International Electronic Journal of Geometry
Volume 6 No.1 pp. 100-111 (2013) ©IEJG

NATURAL METRICS ON T^2M AND HARMONICITY.

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(Communicated by Angel FERRANDEZ)

Abstract. In this paper, we define a natural metric on the tangent bundle of order two T^2M, and we investigate the geometry and the harmonicity of sections as maps from a Riemannian manifold (M, g) onto its tangent bundle T^2M.

1. Introduction

The sections on tangent bundle of order two T^2M (bundle of accelerations on a smooth manifold M), locally, described in detail the second order ordinary differential equations on M (Dodson and Galanis [11] and [2]). These equations have received renewed geometric attention in recent years from interactions with jet fields, linear and nonlinear connections, Lagrangians, Finsler structures and the theory of timedependent Lagrangian particle systems (see [3] ,[4],[17], [18], [19]). As a natural generalization of the works of Ishihara [13], Konderak [14], Oniciuc [15], Boeckx and Vanhecke [5] and Abbassi, Calvaruso and Perrone [1], [6]; In this note, we define a natural metric on the tangent bundle of order two T^2M, and we investigate the geometry and the harmonicity of sections as maps from a Riemannian manifold (M, g) onto its tangent bundle T^2M.

1.1. **Harmonic maps.** Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, then the energy functional is defined by

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g. \tag{1.1}$$

(or over any compact subset $K \subset M$).

A map is called harmonic if it is a critical point of the energy functional E (or $E(K)$ for all compact subsets $K \subset M$). For any smooth variation $\{\phi\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d\phi_t}{dt}\Big|_{t=0}$, we have

$$\frac{d}{dt}E(\phi_t)\Big|_{t=0} = -\int_M h(\tau(\phi), V)\, dv_g, \tag{1.2}$$

2000 *Mathematics Subject Classification.* 53A45, 53C20, 58E20.
Key words and phrases. Horizontal lift, vertical lift, natural metric, λ-lift, harmonic section.
The author is supported by Algerian agency of research ANDRU.

where

$$\tau(\phi) = trace_g \nabla d\phi. \tag{1.3}$$

is the tension field of ϕ. Then we have

Theorem 1.1. *A smooth map $\phi : (M^m, g) \to (N^n, h)$ is harmonic if and only if*

$$\tau(\phi) = 0. \tag{1.4}$$

If $(x^i)_{1\leq i\leq m}$ and $(y^\alpha)_{1\leq\alpha\leq n}$ denote local coordinates on M and N respectively, then equation (1.4) takes the form

$$\tau(\phi)^\alpha = \left(\Delta\phi^\alpha + g^{ij}\,{}^N\Gamma^\alpha_{\beta\gamma}\frac{\partial\phi^\beta}{\partial x^i}\frac{\partial\phi^\gamma}{\partial x^j}\right) = 0, \quad 1 \leq \alpha \leq n, \tag{1.5}$$

where $\Delta\phi^\alpha = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial\phi^\alpha}{\partial x^j}\right)$ is the Laplace operator on (M^m, g) and ${}^N\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols on N. One can refer to [12], [13], [15] and [16] for background on harmonic maps.

2. Some results on TM.

2.1. **Horizontal and vertical lifts on TM.** Let (M, g) be an m-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=1...n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1...n}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.
We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by :

$$\begin{aligned}\mathcal{V}_{(x,u)} &= Ker(d\pi_{(x,u)}) = \{a^i\frac{\partial}{\partial y^i}|_{(x,u)};\quad a^i \in \mathbb{R}\}\\ \mathcal{H}_{(x,u)} &= \{a^i\frac{\partial}{\partial x^i}|_{(x,u)} - a^iu^j\Gamma^k_{ij}\frac{\partial}{\partial y^k}|_{(x,u)};\quad a^i \in \mathbb{R}\},\end{aligned}$$

where $(x, u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.
Let $X = X^i\frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i\frac{\partial}{\partial y^i} \tag{2.1}$$

$$X^H = X^i\frac{\delta}{\delta x^i} = X^i\{\frac{\partial}{\partial x^i} - y^j\Gamma^k_{ij}\frac{\partial}{\partial y^k}\} \tag{2.2}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=1...n}$ is a local adapted frame in TTM.

Remark 2.1. .

