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CONSTANT SCALAR CURVATURE OF THREE DIMENSIONAL SURFACES OBTAINED BY THE EQUIFORM MOTION OF A SPHERE

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Abstract. In this paper we consider the equiform motion of a sphere in Euclidean space $\mathbf{E}^7$. We study and analyze the corresponding kinematic three-dimensional surface under the hypothesis that its scalar curvature $\mathbf{K}$ is constant. Under this assumption, we prove that $|\mathbf{K}| < 2$.

1. Introduction

An equiform transformation in the n-dimensional Euclidean space $\mathbf{E}^n$ is an affine transformation whose linear part is composed of an orthogonal transformation and a homothetical transformation. Such an equiform transformation maps points $\mathbf{x} \in \mathbf{E}^n$ according to the rule

$$\mathbf{x} \longmapsto s\mathcal{A}\mathbf{x} + \mathbf{d}, \quad \mathcal{A} \in SO(n), \;\; s \in \mathbf{R} - \{0\}, \;\; \mathbf{d} \in \mathbf{E}^n. \tag{1.1}$$

The number s is called the scaling factor. An equiform motion is defined if the parameters of (1.1), including s, are given as functions of a time parameter t. Then a smooth one-parameter equiform motion moves a point $\mathbf{x}$ via $\mathbf{x}(t) = s(t)\mathcal{A}(t)\mathbf{x}(t)+ \mathbf{d}(t)$. The kinematic corresponding to this transformation group is called equiform kinematic, see [2, 4].

Under the assumption of the constancy of the scalar curvature, kinematic surfaces obtained by the motion of a circle have been obtained in [1]. In a similar context, one can consider hypersurfaces in space forms generated by one-parameter family of spheres and having constant curvature: [3, 5, 6, 7].

In this paper we consider the equiform motions of a sphere $\mathbf{k}_0$ in $\mathbf{E}^n$. The point paths of the sphere generate a 3-dimensional surface, containing the positions of the starting sphere $\mathbf{k}_0$. The first order properties of these surfaces for the points of these spheres have been studied for arbitrary dimensions $n \geq 3$ [1]. We restrict our

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considerations to dimension $n = 7$ because, at any moment, the infinitesimal transformations of the motion map the points of the sphere $\mathbf{k}_0$ to the velocity vectors, whose end points will form an affine image of $\mathbf{k}_0$ (in general a sphere) that span a subspace $\mathbf{W}$ of $\mathbf{E}^n$ with $n \leq 7$ [8].

On other hand, in the case of cyclic surface foliated by circle, for fixed t, we have a circle in a fixed frame and its image is an ellipse in moving frame. Then, we need at least space of dimension 5 (2 for the circle, 2 for an ellipse and one dimension for skew) [1]. In the present paper, for fixed t, we have a sphere in a fixed frame and its image is an ellipsoid in moving frame. Then, we need at least space of dimension 7 (3 for the sphere, 3 for an ellipsoid and one dimension for skew), see [8]).

Let $\mathbf{x}(\theta, \phi)$ be a parameterization of $\mathbf{k}_0$ and let $\mathbf{X}(t, \theta, \phi)$ be the resultant 3-surface by the equiform motion. We consider a certain position of the moving space given by $t = 0$, and we would like to obtain information about the motion at least during a certain period around $t = 0$ if we know its characteristics for one instant. Then we restrict our study to the properties of the motion for the limit case $t \to 0$. A first choice is then approximate $\mathbf{X}(t, \theta, \phi)$ by the first derivative of the trajectories. Solliman, et al. studied 3-dimensional surfaces in $\mathbf{E}^7$ generated by equiform motions of a sphere proving that, in general, they are contained in a canal hypersurface [8].

The purpose of this paper is to describe the kinematic surfaces obtained by the motion of a sphere and whose scalar curvature $\mathbf{K}$ is constant. As a consequence of our results, we prove:

A kinematic three-dimensional surface obtained by the equiform motion of a sphere and with constant scalar curvature $\mathbf{K}$ *satisfies* $|\mathbf{K}| < 2$.

Moreover, we show the description of the motion of such 3-surface by giving the equations that determine the kinematic geometry.

2. The representation of a kinematic surface

In two copies $\sum^0$, $\sum$ of Euclidean 7-space $\mathbf{E}^7$, we consider a unit sphere $\mathbf{k}_0$ centered at the origin of the 3-space $\varepsilon_0 = [x_1 x_2 x_3]$ and represented by

(2.1)
$$\mathbf{x}(\theta, \phi) = \Big(\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), \sin(\phi), 0, 0, 0, 0\Big)^{\mathrm{T}}, \ \ \theta \in [0, 2\pi], \ \ \phi \in [0, \pi].$$

