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RIEMANNIAN SUBMERSIONS FROM FRAMED METRIC MANIFOLDS

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Abstract. In this paper, we first define the concept of framed submersions between framed metric manifolds, then we provide an example and show that the vertical and horizontal distributions of such submersions are invariant with respect to the framed metric structure of the total manifold. Moreover, we obtain various properties of the O'Neill's tensors for such submersions and find the integrability of the horizontal distribution. We also find necessary and sufficient conditions for a framed submersion to be totally geodesic. The study is focused on fundamental properties and the transference of structures defined on the total manifold.

1. Introduction

The theory of Riemannian submersion was introduced by O'Neill and Gray in [12] and [9], respectively. Presently, there is an extensive literature on the Riemannian submersions with different conditions imposed on the total space and on the fibres. Riemannian submersions were considered between almost complex manifolds by Watson in [20] under the name of almost Hermitian submersion. He showed that if the total manifold is a Kähler manifold, the base manifold is also a Kähler manifold. Riemannian submersions between almost contact manifolds were studied by Chinea in [3] under the name of almost contact submersions. Since then, Riemannian submersions have been used as an effective tool to describe the structure of a Riemannian manifold equipped with a differentiable structure. For instance, Riemannian submersions have been also considered for quaternionic Kähler manifolds[10]. This kind of submersions have been studied with different names by many authors(see [7], [8],[13], [19] and more).

On the other hand, let (M, g) be a Riemannian manifold equipped with a framed metric structure, i.e. an endomorphism φ of the tangent bundle such that $\varphi^3+\varphi = 0$ and which is compatible with g; the compatibility means that for each $X, Y \in TM$ we have $g(\varphi X, Y) = -g(X, \varphi Y)$ [22]. Moreover we assume that the kernel of φ is of constant rank and parallelizable, i.e. there exist global vector fields $\xi_1, ..., \xi_s$

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spanning kerφ. Such manifolds are necessarily of dimension $2m+s$ where $2m$ is the rank of φ. The study of such manifolds was started by Blair, Goldberg and Yano ([1], [5], [6]). Later such structures were considered by other authors([2],[11],[16], [17]). In this paper, we define framed submersions between almost framed metric manifolds and study the geometry of such submersions. We observe that framed submersion has also rich geometric properties.

The paper is organized as follows. In section 2, we collect basic definitions, some formulas and results for later use. In section 3, we introduce the notion of framed submersions and give an example of framed submersion. Moreover, we investigate properties of O'Neill's tensors and show that such tensors have nice algebraic properties for framed submersions. We find the integrability of the horizontal distribution. We also find necessary and sufficient conditions for a framed submersion to be totally geodesic. Finally, section 4 is focused on the transference of structures defined on the total manifold.

2. Preliminaries

In this section we are going to recall main definitions and properties of framed metric manifolds and Riemannian submersions.
Let M be a $(2m+s)$- dimensional framed metric manifold [21](or almost s-contact metric manifold[18]) with a framed metric structure $(\varphi, \xi_j, \eta_j, g)$, $j \in \{1, ..., s\}$, that is, φ is a $(1,1)-$tensor field defining an $f-$structure of rank $2m$; $\xi_1, ..., \xi_s$ are s vector fields; $\eta_1, ..., \eta_s$ are s 1-forms and g is a Riemannian metric on M such that

$$\varphi^2 = -I + \sum_{j=1}^{s} \eta_j \otimes \xi_j, \quad \eta_j(\xi_i) = \delta_i^j, \quad \varphi(\xi_j) = 0, \quad \eta_j \circ \varphi = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{j=1}^{s} \eta_j(X)\eta_j(Y), \tag{2.2}$$

$$\Phi(X, Y) = g(X, \varphi Y) = -\Phi(Y, X), \tag{2.3}$$

$$g(X, \xi_j) = \eta_j(X) \tag{2.4}$$

for all $X, Y \in \Gamma(TM)$ and $i, j \in \{1, ..., s\}$[21].