(1) if $w = w^i\frac{\partial}{\partial x^i} + \overline{w}^j\frac{\partial}{\partial y^j} \in T_{(x,u)}TM$, then its horizontal and vertical parts are defined by

$$w^h = w^i\frac{\partial}{\partial x^i} - w^iu^j\Gamma^k_{ij}\frac{\partial}{\partial y^k} \in \mathcal{H}_{(x,u)}$$

$$w^v = \{\overline{w}^k + w^i u^j \Gamma_{ij}^k\} \frac{\partial}{\partial y^k} \in \mathcal{V}_{(x,u)}$$

(2) if $u = u^i \frac{\partial}{\partial x^i} \in T_x M$ then its vertical and horizontal lifts are defined by

$$\begin{aligned} u^V &= u^i \frac{\partial}{\partial y^i} \in \mathcal{V}_{(x,u)} \in \mathcal{H}_{(x,u)} \\ u^H &= u^i \{ \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \}. \end{aligned}$$

2.2. **β-metric on TM.**

Definition 2.1. Let (M,g) be a Riemannian manifold and $\beta \in \mathbb{R}_+$. On the tangent bundle TM, we define a β-metric noted $\widetilde{g}$ by

(1) $\widetilde{g}(X^H, Y^H) = \frac{\varepsilon}{2} g(X,Y) \circ \pi$
(2) $\widetilde{g}(X^H, Y^V) = 0$
(3) $\widetilde{g}_{(x,u)}(X^V, Y^V) = \frac{1}{\alpha}(g_x(X,Y) + \beta g_x(X,u) g_x(X,u))$

where $X, Y \in \Gamma(TM)$, $(x,u) \in TM$, $r = g(u,u)$, $\alpha = 1 + \beta g_x(u,u)$ and $\varepsilon \in \{1,2\}$.

Note that, if $\beta = 0$ (resp $\beta = 1$) and $\varepsilon = 2$, then $\widetilde{g}$ is the Sasaki metric [20] (resp the Cheeger-Gromoll metric [7]).

In the sequel we take $\varepsilon = 1$.

Lemma 2.1. *Let (M,g) be a Riemannian manifold, then for all $x \in M$ and $u = u^i \frac{\partial}{\partial x^i} \in T_x M$, we have the following*

(1) $X^H(g(u,u))_{(x,u)} = 0$
(2) $X^H(g(Y,u))_{(x,u)} = g(\nabla_X Y, u)_x$
(3) $X^V(g(u,u)_{(x,u)} = 2g(X,u)_x$
(4) $X^V(g(Y,u)_{(x,u)} = g(X,Y)_x$

Proof. Localy, if $U : x \in M \to U_x = u^i \frac{\partial}{\partial x^i} \in TM$ be a local vector field constant on each fiber $T_x M$, then from formulas (2.1) and (2.2) we obtain :

$$\begin{aligned} 1. \quad X^H(g(u,u))_{(x,u)} &= [X^i \frac{\partial}{\partial x^i} g_{st} y^s y^t - \Gamma_{ij}^k X^i y^j \frac{\partial}{\partial y^k} g_{st} y^s y^t]_{(x,u)} \\ &= X(g(U,U)_x - 2(\Gamma_{ij}^k X^i y^j g_{sk} y^s)_{(x,u)} \\ &= (X(g(U,U)_x - 2g(U, \nabla_X U))_x \\ &= 0. \\ 2. \quad X^H(g(u,u))_{(x,u)} &= [X^i \frac{\partial}{\partial x^i} g_{st} Y^s y^t - \Gamma_{ij}^k X^i y^j \frac{\partial}{\partial y^k} g_{st} Y^s y^t]_{(x,u)} \\ &= X(g(Y,U)_x - (\Gamma_{ij}^k X^i y^j g_{sk} Y^s)_{(x,u)} \\ &= (X(g(Y,U)_x - g(Y, \nabla_X U))_x \\ &= g(\nabla_X Y, U))_x. \\ 3. \quad X^V(g(u,u))_{(x,u)} &= [X^i \frac{\partial}{\partial y^i} g_{st} y^s y^t]_{(x,u)} = 2X^i g_{it} u^t = 2g(X,u)_x \\ 4. \quad X^V(g(Y,u))_{(x,u)} &= [X^i \frac{\partial}{\partial y^i} g_{st} Y^s y^t]_{(x,u)} = X^i g_{si} Y^s = g(X,Y)_x \end{aligned}$$

Theorem 2.1. *Let (M,g) be a Riemannian manifold and $\widetilde{g}$ be a β-metric relative to g on TM. If ∇ (resp $\widetilde{\nabla}$) denote the Levi-Civita connection of (M,g) (resp $(TM,\widetilde{g})$), then we have:*