Under a one-parameter equiform motion of moving space $\sum^0$ with respect to a fixed space $\sum$ the general representation of the motion of this surface in $\mathbf{E}^7$ is given by

$$\mathbf{X}(t, \theta, \phi) = s(t)\mathbf{A}(t)\mathbf{x}(\theta, \phi) + \mathbf{d}(t), \qquad t \in I \subset \mathbf{R}.$$

Here $\mathbf{d}(t) = \Big(b_i(t)\Big)^{\mathrm{T}} : i = 1, 2, ..., 7$ describes the position of the origin of $\sum^0$ at time t, $\mathbf{A}(t) = \Big(a_{ij}(t)\Big)^{\mathrm{T}} : i, j = 1, 2, ..., 7$ is an orthogonal matrix and $s(t)$ provides the scaling factor of the moving system. With $s = \text{const.} \neq 0$ (sufficient to set $s = 1$), we have an ordinary Euclidean rigid body motion. For varying t and fixed $\mathbf{x}(\theta, \phi)$, equation (2.1) gives a parametric representation of the surface (or trajectory) of $\mathbf{x}(\theta,\phi)$. Moreover, we assume that all involved functions are at least of class $\mathbf{C}^1$. Using Taylor's expansion up to the first order, the representation of the motion is given by

$$\mathbf{X}(t,\theta,\phi)=\Big[s(0)\mathbf{A}(0)+t\Big(\dot{s}(0)\mathbf{A}(0)+s(0)\dot{\mathbf{A}}(0)\Big)\Big]\mathbf{x}(\theta,\phi)+\mathbf{d}(0)+t\dot{\mathbf{d}}(0),$$

where (.) denotes differentiation with respect to the time t. Assuming that the moving frames $\sum^0$ and $\sum$ coincide at the zero position ($t=0$), we have

$$\mathbf{A}(0)=\mathbf{I},\quad s(0)=1,\quad \text{and}\quad \mathbf{d}(0)=0.$$

Thus we have

$$\mathbf{X}(t,\theta,\phi)=\Big[\mathbf{I}+t\Big(s'\mathbf{I}+\Omega\Big)\Big]\mathbf{x}(\theta,\phi)+t\mathbf{d}',$$

where $\Omega=\dot{\mathbf{A}}(0)=(\omega_i)$, $i=1,2,...,21$ is a skew symmetric matrix, $s'=\dot{s}(0)$, $\mathbf{d}'=\dot{\mathbf{d}}(0)$ and all values of s,b_i and their derivatives are computed at $t=0$. With respect to these frames, the representation of the motion up to the first order is

$$\begin{pmatrix}\mathbf{X}_1\\\mathbf{X}_2\\\mathbf{X}_3\\\mathbf{X}_4\\\mathbf{X}_5\\\mathbf{X}_6\\\mathbf{X}_7\end{pmatrix}=\begin{pmatrix}1+s't&\omega_1 t&\omega_2 t&\omega_3 t&\omega_4 t&\omega_5 t&\omega_6 t\\-\omega_1 t&1+s't&\omega_7 t&\omega_8 t&\omega_9 t&\omega_{10} t&\omega_{11} t\\-\omega_2 t&-\omega_7 t&1+s't&\omega_{12} t&\omega_{13} t&\omega_{14} t&\omega_{15} t\\-\omega_3 t&-\omega_8 t&-\omega_{12} t&1+s't&\omega_{16} t&\omega_{17} t&\omega_{18} t\\-\omega_4 t&-\omega_9 t&-\omega_{13} t&-\omega_{16} t&1+s't&\omega_{19} t&\omega_{20} t\\-\omega_5 t&\omega_{10} t&-\omega_{14} t&-\omega_{17} t&-\omega_{19} t&1+s't&\omega_{21} t\\-\omega_6 t&-\omega_{11} t&-\omega_{15} t&-\omega_{18} t&-\omega_{20} t&-\omega_{21} t&1+s't\end{pmatrix}\tag{2.2}$$
$$\times\begin{pmatrix}\cos(\theta)\cos(\phi)\\\sin(\theta)\cos(\phi)\\\sin(\phi)\\0\\0\\0\\0\end{pmatrix}+t\begin{pmatrix}b'_1\\b'_2\\b'_3\\b'_4\\b'_5\\b'_6\\b'_7\end{pmatrix},$$