A framed metric structure is called normal[21]if

$$[\varphi, \varphi] + 2d\eta_j \otimes \xi_j = 0, \tag{2.5}$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \tag{2.6}$$

A framed metric manifold $(M^{2m+s}, g, \varphi, \xi_j, \eta_j)$ is called

(a) *almost $\mathcal{S}$-manifold*, if $d\eta_j = \Phi$;
(b) *$\mathcal{S}$-manifold*, if $d\eta_j = d\Phi$ and normal;
(c) *$\mathcal{K}$-manifold*, if $d\Phi = 0$ and normal;
(d) *almost $\mathcal{C}$-manifold*, if $d\eta_j = 0$, $d\Phi = 0$;
(e) *$\mathcal{C}$-manifold*, if $d\eta_j = 0$, $d\Phi = 0$ and normal[15].

On $\mathcal{S}-$manifolds we have

$$(2.7)\qquad \begin{aligned}(\nabla_X\Phi)(Y,Z) &= \frac{1}{2}\sum_{i=1}^{s}[\eta_i(Y)g(X,Z)-n_i(Z)g(X,Y)]\\ &- \frac{1}{2}\sum_{i,j=1}^{s}\eta_j(X)[\eta_i(Y)\eta_j(Z)-\eta_i(Z)\eta_j(Y)],\end{aligned}$$

where ∇ denotes the Levi-Civita connection of the Riemannian metric g[14].

It is easy to see that if M is a framed metric manifold, then the following identities are well known:

$$(2.8)\qquad N^{(1)}(X,Y)=[\varphi,\varphi](X,Y)+2\sum_{j=1}^{s}d\eta_j(X,Y)\xi_j,$$

$$(2.9)\qquad (\nabla_X\varphi)Y=\nabla_X\varphi Y-\varphi(\nabla_XY),$$

$$(2.10)\qquad (\nabla_X\Phi)(Y,Z)=g(Y,(\nabla_X\varphi)Z)=-g(Z,(\nabla_X\varphi)Y),$$

$$(2.11)\qquad (\nabla_X\eta_j)Y=g(Y,\nabla_X\xi_j).$$

Let (M,g) and (B,g') be two Riemannian manifolds. A surjective $C^\infty-$map $\pi:M\to B$ is a $C^\infty-$submersion if it has maximal rank at any point of M. Putting $\mathcal{V}_x=Ker\pi_{*x}$, for any $x\in M$, we obtain an integrable distribution $\mathcal{V}$, which is called vertical distribution and corresponds to the foliation of M determined by the fibres of π. The complementary distribution $\mathcal{H}$ of $\mathcal{V}$, determined by the Riemannian metric g, is called horizontal distribution. A $C^\infty-$submersion $\pi:M\to B$ between two Riemannian manifolds (M,g) and (B,g') is called a Riemannian submersion if, at each point x of M, π_{*x} preserves the length of the horizontal vectors. A horizontal vector field X on M is said to be basic if X is $\pi-$related to a vector field X' on B. It is clear that every vector field X' on B has a unique horizontal lift X to M and X is basic.

We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields. A Riemannian submersion $\pi:M\to B$ determines two $(1,2)$ tensor fields T and A on M, by the formulas:

$$(2.12)\qquad T(E,F)=T_EF=h\nabla_{vE}vF+v\nabla_{vE}hF$$

and

$$(2.13)\qquad A(E,F)=A_EF=v\nabla_{hE}hF+h\nabla_{hE}vF$$

for any $E,F\in\Gamma(TM)$, where v and h are the vertical and horizontal projections (see [4]). From (2.12) and (2.13), one can obtain

$$(2.14)\qquad \nabla_UX=T_UX+h(\nabla_UX);$$

$$(2.15)\qquad \nabla_XU=v(\nabla_XU)+A_XU;$$

$$(2.16)\qquad \nabla_XY=A_XY+h(\nabla_XY),$$

for any $X,Y\in\Gamma(\mathcal{H})$, $U\in\Gamma(\mathcal{V})$. Moreover, if X is basic then

$$h(\nabla_U X) = h(\nabla_X U) = A_X U. \tag{2.17}$$

We note that for $U, V \in \Gamma(\mathcal{V})$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X, Y \in \Gamma(\mathcal{H})$, $A_X Y = \frac{1}{2}v[X, Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$. It is known that A is alternating on the horizontal distribution: $A_X Y = -A_Y X$, for $X, Y \in \Gamma(\mathcal{H})$ and T is symmetric on the vertical distribution: $T_U V = T_V U$, for $U, V \in \Gamma(\mathcal{V})$.