$$
\begin{aligned}
1.\quad (\widetilde{\nabla}_{X^H}Y^H)_p &= (\nabla_X Y)^H - \frac{1}{2}(R(X,Y)u)^V,\\
2.\quad (\widetilde{\nabla}_{X^H}Y^V)_p &= (\nabla_X Y)^V + \frac{1}{2\alpha}(R(u,Y)X)^H\\
3.\quad (\widetilde{\nabla}_{X^V}Y^H)_p &= \frac{1}{2\alpha}(R_x(u,X)Y))^H\\
4.\quad (\widetilde{\nabla}_{X^V}Y^V)_p &= -\frac{\beta}{\alpha}\Big[\widetilde{g}(X^V,U^V)Y^V + \widetilde{g}(Y^V,U^V)X^V + \beta\widetilde{g}(X^V,U^V)\widetilde{g}(Y^V,U^V)U^V\\
&\quad -(1+\alpha)\widetilde{g}(X^V,Y^V)U^V\Big]_p\\
&= -\frac{\beta}{\alpha^2}\Big[\alpha(g(X,U)Y^V + g(Y,U)X^V) - \beta g(X,U)g(Y,U)U^V\\
&\quad -(1+\alpha)g(X,Y)U^V\Big]_p
\end{aligned}
$$

for all vector fields $X,Y\in\Gamma(TM)$ and $p=(x,u)\in TM$, where R denote the curvature tensor of (M,g),.

The proof of Theorem 2.1 follows directly from Kozul formula, Lemma 2.1 and the following formulas:

$$
\begin{aligned}
1.\quad \widetilde{g}(X^V,U) &= \frac{1}{\alpha}[g(X,u)+\beta g(X,u)g(u,u)] = g(X,u)\\
2.\quad X^V(\widetilde{g}(Y^V,Z^V) &= -\frac{2\beta}{\alpha^2}g(X,U)[g(Y,Z)+\beta g(Y,U)g(Z,U)]\\
&\quad +\frac{\beta}{\alpha}[g(X,Y)g(Z,U)+g(X,Z)g(Y,U)]
\end{aligned}
$$

3. Natural Metric on T^2M.

3.1. **Some results on T^2M.** Let (M,g) be a Riemannian manifold and ∇ its Levi-Civita connection. The tangent bundle of order 2 is the natural bundle of 2-jets of differentiable curves, defined by:

$$T^2M = \{j_0^2\gamma \quad ; \quad \gamma:\mathbb{R}_0\to M, \text{ is a smooth curve at } 0\in\mathbb{R}\}$$

Theorem 3.1 ([11]). *If $TM\oplus TM$ denotes the Whitney sum, then*

$$
\begin{aligned}
S: T^2M &\to TM\oplus TM\\
j_0^2\gamma &\mapsto (\dot\gamma(0),(\nabla_{\dot\gamma(0)}\dot\gamma)(0))
\end{aligned}
\tag{3.1}
$$

is a diffeomorphism of natural bundles.

In the induced coordinate, we have

$$S:(x^i,y^i,z^i)\mapsto(x^i,y^i,z^i+y^jy^k\Gamma^i_{jk}) \tag{3.2}$$

Definition 3.1 ([9])**.** Let T^2M be a tangent bundle of order 2 endowed with the vectorial structure induced by the diffeomorphism S. For any section $\sigma \in \Gamma(T^2M)$, we define two vector fields on M by:

$$X_\sigma = P_1 \circ S \circ \sigma \tag{3.3}$$
$$Y_\sigma = P_2 \circ S \circ \sigma \tag{3.4}$$

where P_1 and P_2 denotes the first and the second projection from $TM \oplus TM$ onto TM.

3.2. **λ-lifts on T^2M.**

Definition 3.2 ([11])**.** Let (M, g) be a Riemannian manifold and $X \in \Gamma(TM)$ be a vector field on M. For $\lambda = 0, 1, 2$, the λ-lift of X to T^2M is defined by

$$\begin{aligned} X^0 &= S_*^{-1}(X^H, X^H) \\ X^1 &= S_*^{-1}(X^V, 0) \\ X^2 &= S_*^{-1}(0, X^V) \end{aligned} \tag{3.5}$$

From Definition 3.2 we obtain:

Theorem 3.2. *Let (M, g) be a Riemannian manifold. If R denote the tensor curvature of (M, g) , then for all vector fields $X, Y \in \Gamma(TM)$ and $p \in T^2M$ we have*

(1) $[X^0, Y^0]_p = [X, Y]^0_p - (R_x(X, Y)u)^1 - (R_x(X, Y)w)^2$
(2) $[X^0, Y^i] = (\nabla_X Y)^i$
(3) $[X^i, Y^j] = 0.$

where $(x, u, w) = S(p)$ and $i, j = 1, 2$.

3.3. **Natural-metric on T^2M.**

Definition 3.3. Let (M, g) be a Riemannian manifold and $\beta_1, \beta_2 \in \mathbb{R}_+$. We define a natural metric G on the tangent bundle of order two T^2M by

$$G = S_*^{-1}(\widetilde{g_1} \oplus \widetilde{g_2}) \tag{3.6}$$

where $\widetilde{g_1}$ (resp $\widetilde{g_2}$) denote the β_1-metric (resp β_2-metric) on TM.