or in the equivalent form

$$\begin{pmatrix}\mathbf{X}_1\\\mathbf{X}_2\\\mathbf{X}_3\\\mathbf{X}_4\\\mathbf{X}_5\\\mathbf{X}_6\\\mathbf{X}_7\end{pmatrix}=t\begin{pmatrix}b'_1\\b'_2\\b'_3\\b'_4\\b'_5\\b'_6\\b'_7\end{pmatrix}+\cos(\theta)\cos(\phi)\begin{pmatrix}1+s'\,t\\-\omega_1 t\\-\omega_2 t\\-\omega_3 t\\-\omega_4 t\\-\omega_5 t\\-\omega_6 t\end{pmatrix}+\sin(\theta)\cos(\phi)\begin{pmatrix}\omega_1 t\\1+s'\,t\\-\omega_7 t\\-\omega_8 t\\-\omega_9 t\\-\omega_{10} t\\-\omega_{11} t\end{pmatrix}\tag{2.3}$$
$$+\sin(\phi)\begin{pmatrix}\omega_2 t\\\omega_7 t\\1+s'\,t\\-\omega_{12} t\\-\omega_{13} t\\-\omega_{14} t\\-\omega_{15} t\end{pmatrix}$$
$$=t\,\vec{\mathbf{b}}+\cos(\theta)\cos(\phi)\,\vec{\mathbf{a}}_0+\sin(\theta)\cos(\phi)\,\vec{\mathbf{a}}_1+\sin(\phi)\,\vec{\mathbf{a}}_2.\tag{2.4}$$

For any fixed t in equation (2.3), we generally get an ellipsoid for $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$ centered at the point $t(b'_1, b'_2, b'_3, b'_4, b'_5, b'_6, b'_7)$. The latter ellipsoid turns to a 2-dimensional sphere if $\vec{\mathbf{a}}_0$, $\vec{\mathbf{a}}_1$, and $\vec{\mathbf{a}}_2$ form an orthonormal basis. This gives the following conditions:

$$\sum_{i=2}^{6} \omega_i\, \omega_{i+5} = \omega_1\, \omega_7 - \sum_{i=3}^{6} \omega_i\, \omega_{i+9} = \omega_1\, \omega_2 + \sum_{i=8}^{11} \omega_i\, \omega_{i+4} = 0, \tag{2.5}$$

$$\sum_{i=2}^{6} \omega_i^2 = \sum_{i=7}^{11} \omega_i^2, \qquad \omega_1^2 + \sum_{i=3}^{6} \omega_i^2 = \omega_7^2 + \sum_{i=12}^{15} \omega_i^2. \tag{2.6}$$

3. Scalar curvature of the kinematic surface

In this section we shall compute the scalar curvature of 3-surfaces in $\mathbf{E}^7$ generated by equiform motions of a sphere which satisfies the conditions (2.5)-(2.6). The tangents to the parametric curves $t =$ const., $\theta =$ const., and $\phi =$ const. at the zero position are

$$\mathbf{X}_t = \Big[s'\mathbf{I} + \Omega\Big]\mathbf{x} + \mathbf{d}', \ \ \mathbf{X}_\theta = \Big[\mathbf{I} + \Big(s'\mathbf{I} + \Omega\Big)t\Big]\mathbf{x}_\theta, \ \ \mathbf{X}_\phi = \Big[\mathbf{I} + \Big(s'\mathbf{I} + \Omega\Big)t\Big]\mathbf{x}_\phi.$$

The coordinate functions of the first fundamental form of $\mathbf{X}(t, \theta, \phi)$ are

$$\begin{cases} g_{11} = \mathbf{X}_t^{\mathrm{T}}\, \mathbf{X}_t, & g_{12} = \mathbf{X}_t^{\mathrm{T}}\, \mathbf{X}_\theta, \quad g_{13} = \mathbf{X}_t^{\mathrm{T}}\, \mathbf{X}_\phi, \\ g_{22} = \mathbf{X}_\theta^{\mathrm{T}}\, \mathbf{X}_\theta, & g_{23} = \mathbf{X}_\theta^{\mathrm{T}}\, \mathbf{X}_\phi, \quad g_{33} = \mathbf{X}_\phi^{\mathrm{T}}\, \mathbf{X}_\phi. \end{cases}$$