We now recall the following result which will be useful for later.

Lemma 2.1. *(see [4],[12]). If $\pi : M \to B$ is a Riemannian submersion and X, Y basic vector fields on M, $\pi-$related to X' and Y' on B, then we have the following properties*

1. *$h[X, Y]$ is a basic vector field and $\pi_* h[X, Y] = [X', Y'] \circ \pi$;*
2. *$h(\nabla_X Y)$ is a basic vector field $\pi-$related to $(\nabla'_{X'} Y')$, where ∇ and ∇' are the Levi-Civita connection on M and B;*
3. *$[E, U] \in \Gamma(\mathcal{V})$, for any $U \in \Gamma(\mathcal{V})$ and for any basic vector field E.*

3. Framed Submersions

In this section, we define the notion of framed submersion, give an example and study the geometry of such submersions. We now define a $(\varphi, \varphi')-$holomorphic map between two framed metric manifolds.

Definition 3.1. Let M^{2m+s} and B^{2n+s} be manifolds carrying the framed metric manifolds structures $(\varphi, (\xi_j, \eta_j)_{j=1}^s, g)$ and $(\varphi', (\xi'_j, \eta'_j)_{j=1}^s, g')$ respectively. A mapping $\pi : M \to B$ is said to be a $(\varphi, \varphi')-$holomorphic map if $\pi_* \circ \varphi = \varphi' \circ \pi_*$.

By using the above definition, we are ready to give the following notion.

Definition 3.2. A Riemannian submersion $\pi : M^{2m+s} \to B^{2n+s}$ between the framed metric manifolds M^{2m+s} and B^{2n+s} is called a framed submersion if:

(i) $\pi_* \xi_j = \xi'_j$, $j = 1, 2, ..., s$
(ii) $\pi_* \circ \varphi = \varphi' \circ \pi_*$.

We now give an example for framed submersion.

Example 3.1. Consider the following submersion defined by

$$\begin{aligned} \pi : R^{4+2} &\to R^{2+2} \\ (x_1, x_2, y_1, y_2, z_1, z_2) &\to \left(\frac{x_1 + x_2}{\sqrt{2}}, \frac{y_1 + y_2}{\sqrt{2}}, z_1, z_2\right). \end{aligned}$$

Then, the kernel of π_* is

$$\mathcal{V} = Ker\pi_* = Span\{V_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, V_2 = -\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\}$$

and the horizontal distribution is spanned by

$$\mathcal{H} = (Ker\pi_*)^\perp = Span\{X = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, Y = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}, \xi_1 = \frac{\partial}{\partial z_1}, \xi_2 = \frac{\partial}{\partial z_2}\}.$$

Hence, we have

$$g(X,X) = g'(\pi_* X, \pi_* X) = 4, \quad g(Y,Y) = g'(\pi_* Y, \pi_* Y) = 4$$

and

$$g(\xi_1,\xi_1) = g'(\pi_*\xi_1, \pi_*\xi_1) = 1, \quad g(\xi_2,\xi_2) = g'(\pi_*\xi_2, \pi_*\xi_2) = 1.$$

Thus, π is a Riemannnian submersion. Moreover, we can easily obtain that π satisfies

$$\pi_*\xi_1 = \xi_1', \quad \pi_*\xi_2 = \xi_2'$$

and

$$\pi_*\varphi X = \varphi'\pi_* X, \quad \pi_*\varphi Y = \varphi'\pi_* Y.$$

Thus, π is a framed submersion.

The following result can be proved in a standard way.

Proposition 3.1. *Let $\pi : M \to B$ be a framed submersion from a framed metric manifold M onto a framed metric manifold B. If X, Y are basic vector fields on M, $\pi-$related to X', Y' on B, then, we have*

(i) *$h(\nabla_X\varphi)Y$ is the basic vector field $\pi-$related to $({\nabla'}_{X'}\varphi')Y'$;*
(ii) *φX is the basic vector field $\pi-$related to $\varphi' X'$.*

Next proposition shows that a framed submersion puts some restrictions on the distributions $\mathcal{V}$ and $\mathcal{H}$.

