5.1. *$\phi = \phi(s)$ is a general helix in H_3 if and only if*

1)
$$[\nabla_T T, \nabla_T^2 T, \nabla_T^3 T] = 0,$$

2)
$$[\nabla_T B, \nabla_T^2 B, \nabla_T^3 B] = 0,$$

3)
$$[\nabla_T N, \nabla_T^2 N, \nabla_T^3 N] = 0.$$

Proof. From (2.6) and

$$\begin{pmatrix} \nabla_T T \\ \nabla_T^2 T \\ \nabla_T^3 T \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k^2 & k' & \tau k \\ -3kk' & -k'' + k^3 + k\tau^2 & 2\tau k' + k\tau' \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

we have

$$\begin{aligned} [\nabla_T T, \nabla_T^2 T, \nabla_T^3 T] &= -k^5 \left(\frac{\tau}{k}\right)' \\ &= 0. \end{aligned}$$

The proof is completed.

2) From (2.6) and

$$\begin{pmatrix} \nabla_T B \\ \nabla_T^2 B \\ \nabla_T^3 B \end{pmatrix} = \begin{pmatrix} 0 & -\tau & 0 \\ k\tau & -\tau' & -\tau^2 \\ k'\tau + 2k\tau' & -\tau'' + \tau^3 + \tau k^2 & -3\tau\tau' \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

we have

$$\begin{aligned} [\nabla_T B, \nabla_T^2 B, \nabla_T^3 B] &= -\tau^5 \left(\frac{k}{\tau}\right)' \\ &= 0. \end{aligned}$$

The proof is completed.

2) From (2.6) and

$$\begin{pmatrix} \nabla_T N \\ \nabla_T^2 N \\ \nabla_T^3 N \end{pmatrix} = \begin{pmatrix} -k & 0 & \tau \\ -k' & -k^2 - \tau^2 & \tau' \\ -k'' + k^3 + k\tau^2 & -3\tau\tau' - 3kk' & \tau'' - \tau^3 - \tau k^2 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

we have

$$\begin{aligned} [\nabla_T N, \nabla_T^2 N, \nabla_T^3 N] &= \tau^5 \left(\frac{\left(\frac{k}{\tau}\right)'}{\tau}\right)' + k^5 \left(\frac{\left(\frac{\tau}{k}\right)'}{k}\right)' \\ &= 0. \end{aligned}$$

The proof is completed. □

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