Under the conditions (2.5)-(2.6), we obtain

$$\begin{aligned}
g_{11} &= \gamma + \alpha_5 \cos(2\phi) + \alpha_8 \sin(\phi) + 2\cos(\phi)\Big[\cos(\phi)\Big(\alpha_4 \cos(2\theta) + \alpha_1 \sin(2\theta)\Big) \\
&+ \sin(\theta)\Big(\alpha_7 + \alpha_2 \sin(\phi)\Big) + \cos(\theta)\Big(\alpha_6 - 2\alpha_3 \sin(\phi)\Big)\Big], \\
g_{12} &= \cos(\phi)\Big[2\,t\, \cos(\phi)\Big(\alpha_1 \cos(2\theta) - \alpha_4 \sin(2\theta)\Big) - \omega_1 \cos(\phi) \\
&- \sin(\theta)\Big[t\Big(\alpha_6 - 2\alpha_3 \sin(\phi)\Big) + b'_1 + \omega_2 \sin(\phi)\Big] \\
&+ \cos(\theta)\Big[t\Big(\alpha_7 + 2\alpha_2 \sin(\phi)\Big) + b'_2 + \omega_7 \sin(\phi)\Big]\Big], \\
g_{13} &= 2t\cos(2\phi)\Big(\alpha_2 \sin(\theta) - \alpha_3 \cos(\theta)\Big) - t\sin(2\phi)\Big(\alpha_5 + \alpha_4 \cos(2\theta) \\
&+ \alpha_1 \sin(2\theta)\Big) - \sin(\phi)\Big[\Big(b'_1 + t\,\alpha_6\Big)\cos(\theta) + \Big(b'_2 + t\,\alpha_7\Big)\sin(\theta)\Big], \\
g_{22} &= \cos^2(\phi)\Big[1 + 2t\,s' + 2t^2\Big(\delta - \alpha_4 \cos(2\theta) - \alpha_6 \sin(2\theta)\Big)\Big], \\
g_{23} &= t^2\Big[2\cos^2(\phi)\Big(\alpha_2 \cos(\theta) + \alpha_3 \sin(\theta)\Big) + \sin(2\phi)\Big(\alpha_4 \sin(2\theta) - \alpha_1 \cos(2\theta)\Big)\Big], \\
g_{33} &= 1 + 2t\,s' + t^2\Big[\gamma - \beta - \alpha_5 \cos(2\phi) + 2\sin^2(\phi)\Big(\alpha_4 \cos(2\theta) + \alpha_1 \sin(2\theta)\Big) \\
&+ 2\sin(2\phi)\Big(\alpha_3 \cos(\theta) - \alpha_2 \sin(\theta)\Big)\Big],
\end{aligned}$$

where

$$\begin{cases} \alpha_1 = \frac{1}{2}\Big[\sum_{i=2}^{6}\omega_i\,\omega_{i+5}\Big],\\ \alpha_2 = \frac{1}{2}\Big[\omega_1\,\omega_2 + \sum_{i=8}^{11}\omega_i\,\omega_{i+4}\Big],\\ \alpha_3 = \frac{1}{2}\Big[\omega_1\,\omega_7 - \sum_{i=3}^{6}\omega_i\,\omega_{i+9}\Big],\\ \alpha_4 = \frac{1}{4}\Big[\sum_{i=2}^{6}\Big(\omega_i^2 - \omega_{i+5}^2\Big)\Big],\\ \alpha_5 = \frac{1}{4}\Big[\omega_1^2 - 2\,\omega_2^2 - 2\,\omega_7^2 + \sum_{i=1}^{11}\omega_i^2 - 2\Big(\sum_{i=12}^{15}\omega_i^2\Big)\Big],\\ \alpha_6 = b_1'\,s' - \sum_{i=2}^{7} b_i'\,\omega_{i-1},\\ \alpha_7 = b_1'\,\omega_1 + b_2'\,s' - \sum_{i=3}^{7} b_i'\,\omega_{i+4},\\ \alpha_8 = 2\Big[b_1'\,\omega_2 + b_2'\,\omega_7 + b_3'\,s' - \sum_{i=4}^{7} b_i'\,\omega_{i+8}\Big],\\ \beta = \sum_{i=1}^{7} b_i'^2,\\ \gamma = \beta + s'^2 + \frac{1}{4}\Big[2(\omega_1^2 + \omega_2^2 + \omega_7^2) + \sum_{i=2}^{15}\omega_i^2 + \sum_{i=12}^{15}\omega_i^2\Big],\\ \delta = \frac{1}{4}\Big[2\,(s'^2 + \omega_1^2) + \sum_{i=2}^{11}\omega_i^2\Big]. \end{cases}$$

The conditions (2.5)-(2.6) lead to the following relations

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0, \quad \gamma = \beta + 2\delta.$$

In order to calculate the scalar curvature, we need to compute the Christoffel symbols of the second kind, which are defined as

$$\Gamma_{ij}^{l} = \frac{1}{2} g^{lm}\Big[\frac{\partial g_{im}}{\partial x_j} + \frac{\partial g_{jm}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_m}\Big], \tag{3.1}$$

where i, j, l are indices that take the values $1, 2, 3$, $x_1 = t, x_2 = \theta, x_3 = \phi$, and $\Big(g^{lm}\Big)$ is the inverse matrix of $\Big(g_{ij}\Big)$. Then the scalar curvature of the surface $\mathbf{X}(t, \theta, \phi)$ is

$$\mathbf{K}(t, \theta, \phi) = g^{ij}\Big[\frac{\partial \Gamma_{ij}^{l}}{\partial x_l} - \frac{\partial \Gamma_{il}^{l}}{\partial x_j} + \Gamma_{ij}^{l}\,\Gamma_{lm}^{m} - \Gamma_{il}^{m}\,\Gamma_{jm}^{l}\Big].$$

At the zero position ($t = 0$), the scalar curvature of $\mathbf{X}(t, \theta, \phi)$ is given by

$$\mathbf{K} = \mathbf{K}(0, \theta, \phi) \;=\; \frac{P\Big(\cos(n_1\theta \pm m_1\phi), \sin(n_1\theta \pm m_1\phi)\Big)}{Q\Big(\cos(n_2\theta \pm m_2\phi), \sin(n_2\theta \pm m_2\phi)\Big)}. \tag{3.2}$$