Proposition 3.2. *Let $\pi : M \to B$ be a framed submersion from a framed metric manifold M onto a framed metric manifold B. Then, the horizontal and vertical distributions are $\varphi-$ invariant.*

Proof. Consider a vertical vector field U; it is known that $\pi_*(\varphi U) = \varphi'(\pi_* U)$. Since U is vertical and π is a Riemannian submersion, we have $\pi_* U = 0$ from which $\pi_*(\varphi U) = 0$ follows and implies that φU is vertical, being in the kernel of π_*. As concerns the horizontal distribution, let X be a horizontal vector field. We have $g(\varphi X, U) = -g(X, \varphi U) = 0$ because φU is vertical and X is horizontal. From $g(\varphi X, U) = 0$ we deduce that φX is orthogonal to U and then φX is horizontal. □

Proposition 3.3. *Let $\pi : M \to B$ be a framed submersion from a framed metric manifold M onto a framed metric manifold B. Then, we have*

(i) $\pi^*\Phi' = \Phi$;
(ii) $\pi^*\eta_j' = \eta_j,\ j = 1, ..., s.$

Proof. (i) If X and Y are basic vector fields on M, $\pi-$related to X', Y' on B, then using the definition of a framed submersion, we have

$$\begin{aligned}(\pi^*\Phi')(X,Y) &= \Phi'(\pi_* X, \pi_* Y) = g'(\pi_* X, \varphi'\pi_* Y) = g'(\pi_* X, \pi_*\varphi Y) \\ &= (\pi^* g')(X, \varphi Y) = g(X, \varphi Y) = \Phi(X,Y)\end{aligned}$$

which gives the proof of assertion(i).

(ii) Let X be basic. Let us consider the case of $\pi_*\eta_j'$. We have

$$(\pi_*\eta_j')(X) = \eta_j'(\pi_* X) = g'(\pi_* X, \xi_j') = g'(\pi_* X, \pi_*\xi_j) = (\pi^* g')(X, \xi_j).$$

Since π is a Riemannian submersion, we have $\pi^* g' = g$ so that

$$(\pi^* g')(X, \xi_j) = g(X, \xi_j) = \eta_j(X)$$

and therefore $(\pi^*\eta_j')(X) = \eta_j(X)$ which implies $\pi^*\eta_j' = \eta_j$ as claimed. □

Since each ξ_j is horizontal, we have $\eta_j(U) = 0$ for any vertical vector field U and this implies $\mathcal{V}_p \subset ker\eta_{jp}$, for any $p \in M$.

We now check the properties of the tensor fields T and A for a framed submersion, we will see that such tensors have extra properties for such submersions.

Proposition 3.4. *Let $\pi : M \to B$ be a framed submersion. If the total space is a $\mathcal{S}-$manifold or $\mathcal{K}-$manifold, then we have*

(i) $A_X\varphi Y = \varphi A_X Y$;

(ii) $A_{\varphi X}Y = \varphi A_X Y$;

(iii) $A_{\varphi X}\varphi Y = -A_X Y$;

(iv) $A_X\varphi X = 0$;

(v) $T_U\varphi V = \varphi T_U V$;

(vi) $T_{\varphi U}V = \varphi T_U V$,

for $X, Y \in \Gamma(\mathcal{H})$ and $U, V \in \Gamma(\mathcal{V})$.

Proof. We only prove (i), the other assertions can be obtained in a similar way. From (2.7) we obtain

$$(\nabla_X\Phi)(Y, U) = 0.$$

By using (2.10) we have $g((\nabla_X\varphi)Y, U) = 0$.
Thus, since the vertical and the horizontal distributions are $\varphi-$invariant, from (2.16) we obtain

$$g(A_X\varphi Y - \varphi A_X Y, U) = 0.$$

Then, we have $A_X\varphi Y = \varphi A_X Y$. □

We now investigate the integrability of the horizontal distribution $\mathcal{H}$.