From Definitions 3.3 and formulae (3.5), we have

Proposition 3.1. *If $p \in T^2M$, then for all vector fields $X, Y \in \Gamma(TM)$ and $i, j \in \{0, 1, 2\}$ $(i \neq j)$, we obtain*

(1) $G_p(X^0, Y^0) = g(X, Y)_x$
(2) $G_p(X^i, Y^j) = 0$
(3) $G_p(X^1, Y^1) = \frac{1}{\alpha_1}(g(X, Y) + \beta_1 g(X, u)g(Y, u))_x$
(4) $G_p(X^2, Y^2) = \frac{1}{\alpha_2}(g(X, Y) + \beta_2 g(X, w)g(Y, w))_x$

where $S(p) = (x, u, w) \in T_xM \oplus T_xM$, $\alpha_1 = 1 + \beta_1 g(u, u)$ and $\alpha_2 = 1 + \beta_2 g(w, w)$

Note that, if $\beta_1 = \beta_2 = 0$ then G is the Diagonal metric on T^2M.

Theorem 3.3. *Let (M, g) be a Riemannian manifold and $\beta_1, \beta_2 \in \mathbb{R}_+$. If $\widetilde{\nabla}$ denote the Levi-Civita connection of (T^2M, G), then for $p \in T^2M$, $X, Y \in \Gamma(TM)$ and $i, j = 1, 2$ $(i \neq j)$ we have:*

$$
\begin{aligned}
1.\quad (\widetilde{\nabla}_{X^0}Y^0)_p &= (\nabla_X Y)^0 - \frac{1}{2}(R(X,Y)u)^1 - \frac{1}{2}(R(X,Y)w)^2,\\
2.\quad (\widetilde{\nabla}_{X^0}Y^1)_p &= (\nabla_X Y)^1 + \frac{1}{2\alpha_1}(R(u,Y)X)^0\\
3.\quad (\widetilde{\nabla}_{X^0}Y^2)_p &= (\nabla_X Y)^2 + \frac{1}{2\alpha_2}(R(w,Y)X)^0\\
4.\quad (\widetilde{\nabla}_{X^1}Y^0)_p &= \frac{1}{2\alpha_1}(R(u,X)Y))^0\\
5.\quad (\widetilde{\nabla}_{X^2}Y^0)_p &= \frac{1}{2\alpha_2}(R(w,X)Y))^0\\
6.\quad (\widetilde{\nabla}_{X^1}Y^1)_p &= -\frac{\beta_1}{\alpha_1^2}\Big[\alpha_1\Big(g(X_x,u)Y^1 + g(Y_x,u)X^1\Big) - \beta_1 g(X_x,u)g(Y_x,u)u^1\\
&\quad -(1+\alpha_1)g(X_x,Y_x)u^1\Big]\\
7.\quad (\widetilde{\nabla}_{X^2}Y^2)_p &= -\frac{\beta_2}{\alpha_2^2}\Big[\alpha_2\Big(g(X_x,w)Y^2 + g(Y_x,w)X^2\Big) - \beta_2 g(X_x,w)g(Y_x,w)w^2\\
&\quad -(1+\alpha_2)g(X_x,Y_x)w^2\Big]\\
8.\quad (\widetilde{\nabla}_{X^i}Y^j)_p &= 0.
\end{aligned}
$$

where ∇ and R denotes the Levi-Civita connection and the curvature tensor respectively of (M,g).

Using Proposition 3.1 and Kozul formula the Theorem 3.3 follows.

Lemma 3.1. *Let (M,g) be a Riemannian manifold . If $X,Y \in \Gamma(TM)$ are a vector fields and $(x,u) \in TM$ such that $X_x = u$, then we have*

$$d_xX(Y_x) = Y^H_{(x,u)} + (\nabla_Y X)^V_{(x,u)}.$$

Proof. : Let (U,x^i) be a local chart on M in $x \in M$ and $(\pi^{-1}(U),x^i,y^j)$ be the induced chart on TM, if $X_x = X^i(x)\frac{\partial}{\partial x^i}|_x$ and $Y_x = Y^i(x)\frac{\partial}{\partial x^i}|_x$, then

$$d_xX(Y_x) = Y^i(x)\frac{\partial}{\partial x^i}|_{(x,X_x)} + Y^i(x)\frac{\partial X^k}{\partial x^i}(x)\frac{\partial}{\partial y^k}|_{(x,X_x)},$$

thus the horizontal part is given by

$$
\begin{aligned}
(d_xX(Y_x))^h &= Y^i(x)\frac{\partial}{\partial x^i}|_{(x,X_x)} - Y^i(x)X^j(x)\Gamma^k_{ij}(x)\frac{\partial}{\partial y^k}|_{(x,X_x)}\\
&= Y^H_{(x,X_x)}
\end{aligned}
$$

and the vertical part is given by

$$
\begin{aligned}
(d_xX(Y_x))^v &= \{Y^i(x)\frac{\partial X^k}{\partial x^i}(x) + Y^i(x)X^j(x)\Gamma^k_{ij}(x)\}\frac{\partial}{\partial y^k}|_{(x,X_x)}\\
&= (\nabla_Y X)^V_{(x,X_x)}.
\end{aligned}
$$