This quotient writes then as

$$P\Big(\cos(n_1\theta \pm m_1\phi), \sin(n_1\theta \pm m_1\phi)\Big) - \mathbf{K}\,Q\Big(\cos(n_2\theta \pm m_2\phi), \sin(n_2\theta \pm m_2\phi)\Big) = 0. \tag{3.3}$$

The assumption on the constancy of the scalar curvature $\mathbf{K}$ implies that equation (3.3) is a linear combination of the functions $\cos(n\,\theta \pm m\,\phi)$, $\sin(n\,\theta \pm m\,\phi)$. Because these functions are linearly independent , the corresponding coefficients must vanish. Throughout this work, we have employed the Mathematica programm in order to compute the explicit expressions of these coefficients.

Assumption 3.1. *Without loss of generality, we assume that the two conditions (2.5)-(2.6) are satisfied and there are no translation motions in the plane which contain the starting sphere, i.e.,*

$$b'_1 = b'_2 = b'_3 = 0.$$

3.1. **Kinematic surfaces with zero scalar curvature.** We assume that $\mathbf{K} = 0$. From the expression (3.2), we have

$$\begin{aligned} P\Big(\cos(n_1\theta \quad &\pm \quad m_1\phi), \sin(n_1\theta \pm m_1\phi)\Big) \\ &= \sum_{i=0}^{12}\sum_{j=-12}^{12}\Big(A_{i,j}\cos(i\,\theta + j\,\phi) + B_{i,j}\sin(i\,\theta + j\,\phi)\Big) = 0. \end{aligned}$$

In this case, a straightforward computation shows that the coefficients of $\cos(12\phi)$, $\cos(6\theta + 12\phi)$ and $\sin(6\theta + 12\phi)$ are

$$\begin{aligned} A_{0,12} &= \frac{3}{8192}\Big[16\,\omega_1^6 - 120\,\omega_1^4\left(\omega_2^2 + \omega_7^2\right) + 9\,\omega_1^2\left(\omega_2^2 + \omega_7^2\right)^2 - 5\left(\omega_2^2 + \omega_7^2\right)^3\Big], \\ A_{6,12} &= \frac{3}{32768}\Big[\omega_2^6 - 15\,\omega_2^4\,\omega_7^2 + 15\,\omega_2^2\,\omega_7^4 - \omega_7^6\Big], \\ B_{6,12} &= \frac{3}{16384}\,\omega_2\,\omega_7\Big(3\,\omega_2 - \omega_7^2\Big)\Big(\omega_2^2 - 3\,\omega_7^2\Big). \end{aligned}$$

By solving the three equations $A_{0,12} = 0$, $A_{6,+12} = 0$ and $B_{6,+12} = 0$, we get

$$\omega_1 = \omega_2 = \omega_7 = 0.$$

Then

$$\begin{aligned} B_{0,9} &= \frac{3}{256}\,\alpha_8\left[\alpha_8^2 - 6\left(\alpha_6^2 + \alpha_7^2\right)\right], \\ A_{3,9} &= \frac{3}{256}\,\alpha_6\left(3\,\alpha_7^2 - \alpha_6^2\right), \\ B_{3,9} &= \frac{3}{256}\,\alpha_7\left(\alpha_7^2 - 3\,\alpha_6^2\right). \end{aligned}$$

The three equations $B_{0,9} = 0$, $A_{3,9} = 0$, and $B_{3,9} = 0$ imply

$$\alpha_6 = \alpha_7 = \alpha_8 = 0.$$

From these values, equation $A_{0,6} = 0$ leads to

$$\Big(\beta + 2\,\delta\Big)\Big(\beta + s'^2 - 2\,\delta\Big) = 0.$$

It is worth to point out that the quantities β and δ are positive and thus we obtain the following condition

$$\delta = \frac{1}{2}\Big(\beta + s'^2\Big).$$

At this time, the explicit computations of coefficients imply that all $A_{i,j}$ and $B_{i,j}$ are equal zero. So, we have the following:

Theorem 3.1. *A kinematic 3-surface in* $\mathbf{E}^7$ *foliated by spheres and with zero constant scalar curvature satisfies*

$$\begin{aligned} \omega_1 &= \omega_2 = \omega_7 = 0, \\ \sum_{i=4}^{7} b'_i\,\omega_{i-1} &= \sum_{i=4}^{7} b'_i\,\omega_{i+4} = \sum_{i=4}^{7} b'_i\,\omega_{i+8} = 0, \\ \sum_{i=3}^{6} \omega_i^2 &= \sum_{i=4}^{7} b_i'^2. \end{aligned}$$