Theorem 3.1. *Let $\pi : M \to B$ be a framed submersion. If the total space is a $\mathcal{S}-$manifold or a $\mathcal{K}-$manifold, then the horizontal distribution is integrable.*

Proof. Let X be basic vector field on M, and U vertical and Y horizontal. By using Proposition 3.4, then we have

$$g(A_{\varphi X}Y, U) = g(A_X\varphi Y, U) = -g(A_X U, \varphi Y).$$

From (2.17) we have $h\nabla_U X = h\nabla_X U = A_X U$. Hence we obtain

$$g(A_{\varphi X}Y, U) = -g(\varphi Y, h\nabla_U X) = g(Y, \varphi\nabla_U X). \tag{3.1}$$

On the other hand, from (2.7)we have $(\nabla_U\Phi)(X, Y) = g((\nabla_U\varphi)X, Y) = 0$. By using (2.14), we get

$$\begin{aligned} 0 &= g(Y, h\nabla_U\varphi X + T_U\varphi X - h\varphi\nabla_U X - \varphi T_U X) \\ &= g(Y, h(\nabla_U\varphi X - \varphi\nabla_U X)). \end{aligned}$$

Hence we get

$$h\nabla_U\varphi X = h\varphi\nabla_U X. \tag{3.2}$$

Then, from (3.1) and (3.2) we obtain

$$\begin{aligned} g(A_{\varphi X}Y,U) &= g(Y,h\nabla_U\varphi X) \\ &= g(Y,h\nabla_{\varphi X}U) \\ &= g(Y,A_{\varphi X}U). \end{aligned}$$

Since A is skew-symmetric operator, we get $g(A_{\varphi X}Y,U)=0$. This proves the assertion. □

Theorem 3.2. *Let $\pi : M \to B$ be a framed submersion from an almost $\mathcal{C}-$manifold M onto a framed metric manifold B. Then, the horizontal distribution is integrable.*

Proof. Let X and Y be basic vector fields. It suffices to prove that $v([X,Y])=0$, for basic vector fields on M. Since M is an almost $\mathcal{C}-$manifold, it implies $d\Phi(X,Y,V)=0$, for any vertical vector V. Then, one obtains

$$\begin{aligned} X(\Phi(Y,V)) - Y(\Phi(X,V)) + V(\Phi(X,Y)) \\ -\Phi([X,Y],V) + \Phi([X,V],Y) - \Phi([Y,V],X) = 0. \end{aligned}$$

Since $[X,V],[Y,V]$ are vertical and the two distributions are $\varphi-$invariant, the last two and the first two terms vanish. Thus, one gets

$$g([X,Y],\varphi V) = V(g(X,\varphi Y)).$$

On the other hand, if X is basic then $h(\nabla_V X) = h(\nabla_X V) = A_X V$, thus we have

$$\begin{aligned} V(g(X,\varphi Y)) &= g(\nabla_V X,\varphi Y) + g(\nabla_V\varphi Y,X) \\ &= g(A_X V,\varphi Y) + g(A_{\varphi Y}V,X). \end{aligned}$$

Since, A is skew-symmetric and alternating operator, we get $V(g(X,\varphi Y))=0$. This proves the assertion. □

Since for a $\mathcal{C}-$manifold $d\Phi=0$, applying Theorem 3.2, we have the following result.

Corollary 3.1. *Let $\pi : M \to B$ be a framed submersion from a $\mathcal{C}-$manifold M onto a framed metric manifold B. Then, the horizontal distribution is integrable.*

Theorem 3.3. *Let $\pi : M \to B$ be a framed submersion from a $\mathcal{K}-$manifold M onto a framed metric manifold B. If X horizontal vector field is an infinitesimal automorphism of $\varphi-$tensor field, then the fibres are totally geodesic.*

Proof. Let W and V be vertical vector fields on M, X horizontal. Since M is a $\mathcal{K}-$manifold, it implies $d\Phi=0$. Then, we obtain:

$$\begin{aligned} d\Phi(W,\varphi V,X) = W(\Phi(\varphi V,X)) - \varphi V(\Phi(W,X)) + X(\Phi(W,\varphi V)) \\ -\Phi([W,\varphi V],X) + \Phi([W,X],\varphi V) - \Phi([\varphi V,X],W) = 0. \end{aligned}$$