Lemma 3.2. *Let (M,g) be a Riemannian manifold . If $Z \in \Gamma(TM)$ and $\sigma \in \Gamma(T^2M)$, then we have*

$$d_x\sigma(Z_x) = Z^0_p + (\nabla_Z X_\sigma)^1_p + (\nabla_Z Y_\sigma)^2_p. \tag{3.7}$$

where $p = \sigma(x)$.

Proof. : Using Lemma 3.1, we obtain

$$\begin{aligned} d_x\sigma(Z) &= dS^{-1}(dX_\sigma(Z), dY_\sigma(Z))_{S(p)} \\ &= dS^{-1}(Z^H, Z^H)_{S(p)} + dS^{-1}((\nabla_Z X_\sigma)^V, (\nabla_Z Y_\sigma)^V)_{S(p)} \\ &= Z_p^0 + (\nabla_Z X_\sigma)_p^1 + (\nabla_Z Y_\sigma)_p^2 \end{aligned}$$

Lemma 3.3. *Let (M, g) be a Riemannian n-dimensional manifold and (T^2M, G) be its tangent bundle of order two equipped with the natural metric. if $\sigma \in \Gamma(T^2M)$, then the energy density associated to σ is given by*

$$\begin{aligned} e(\sigma) &= \frac{n}{2} + \frac{1}{2\alpha_1} trace_g g(\nabla X_\sigma, \nabla X_\sigma) + \frac{1}{2\alpha_2} trace_g g(\nabla Y_\sigma, \nabla Y_\sigma) \\ &+ \frac{\beta_1}{2\alpha_1} trace_g [g(\nabla X_\sigma, X_\sigma)]^2 + \frac{\beta_2}{2\alpha_2} trace_g [g(\nabla Y_\sigma, Y_\sigma)]^2. \end{aligned}$$

Proof. Let $p = S^{-1}(x, u, w) \in T^2M$ and $(e_1, ..., e_n)$ be a local orthonormal frame on M at x, then

$$2e(\sigma)_p = \sum_{i=1}^{n} G_p(d\sigma(e_i), d\sigma(e_i))$$

Using formula 3.7 , we obtain

$$\begin{aligned} 2e(\sigma)_p &= \sum_{i=1}^{n} G_p(e_i^0, e_i^0) + \sum_{i=1}^{n} G_p(\nabla_{e_i} X_\sigma)^1, (\nabla_{e_i} X_\sigma)^1) \\ &+ \sum_{i=1}^{n} G_p(\nabla_{e_i} Y_\sigma)^1, (\nabla_{e_i} Y_\sigma)^1) \end{aligned}$$

Taking account that $(X_\sigma)_x = u$ and $(Y_\sigma)_x = w$, then from Proposition 3.1, we deduce :

$$\begin{aligned} 2e(\sigma) &= n + \frac{1}{\alpha_1} trace_g g(\nabla X_\sigma, \nabla X_\sigma) + \frac{1}{\alpha_2} trace_g g(\nabla Y_\sigma, \nabla Y_\sigma) \\ &+ \frac{\beta_1}{\alpha_1} trace_g [g(\nabla X_\sigma, X_\sigma)]^2 + \frac{\beta_2}{\alpha_2} trace_g [g(\nabla Y_\sigma, Y_\sigma)]^2. \end{aligned}$$

Theorem 3.4. *Let (M, g) be a Riemannian manifold and (T^2M, G) be its tangent bundle of order two equipped with the natural metric. Then the tension field associated with $\sigma \in \Gamma(T^2M)$ is given by:*

$$\begin{aligned} \tau(\sigma) &= (trace_g A(X_\sigma))^1 + (trace_g B(Y_\sigma))^2 \\ &+ (trace_g \{R(X_\sigma, \nabla_* X_\sigma) * + R(Y_\sigma, \nabla_* Y_\sigma) *\})^0. \end{aligned} \tag{3.8}$$

where $A(X_\sigma)$ and $B(Y_\sigma))$ are a bilinear forms defined by:

$$\begin{aligned} A(X_\sigma) &= \nabla^2 X_\sigma + \frac{(1+\alpha_1)\beta_1}{\alpha_1^2} g(\nabla X_\sigma, \nabla X_\sigma) X_\sigma + \frac{\beta_1^2}{\alpha_1^2} g(\nabla X_\sigma, X_\sigma)^2 X_\sigma \\ &- 2\frac{\beta_1}{\alpha_1} g(\nabla X_\sigma, X_\sigma) \nabla X_\sigma \end{aligned}$$

$$
\begin{aligned}
B(Y_\sigma) &= \nabla^2 Y_\sigma + \frac{(1+\alpha_2)\beta_2}{\alpha_2^2} g(\nabla Y_\sigma, \nabla Y_\sigma) Y_\sigma + \frac{\beta_2^2}{\alpha_2^2} g(\nabla Y_\sigma, Y_\sigma)^2 Y_\sigma \\
&\quad -2\frac{\beta_2}{\alpha_2} g(\nabla Y_\sigma, Y_\sigma)\nabla Y_\sigma
\end{aligned}
$$

Proof. Let $x \in M$ and $\{e_i\}_{i=1}^n$ be a local orthonormal frame on M such that $\nabla_{e_i} e_j = 0$, then by summing over i, we have

$$
\begin{aligned}
\tau(\sigma)_x &= \Big[\widetilde{\nabla}_{d\sigma(e_i)} d\sigma(e_i)\Big]_{\sigma(x)} \\
&= \Big[\widetilde{\nabla}_{e_i^0 + (\nabla_{e_i} X_\sigma)^1 + (\nabla_{e_i} Y_\sigma)^2}\Big(e_i^0 + (\nabla_{e_i} X_\sigma)^1 + (\nabla_{e_i} Y_\sigma)^2\Big)\Big]_{\sigma(x)} \\
&= \Big[\widetilde{\nabla}_{e_i^0} e_i^0 + \widetilde{\nabla}_{e_i^0} (\nabla_{e_i} X_\sigma)^1 + \widetilde{\nabla}_{e_i^0} (\nabla_{e_i} Y_\sigma)^2 + \widetilde{\nabla}_{(\nabla_{e_i} X_\sigma)^1} e_i^0 \\
&\quad + \widetilde{\nabla}_{(\nabla_{e_i} Y_\sigma)^2} e_i^0 + \widetilde{\nabla}_{(\nabla_{e_i} X_\sigma)^1} (\nabla_{e_i} X_\sigma)^1 + \widetilde{\nabla}_{(\nabla_{e_i} Y_\sigma)^2} (\nabla_{e_i} Y_\sigma)^2\Big]_{\sigma(x)}
\end{aligned}
$$

and using Theorem 3.3 the formula (3.8) follows.

Theorem 3.5. *Let (M, g) be a Riemannian manifold and (T^2M, G) be its tangent bundle of order two equipped with the natural metric. A section $\sigma : M \to T^2M$ is harmonic if and only the following conditions are verified*

$$
\begin{aligned}
&trace_g(trace_g A(X_\sigma)) = 0, \\
&trace_g(trace_g B(Y_\sigma)) = 0, \\
&trace_g\{R(X_\sigma, \nabla_* X_\sigma) * + R(Y_\sigma, \nabla_* Y_\sigma) *\} = 0.
\end{aligned}
$$

Corollary 3.1. *Let (M, g) be a Riemannian manifold and (T^2M, G) be its tangent bundle of order two equipped with the natural metric. If $\sigma : M \to T^2M$ is a section such that X_σ and Y_σ are parallel, then σ is harmonic.*

Theorem 3.6. *Let (M, g) be a Riemannian compact manifold and (T^2M, G) be its tangent bundle of order two equipped with the natural metric. Then $\sigma : M \to T^2M$ is a harmonic section if and only if X_σ and Y_σ are parallel (i.e : $\nabla X_\sigma = \nabla Y_\sigma = 0$).*

Proof. . If σ is parallel, from Corollary 3.1, we deduce that σ is harmonic. Inversely. Let σ_t be a compactly supported variation of σ defined by $\sigma_t = (1+t)\sigma$. From Lemma 3.3 we have

$$
\begin{aligned}
e(\sigma_t) &= \frac{n}{2} + \frac{(t+1)^2}{2}\Big[\frac{1}{\alpha_1} trace_g g(\nabla X_\sigma, \nabla X_\sigma) + \frac{1}{\alpha_2} trace_g g(\nabla Y_\sigma, \nabla Y_\sigma) \\
&\quad + \frac{1}{\beta_1} trace_g [g(\nabla X_\sigma, X_\sigma)]^2 + \frac{1}{\beta_2} trace_g [g(\nabla Y_\sigma, Y_\sigma)]^2\Big].
\end{aligned}
$$