3.2. **Kinematic surfaces with non-zero constant scalar curvature.** We assume that the kinematic 3-surface has constant scalar curvature $\mathbf{K} \neq 0$. From (3.1), we have

$$\begin{aligned} P\Big(\cos(n_1\theta &\pm m_1\phi), \sin(n_1\theta \pm m_1\phi)\Big) - \mathbf{K}\,Q\Big(\cos(n_2\theta \pm m_2\phi), \sin(n_2\theta \pm m_2\phi)\Big) \\ &= \sum_{j=-12}^{12}\sum_{i=0}^{12}\Big(A_{i,j}\cos(i\,\theta + j\,\phi) + B_{i,j}\sin(i\,\theta + j\,\phi)\Big) = 0. \end{aligned}$$

In this case, a straightforward computation shows that the coefficients of $\cos(12\phi)$, $\cos(12\theta + 6\phi)$ and $\sin(12\theta + 6\phi)$ are

$$\begin{aligned} A_{0,12} &= \frac{1}{16384}\Big(\mathbf{K}+6\Big)\Big[16\omega_1^6 - 120\,\omega_1^4\Big(\omega_2^2+\omega_7^2\Big) + 90\,\omega_1^2\Big(\omega_2^2+\omega_7^2\Big)^2 - 5\Big(\omega_2^2+\omega_7^2\Big)^3\Big], \\ A_{6,12} &= \frac{1}{65536}\Big(\mathbf{K}+6\Big)\Big[\omega_2^6 - 15\,\omega_2^4\,\omega_7^2 + 15\omega_2^2\,\omega_7^4 - \omega_7^6\Big], \\ B_{6,12} &= \frac{1}{32768}\,\omega_2\,\omega_7\Big(\mathbf{K}+6\Big)\Big(3\,\omega_2 - \omega_2^2\Big)\Big(\omega_2^2 - 3\,\omega_7^2\Big). \end{aligned}$$

We consider the three equations $A_{0,12} = 0$, $A_{6,12} = 0$, and $B_{6,12} = 0$. From here, we discuss two possibilities: $\mathbf{K} = -6$ and $\omega_1 = \omega_2 = \omega_7 = 0$.

(1) **Case $\mathbf{K} = -6$.** A computation of coefficients yields

$$\begin{aligned} A_{5,11} &= \frac{1}{2048}\Big[\alpha_6\Big(\omega_2^4 - 6\,\omega_2^2\,\omega_7^2 + \omega_7^4\Big) - 4\alpha_7\,\omega_2\,\omega_7\Big(\omega_2^2 - \omega_7^2\Big)\Big] = 0 \\ B_{5,11} &= \frac{1}{2048}\Big[\alpha_7\Big(\omega_2^4 - 6\,\omega_2^2\,\omega_7^2 + \omega_7^4\Big) + 4\alpha_6\,\omega_2\,\omega_7\Big(\omega_2^2 - \omega_7^2\Big)\Big] = 0. \end{aligned}$$

We consider two cases: $\alpha_6 = \alpha_7 = 0$ and $\omega_2 = \omega_7 = 0$.

Case (1): We assume $\alpha_6 = \alpha_7 = 0$. The computation of coefficients leads to

$$\begin{aligned} B_{0,11} &= \frac{1}{2048}\,\alpha_8\Big[8\,\omega_1^4 - 24\,\omega_1^2\Big(\omega_2^2+\omega_7^2\Big) + 3\Big(\omega_2^2+\omega_7^2\Big)^2\Big] = 0, \\ A_{4,11} &= \frac{1}{512}\,\alpha_8\,\omega_2\,\omega_7\Big(\omega_7^2 - \omega_2^2\Big) = 0, \\ B_{4,11} &= \frac{1}{2048}\,\alpha_8\Big(\omega_2^4 - 6\,\omega_2^2\,\omega_7^2 + \omega_7^4\Big) = 0, \end{aligned}$$

which implies two subcases: $\alpha_8 = 0$ and $\omega_1 = \omega_2 = \omega_7 = 0$.

Subcase (1.1): If $\alpha_8 = 0$, then we have

$$\begin{aligned} A_{0,10} &= \frac{1}{1024}\left[8\,\omega_1^2 - \omega_1^2\Big(\omega_2^2+\omega_7^2\Big) + 3\left(\omega_2^2+\omega_7^2\right)^2\right]\Big(\beta + s'^2 + 6\,\delta - \omega_1^2 - \omega_2^2 - \omega_7^2\Big) \\ A_{4,10} &= \frac{1}{2048}\,\omega_2\,\omega_7\left(\omega_2^4 - 6\,\omega_2^2\,\omega_7^2 + \omega_7^4\right)\Big(\beta + s'^2 + 6\,\delta - \omega_1^2 - \omega_2^2 - \omega_7^2\Big) \\ B_{4,10} &= \frac{1}{512}\,\omega_2\,\omega_7\left(\omega_7^2 - \omega_2^2\right)\Big(\beta + s'^2 + 6\,\delta - \omega_1^2 - \omega_2^2 - \omega_7^2\Big). \end{aligned}$$