Since $[W,\varphi V]$ is vertical and the two distributions are $\varphi-$invariant, the first two terms vanish. Thus, one gets:

$$X(\Phi(W,\varphi V)) + \Phi([W,X],\varphi V) - \Phi([\varphi V,X],W) = 0.$$

Thus, we have

$$\begin{aligned} 0 &= Xg(W,V) + g([W,X],V) + g([\varphi V,X],\varphi W) \\ 0 &= g(\nabla_X V,W) + g(\nabla_W X,V) + g(\varphi[X,\varphi V],W) \\ 0 &= g([X,V]+\nabla_V X,W) + g(\nabla_W X,V) + g(\varphi[X,\varphi V],W). \end{aligned}$$

On the other hand, if X horizontal vector field is an infinitesimal automorphism of $\varphi-$tensor field, then we have

$$[X, \varphi V] = \varphi[X, V] \Rightarrow -[X, V] = \varphi[X, \varphi V].$$

Thus, we obtain

$$g(T_V X, W) + g(T_W X, V) = 0.$$

Since T_V is skew-symmetric operator, we get $g(T_V W, X) = 0$. This proves the assertion. □

From Theorem 3.3, we have the following results.

Corollary 3.2. *Let $\pi : M \to B$ be a framed submersion from an almost $\mathcal{C}-$manifold M onto a framed metric manifold B. If X horizontal vector field is an infinitesimal automorphism of $\varphi-$tensor field, then the fibres are totally geodesic.*

Corollary 3.3. *Let $\pi : M \to B$ be a framed submersion from a $\mathcal{C}-$manifold M onto a framed metric manifold B. If X horizontal vector field is an infinitesimal automorphism of $\varphi-$tensor field, then the fibres are totally geodesic.*

Corollary 3.4. *Let $\pi : M \to B$ be a framed submersion from a $\mathcal{S}-$manifold M onto a framed metric manifold B. If X horizontal vector field is an infinitesimal automorphism of $\varphi-$tensor field, then the fibres are totally geodesic.*

4. Transference of Structures

In this section, we investigate what kind of framed structures are defined on the base manifold, when the total manifold has some special framed structures.

We now recall that an almost Hermitian manifold (M, J, g) is an almost complex manifold (M, J) with a $J-$invariant Riemannian metric g. The $J-$invariance of g means that $g(JX, JY) = g(X, Y)$, for any $X, Y \in \chi(M)$[20].

As the fibres of a framed submersion is an invariant submanifold of M with respect to φ, we have the following.

Proposition 4.1. *Let $\pi : (M^{2m+s}, \varphi, \xi, \eta, g) \to (B^{2n+s}, \varphi', \xi', \eta', g')$ be a framed submersion from a framed metric manifold M onto a framed metric manifold B. Then, the fibres are almost Hermitian manifolds.*

Proof. Denoting by F the fibres, it is clear that $dimF = 2(m - n) = 2r$, where $r = m - n$. On $(F^{2r}, \hat{g})$, setting $J = \hat{\varphi}$ and $g|_F = \hat{g}$ we have to show that $(J, \hat{g})$ is an almost Hermitian structure. Indeed, by using the definition of a framed metric structure we get

$$J^2 U = \varphi^2 U = -U + \sum_{j=1}^{s} \eta_j(U)\xi_j.$$

Since $\eta_j(U) = 0$, we have $J^2 U = -U$. On the other hand,

$$g(JV, JU) = -g(V, J^2 U) = g(V, U)$$

which achieves the proof. □

In the sequel, we show that base space is a normal if the total space is a normal.