If σ is a critical point of the energy functional we have :

$$
\begin{aligned}
0 &= \frac{d}{dt} E(\sigma_t)_{|t=0}, \\
&= \int_M \Big[\frac{1}{\alpha_1} trace_g g(\nabla X_\sigma, \nabla X_\sigma) + \frac{1}{\alpha_2} trace_g g(\nabla Y_\sigma, \nabla Y_\sigma) \\
&\quad + \frac{\beta_1}{\alpha_1} trace_g [g(\nabla X_\sigma, X_\sigma)]^2 + \frac{\beta_2}{\alpha_2} trace_g [g(\nabla Y_\sigma, Y_\sigma)]^2\Big] dv_{g^D}
\end{aligned}
$$

then

$$g(\nabla X_\sigma, \nabla X_\sigma) = g(\nabla Y_\sigma, \nabla Y_\sigma) = g(\nabla X_\sigma, X_\sigma) = g(\nabla Y_\sigma, Y_\sigma) = 0$$

Example 3.1. Let $\varphi, \psi : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth functions. If $\mathbb{R}^n$ is equipped with the euclidean metric $<,>$, then $T\mathbb{R}^n = \mathbb{R}^{3n}$ and the section $\sigma = (\varphi, \psi)$ is harmonic if and only φ and ψ are solutions of the following equations:

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \frac{\partial^2 \varphi^s}{(\partial x^i)^2} + \frac{(1+\alpha_1)\beta_1}{\alpha_1^2} \varphi^s \sum_k (\frac{\partial \varphi^k}{\partial x^i})^2 + \frac{\beta_1^2}{\alpha_1^2} \varphi^s \Big(\sum_k \varphi^k \frac{\partial \varphi^k}{\partial x^i} \Big)^2 \\ & -2 \frac{\beta_1}{\alpha_1} \frac{\partial \varphi^s}{\partial x^i} \Big(\sum_k \varphi^k \frac{\partial \varphi^k}{\partial x^i} \Big) \end{aligned}$$

and

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \frac{\partial^2 \psi^s}{(\partial x^i)^2} + \frac{(1+\alpha_2)\beta_2}{\alpha_2^2} \psi^s \sum_k (\frac{\partial \psi^k}{\partial x^i})^2 + \frac{\beta_2^2}{\alpha_2^2} \psi^s \Big(\sum_k \varphi^k \frac{\partial \psi^k}{\partial x^i} \Big)^2 \\ & -2 \frac{\beta_2}{\alpha_2} \frac{\partial \psi^s}{\partial x^i} \Big(\sum_k \psi^k \frac{\partial \psi^k}{\partial x^i} \Big) \end{aligned}$$

for all $s = 1, ..., n$.
in the case of $\beta_1 = \beta_2 = 0$, then $\sigma = (\varphi, \psi)$ is harmonic if and only if φ and ψ are harmonic functions.

Example 3.2. Let $S^1 = \{(x, y) \in \mathbb{R}^2\}$ endowed with the natural metric $dx^2 + dy^2$ and $\sigma : (x, y) \in S^2 \to (x, y, 0, 0, -y, x) \in T^2 S^2$, we have: $X_\sigma = 0$, $Y_\sigma = (-y, x)$ and $\nabla Y_\sigma = 0$. From Theorem (3.6) we deduce that σ is harmonic section.

Example 3.3. If $S^2 \times \mathbb{R}$ is endowed with the product of canonical metrics, then the section $\sigma = (0, \frac{\partial}{\partial t})$ is harmonic.

Remark 3.1. In general, using Corollary (3.1) and Theorem (3.6) we can construct many examples for harmonic sections.

3.4. **Harmonicity conditions of inclusions.**

Lemma 3.4. *Let (M, g) be a Riemannian manifold. If $i : (x, u) \in TM \to S^{-1}(x, 0, u) \in T^2 M$ denote the second inclusion, then for all $X \in \Gamma(TM)$ and $(x, u) \in TM$, we have*

$$d_{(x,u)} i(X^H) = X_p^0 + (\nabla_X U)_p^1 \tag{3.9}$$

$$d_{(x,u)} i(X^V) = X_p^2 \tag{3.10}$$

where $S(p) = (x, 0, u)$ and $U = u^i \frac{\partial}{\partial x^i}$ is a local vector fields constant on each fiber.