The last term in the above three equations is not zero because

$$\beta + s'^2 + 6\,\delta - \omega_1^2 - \omega_2^2 - \omega_7^2 = \sum_{i=4}^{7} b_i'^2 + \omega_2^2 + \sum_{i=8}^{11}\omega_i^2 + 2\Big[2\,s'^2 + \omega_1^2 + \sum_{i=3}^{6}\omega_i^2\Big] > 0.$$

The three equations $A_{0,10} = 0$, $A_{4,10} = 0$, and $B_{4,10} = 0$ lead $\omega_1 = \omega_2 = \omega_7 = 0$. Now, the coefficient $A_{0,6}$ must equal zero, that is,

$$\Big(\beta + 2\,\delta\Big)^2\Big(2\,\beta + 8\,\delta - s'^2\Big) = 0,$$

contradiction.

Subcase (1.2): If $\omega_1 = \omega_2 = \omega_7 = 0$ and $\alpha_8 \neq 0$, the equation $B_{0,9} = 0$ implies that $\alpha_8 = 0$: contradiction.

Case (2): If $\omega_2 = \omega_7 = 0$ and $\alpha_6, \alpha_7 \neq 0$, the computation of coefficients yields

$$\begin{aligned} A_{5,11} &= \frac{1}{2048}\,\alpha_6\,\omega_1^4 = 0, \\ B_{5,11} &= \frac{1}{2048}\,\alpha_7\,\omega_1^4 = 0. \end{aligned}$$

Because $\alpha_6 \neq 0$ and $\alpha_7 \neq 0$, we conclude $\omega_1 = 0$. New computations give

$$\begin{aligned} A_{3,9} &= \frac{9}{256}\alpha_6\left(\alpha_6^2 - 3\,\alpha_7^2\right), \\ B_{3,9} &= \frac{9}{256}\alpha_7\left(3\,\alpha_6^2 - \alpha_7^2\right). \end{aligned}$$

By solving the equations $A_{3,9} = 0$ and $B_{3,9} = 0$, we get $\alpha_6 = \alpha_7 = 0$: contradiction.

Corollary 3.1. *There are no kinematic 3-surfaces in* $\mathbf{E}^7$ *foliated by spheres and with scalar curvature* $\mathbf{K}$ *equal* -6.

(2) **Case $\omega_1 = \omega_2 = \omega_7 = 0$ and $\mathbf{K} \neq -6$.**
A computation of the coefficients yields

$$\begin{aligned} B_{0,9} &= \frac{1}{256}\,\alpha_8\left[\alpha_8^2 - 6\left(\alpha_6^2 + \alpha_7^2\right)\right]\Big(2\,\mathbf{K} + 3\Big) = 0, \\ A_{3,9} &= \frac{1}{256}\,\alpha_6\left(3\,\alpha_7^2 - \alpha_6^2\right)\Big(2\,\mathbf{K} + 3\Big) = 0, \\ B_{3,9} &= \frac{1}{256}\,\alpha_7\left(\alpha_7^2 - 3\,\alpha_6^2\right)\Big(2\,\mathbf{K} + 3\Big) = 0, \end{aligned}$$

which gives two cases: $\mathbf{K}=-\dfrac{3}{2}$ or $\alpha_7=\alpha_6=\alpha_8=0$.

Case (1): Assume $\mathbf{K}=-\dfrac{3}{2}$. Now, we obtain

$$\begin{aligned} A_{0,8} &= \frac{1}{64}\left[\alpha_8^2-\left(\alpha_6^2-\alpha_7^2\right)\right]\left(6\,\delta-\beta-2\,s'^2\right),\\ A_{2,8} &= \frac{1}{64}\left(\alpha_6^2-\alpha_7^2\right)\left(6\,\delta-\beta-2\,s'^2\right),\\ B_{2,8} &= \frac{1}{32}\,\alpha_7\alpha_6\left(6\,\delta-\beta-2\,s'^2\right). \end{aligned}$$

Solving the three equations $A_{0,8}=0$, $A_{2,8}=0$, and $B_{2,8}=0$, we find two cases: $\delta=\dfrac{\beta+2\,s'^2}{6}$ and $\alpha_6=\alpha_7=\alpha_8=0$.

Case (1.1): If $\delta=\dfrac{\beta+2\,s'^2}{6}$, we obtain

$$A_{0,4}=\frac{1}{72}\,(2\beta+s'^2)\left[4(2\,\beta+s'^2)^2-9\left(\alpha_8^2+4(\alpha_6^2+\alpha_7^2)\right)\right],$$

which leads the following condition

$$4(2\,\beta+s'^2)^2=9\left(\alpha_8^2+4(\alpha_6^2+\alpha_7^2)\right).$$

At this point, all coefficients $A_{i,j}$ and $B_{i,j}$ are equal zero.