Theorem 4.1. *Let $\pi : M \to B$ be a framed submersion. If the framed metric structure of M is normal, then the framed metric structure of B is normal.*

Proof. Let X and Y be basic. From (2.8), we have

$$\pi_* N^{(1)}(X,Y) = \pi_*([\varphi,\varphi](X,Y) + \sum_{j=1}^{s} 2d\eta_j(X,Y)\xi_j).$$

On the other hand, $\pi_*\varphi = \varphi'\pi_*$ and $\pi_*\xi_j = \xi'_j$ imply that

$$\begin{array}{rcl} \pi_*([\varphi,\varphi](X,Y)) &=& \pi_*\varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y] \\ &=& [\pi_* X,\pi_* Y] - \eta[X,Y]\pi_*\xi_j + [\pi_*\varphi X,\pi_*\varphi Y] - \varphi'\pi_*[\varphi X,Y] \\ &-& \varphi'\pi_*[X,\varphi Y] \\ &=& [X',Y'] - g'([X',Y'],\xi'_j)\xi'_j + [\varphi' X',\varphi' Y'] - \varphi'[\varphi' X',Y'] \\ &-& \varphi'[X',\varphi' Y']. \end{array}$$

Then, we have

$$\pi_*[\varphi,\varphi](X,Y) = N'(X',Y'). \tag{4.1}$$

In a similar way, since π is a Riemannian submersion, by using Proposition 3.3(ii), we have

$$\pi_* 2d\eta_j \otimes \xi_j = 2d\eta'_j \otimes \xi'_j. \tag{4.2}$$

Now, from (4.1) and (4.2) we obtain

$$\pi_* N^{(1)}(X,Y) = N'^{(1)}(X',Y') = 0.$$

□

Proposition 4.2. *Let $\pi : M \to B$ be a framed submersion. If the total space M is an almost $\mathcal{C}-$manifold or a $\mathcal{K}-$manifold, then the base space B belongs to the same class.*

Proof. Let X, Y and Z be basic vector fields on M, $\pi-$related to X', Y' and Z' on B. Since M is an almost $\mathcal{C}-$manifold, it implies $d\Phi(X,Y,Z) = 0$. Then, we have

$$\begin{gathered} X(\Phi(Y,Z)) - Y(\Phi(X,Z)) + Z(\Phi(X,Y)) \\ -\Phi([X,Y],Z) + \Phi([X,Z],Y) - \Phi([Y,Z],X) = 0. \end{gathered}$$

On the other hand, by direct calculations, we obtain

$$\begin{array}{rcl} 0 &=& g(\nabla_X Y,\varphi Z) + g(Y,\nabla_X\varphi Z) - g(\nabla_Y X,\varphi Z) - g(X,\nabla_Y\varphi Z) \\ &+& g(\nabla_Z X,\varphi Y) + g(X,\nabla_Z\varphi Y) - g([X,Y],\varphi Z) \\ &+& g([X,Z],\varphi Y) - g([Y,Z],\varphi X). \end{array}$$

Then, by using $\pi_*\varphi = \varphi'\pi_*$, we get

$$\begin{aligned} &0 = g'(\nabla'_{X'}Y',\varphi' Z') + g'(Y',\nabla'_{X'}\varphi' Z') - g'(\nabla'_{Y'}X',\varphi' Z') - g'(X',\nabla'_{Y'}\varphi' Z') \\ &+g'(\nabla'_{Z'}X',\varphi' Y') + g'(X',\nabla'_{Z'}\varphi' Y') - g'([X',Y'],\varphi' Z') \\ &+g'([X',Z'],\varphi' Y') - g'([Y',Z'],\varphi' X') \\ &0 = X'(\Phi'(Y',Z')) - Y'(\Phi'(X',Z')) + Z'(\Phi'(X',Y')) \\ &-\Phi'([X',Y'],Z') + \Phi'([X',Z'],Y') - \Phi'([Y',Z'],X') \end{aligned}$$

(4.3) $0 = d\Phi'(X',Y',Z')$

In a similar way, we have

$$0 = d\eta_j = d\eta'_j. \tag{4.4}$$

Thus, from (4.3) and (4.4) if the total space M is an almost $\mathcal{C}-$manifold, then the base space B belongs to the same class.