Proof. locally we have

$$d(S \circ i) = dx^i \otimes \frac{\partial}{\partial x^i} + dy^j \otimes \frac{\partial}{\partial z^j}$$

then

$$\begin{aligned}
1.\quad d_{(x,u)}(S\circ i)(X^H) &= X^i\frac{\partial}{\partial x^i}-\Gamma^j_{sk}X^su^k\frac{\partial}{\partial z^j}\\
&= X^i\frac{\partial}{\partial x^i}-\Gamma^j_{sk}X^su^k\frac{\partial}{\partial y^j}-\Gamma^j_{sk}X^su^k\frac{\partial}{\partial z^j}+\Gamma^j_{sk}X^su^k\frac{\partial}{\partial y^j}\\
&= (X^H_{(x,u)},X^H_{(x,u)})+((\nabla_XU)^V_{(x,u)},0).\\
2.\quad d_{(x,u)}(S\circ i)(X^V) &= X^j\frac{\partial}{\partial z^j}\\
&= (0,X^V).
\end{aligned}$$

Theorem 3.7. *Let (M,g) be a Riemannian manifold and $\overline{g}$ (resp G) be the Sasaki metric on TM (rep natural metric on T^2M), then the tension field of the second inclusion $i:(TM,\overline{g})\to(T^2M,G)$ is given by:*

$$\tau_{(x,u)}(i) = \frac{\beta_2}{\alpha_2}\Big[\frac{\beta_2}{\alpha_2}g(u,u)-2+\frac{n(1+\alpha_2)}{\alpha_2}\Big]u^2$$

Proof. Let $x\in M$ and $\{e_i\}_{i=1}^n$ be a local orthonormal frame on M such that $\nabla_{e_i}e_j=0$.

Using Lemma 3.4 and summing over i, we obtain

$$\begin{aligned}
\tau_{(x,u)}(i) &= \widetilde{\nabla}_{di(e_i^H)}di(e_i^H)+\widetilde{\nabla}_{di(e_i^V)}di(e_i^V)\\
&= \widetilde{\nabla}_{e_i^0}e_i^0+\widetilde{\nabla}_{e_i^0}(\nabla_{e_i}U)^1+\widetilde{\nabla}_{(\nabla_{e_i}U)^1}e_i^0+\widetilde{\nabla}_{(\nabla_{e_i}U)^1}(\nabla_{e_i}U)^1\\
&\quad+\widetilde{\nabla}_{e_i^2}e_i^2
\end{aligned}$$

From Theorem 3.3 and taking into account that $p=S^{-1}(x,0,u)$, then

$$\begin{aligned}
\tau_{(x,u)}(i) &= \widetilde{\nabla}_{e_i^2}e_i^2\\
&= -\frac{\beta_2}{\alpha_2^2}\Big[\alpha_2\Big(g(e_i,u)e_i^2+g(e_i,u)e_i^2\Big)-\beta_2g(e_i,u)g(e_i,u)u^2\\
&\quad-(1+\alpha_2)g(e_i,e_i)u^2\Big]\\
&= -\frac{\beta_2}{\alpha_2^2}\Big[\alpha_2\Big(g(e_i,u)e_i+g(e_i,u)e_i\Big)-\beta_2g(e_i,u)g(e_i,u)u\\
&\quad-(1+\alpha_2)g(e_i,e_i)u\Big]^2\\
&= -\frac{\beta_2}{\alpha_2^2}\Big[2\alpha_2-\beta_2g(u,u)-n(1+\alpha_2)\Big]u^2
\end{aligned}$$

From Theorem 3.7, we have

Corollary 3.2. *Let (M,g) be a Riemannian manifold and $\overline{g}$ (resp G) be the Sasaki metric on TM (rep natural metric on T^2M), then the second inclusion $i:(TM,\overline{g})\to(T^2M,G)$ is harmonic if and only if $\beta_2=0$.*

Similarly we have the following theorem

Theorem 3.8. *Let (M,g) be a Riemannian manifold and , If $J:(x,u)\in(TM,\overline{g})\to S^{-1}(x,u,0)\in(T^2M,G)$ denote the first inclusion, then for all $X\in\Gamma(TM)$ and*

$(x,u) \in TM$, we have

$$d_{(x,u)}i(X^H) = X^0_p + (\nabla_X U)^2_p \tag{3.11}$$

$$d_{(x,u)}i(X^V) = X^1_p \tag{3.12}$$

$$\tau_{(x,u)}(J) = \frac{\beta_1}{\alpha_1}\Big[\frac{\beta_1}{\alpha_1}g(u,u) - 2 + \frac{n(1+\alpha_1)}{\alpha_1}\Big]u^1 \tag{3.13}$$

where $p = S^{-1}(x,u,0)$ and $\overline{g}$ (resp G) denote the Sasaki metric on TM (resp natural metric on T^2M).

Corollary 3.3. *Let (M,g) be a Riemannian manifold . The first inclusion $J : (TM,\overline{g}) \to (T^2M,G)$ is harmonic if and only if $\beta_1 = 0$.*

Corollary 3.4. *Let (M,g) be a Riemannian manifold . Then the first inclusion J and the second inclusion i are harmonic if and only if the natural metric G is diagonal.*

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