Case (1.2): Assume $\alpha_6=\alpha_7=\alpha_8=0$ and $\delta\neq\dfrac{\beta+2\,s'^2}{6}$. The coefficient $A_{0,6}$ is $A_{0,6}=\dfrac{1}{32}\left(\beta+2\,\delta\right)\left[14\,\delta-\beta-4\,s'^2\right]$. From $A_{0,6}=0$, we conclude

$$\delta=\frac{\beta+4\,s'^2}{14}.$$

As a consequence, all coefficients $A_{i,j}$ and $B_{i,j}$, $i=1,2,...,12$, $j=-12,...,12$ are zero.

From the above reasonings, it follows the next:

Theorem 3.2. *A kinematic 3-surface in* $\mathbf{E}^7$ *foliated by spheres and with* $\mathbf{K}=-\dfrac{3}{2}$ *satisfies* $\omega_1=\omega_2=\omega_7=0$ *and one of the following pairs of equations:*

$$\begin{aligned} s'^2+3\sum_{i=3}^{6}\omega_i^2 &= \sum_{i=4}^{7}b_i'^2,\\ 4\Big[s'^2+2\sum_{i=4}^{7}b_i'^2\Big] &= 9\Big[\Big(\sum_{i=4}^{7}b_i'\omega_{i+8}\Big)^2+4\Big(\sum_{i=4}^{7}b_i'\omega_{i-1}\Big)^2+4\Big(\sum_{i=4}^{7}b_i'\omega_{i+4}\Big)^2\Big], \end{aligned}$$

or

$$\begin{aligned} \sum_{i=4}^{7}b_i'\,\omega_{i-1} &= \sum_{i=4}^{7}b_i'\,\omega_{i+4}=\sum_{i=4}^{7}b_i'\,\omega_{i+8}=0,\\ 3\,s'^2+7\sum_{i=3}^{6}\omega_i^2 &= \sum_{i=4}^{7}b_i'^2. \end{aligned}$$

Case (2): Assume $\alpha_6 = \alpha_7 = \alpha_8 = 0$ and $\mathbf{K} \neq -\frac{3}{2}$. In this case we obtain

$$A_{0,6} = \frac{1}{16}\Big(\beta + 2\,\delta\Big)^2\Big[\mathbf{K}\Big(\beta + 2\,\delta\Big) - 2\Big(2\,\delta - \beta - s'^2\Big)\Big] = 0,$$

which yields

$$\mathbf{K} = \frac{2(2\,\delta - \beta - s'^2)}{\beta + 2\delta}. \tag{3.4}$$

From here, all coefficients $A_{i,j}$ and $B_{i,j}$ are equal zero. So, we have the following:

Theorem 3.3. *A kinematic 3-surface in* $\mathbf{E}^7$ *foliated by spheres and with constant scalar curvature*

$$\mathbf{K} = \frac{2\Big[\sum_{i=3}^{6} \omega_i^2 - \sum_{i=4}^{7} b'^2\Big]}{s'^2 + \sum_{i=3}^{6} \omega_i^2 + \sum_{i=4}^{7} b'^2}$$

satisfies

$$\begin{aligned}\omega_1 &= \omega_2 = \omega_7 = 0,\\ \sum_{i=4}^{7} b'_i\,\omega_{i-1} &= \sum_{i=4}^{7} b'_i\,\omega_{i+4} = \sum_{i=4}^{7} b'_i\,\omega_{i+8} = 0.\end{aligned}$$

From the expression (3.4), we can write the quantity β in the form

$$\beta = \frac{2\Big[\Big(2 - \mathbf{K}\Big)\delta - s'^2\Big]}{\mathbf{K} + 2}.$$

As β is positive, we have two cases:

(a) Case $\mathbf{K} + 2 < 0$ and $\Big(2 - \mathbf{K}\Big)\delta - s'^2 < 0$. This implies

$$\mathbf{K} < -2, \quad \text{and} \quad \mathbf{K} > \frac{2\,\delta - s'^2}{\delta} = \frac{2\sum_{i=3}^{6} \omega_i^2}{s'^2 + \sum_{i=3}^{6} \omega_i^2} > 0,$$

which is a contradiction.

(b) Case $\mathbf{K} + 2 > 0$ and $\Big(2 - \mathbf{K}\Big)\delta - s'^2 > 0$. This gives the following condition for $\mathbf{K}$:

$$-2 < \mathbf{K} < \frac{2\delta - s'^2}{\delta} = \frac{2\sum_{i=3}^{6} \omega_i^2}{s'^2 + \sum_{i=3}^{6} \omega_i^2} < 2.$$

As consequence of Theorems 3.2 and 3.3, we have the next statement, which was established in the Introduction.

Corollary 3.2. *A kinematic three-dimensional surface in* $\mathbf{E}^7$ *obtained by the equiform motion of a sphere and with constant scalar curvature* $\mathbf{K}$ *satisfies* $|\mathbf{K}| < 2$.

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