From Theorem 4.1 we have

$$\begin{aligned} \pi_* N^{(1)}(X,Y) &= \pi_*([\varphi,\varphi](X,Y) + \sum_{j=1}^{s} 2d\eta_j(X,Y)\xi_j) \\ &= [\varphi',\varphi'](X',Y') + \sum_{j=1}^{s} 2d\eta'_j(X',Y')\xi'_j \\ &= N'^{(1)}(X',Y') = 0. \end{aligned} \tag{4.5}$$

On the other hand, by direct calculations, we obtain

$$d\Phi(X,Y,Z) = d\Phi'(X',Y',Z') = 0. \tag{4.6}$$

Thus, from (4.5) and (4.6) if the total space M is a $\mathcal{K}-$manifold, then the base space B belongs to the same class. □

We also have the following result which shows that the other structures can be mapped onto the base manifold.

Proposition 4.3. *Let $\pi : M \to B$ be a framed submersion. If M belongs to any of the classes $\mathcal{C}-$manifold, almost $\mathcal{S}-$manifold or $\mathcal{S}-$manifold, then the base space B belongs to the same class.*

References

[1] Blair, D.E., Geometry of manifolds with structural group $\mathcal{U}(n) \times \mathcal{O}(s)$. *J. Differential Geom.* 4 (2)(1970), 155-167.

[2] Cabrerizo, J.L., Fernandez, L.M. and Fernandez, M., The curvature tensor field on $f-$manifold with complemented frames. *An. Univ. 'Al.I.Cuza', Iași, Matematica* 36 (1990), 151-161.

[3] Chinea, D., Almost contact metric submersions. *Rend. Circ. Mat. Palermo, II Ser.* 34 (1985), 89-104.

[4] Falcitelli, M., Ianuș, S. and Pastore, A.M., Riemannian submersions and related topics. World Scientific, 2004.

[5] Goldberg, S.I. and Yano, K., On normal globally framed $f-$manifolds. *Tôhoku Math. Journal* 22 (1970), 362-370.

[6] Goldberg, S.I. and Yano, K., Globally framed $f-$manifolds. *Illinois Math. Journal* 22 (1971), 456-474.

[7] Gündüzalp, Y. and Șahin, B., Paracontact semi-Riemannian submersions. *Turkish J.Math.* 37 (2013), 114-128.

[8] Gündüzalp, Y. and Șahin, B., Para-contact para-complex semi-Riemannian submersions. *Bull. Malays. Math. Sci. Soc.* In Press.

[9] Gray, A., Pseudo-Riemannian almost product manifolds and submersions. *J. Math. Mech.* 16 (1967), 715-737.

[10] Ianuș, S., Mazzocco, R. and Vilcu, G.V., Riemannian submersions from quaternionic manifolds. *Acta Appl. Math.* 104 (2008), 83-89.

[11] Leo, G.D. and Lotta, A., On the structure and symmetry properties of almost $S-$manifolds. *Geom. Dedicata* 110 (2005), 191-211.

[12] O'Neill, B., The fundamental equations of a submersion. *Michigan Math. J.* 13 (1966), 459 469.

[13] Şahin, B., Anti-invariant Riemannian submersions from almost Hermitian manifolds. *Cent. Eur. J. Math.* 8 (2010), 437-447.
[14] Terlizzi, L.D., On invariant submanifolds of $C-$and $S-$manifolds. *Acta Math. Hungar.* 85 (1999), 229-239.
[15] Terlizzi, L.D., Scalar and $\varphi-$sectional curvature of a certain type of metric $f-$structures. *Mediterr. j. math.* 3 (2006),533-547.
[16] Vaisman, I., Generalized Hopf manifolds. *Geom. Dedicata* 13 (1982), 231-255.
[17] Vaisman, I., A survey of generalized Hopf manifolds. *Rend. Sem. Math., Univ. Politec. Torino* (1984), special issue.
[18] Vanzura, J., Almost s-contact structures. *Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat.* 26 (1972), 97-115.
[19] Vilcu, G.V., 3-submersions from QR-hypersurfaces of quaternionic Kähler manifolds. *Ann. Polon. Math.* 98 (2010), 301-309.
[20] Watson, B., Almost Hermitian submersions. *J. Differential Geom.* 11 (1976), 147-165.
[21] Yano, K. and Kon, M., Structures on manifolds. World Scientific, 1984.
[22] Yano, K., On a structure defined by a tensor field f satisfying $f^3 + f = 0$. *Tensor* 14 (1963),99-109.

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