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CENTERS OF CURVATURE FOR UNWRAPPINGS OF PLANE INTERSECTIONS OF TAME DEVELOPABLE SURFACES

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Abstract. We call tame all those developable surfaces that split into parts recognizable as generalized cones, generalized cylinders, planes and tangent surfaces of curves. For them we prove that there exist only two isometries of any small region U around any regular point A, which map U on the tangent plane fixing the ruling ℓ through A (rulings exist since developable surfaces are also ruled). We choose the most natural one among these isometries to call as the unwrapping of U on the tangent plane along ℓ.

Let γ be the section curve of a tame developable surface F by a plane ρ, and K the center of curvature of γ at some point A. Let also $\overline{\gamma}$ be the unwrapping of γ on an unwrapping on the tangent plane of a neighborhood of A in F along the ruling through A. Call $\overline{K}$ the center of curvature of $\overline{\gamma}$ at A. Our main result is that $\overline{K}$ projects orthogonally on ρ onto K.

1. The meaning of unwrappings. Statements of the results.

We shall call tame the most natural among developable surfaces (Definition 1.1), and we shall define their unwrappings as the most natural among their developements (Definition 1.2). In Proposition 1.1 we shall show that for any tame developable surface there exists a unique unwrapping along any fixed ruling. The main result of the paper, given in Proposition 1.2, says that under reasonable assumptions the center of curvature K of a plane-section curve of a tame developable surface, and the center of curvature $\overline{K}$ of the curve's image under an unwrapping are nicely related when considered at a common point: K is just the orthogonal projection of $\overline{K}$ on the plane of the section curve (Figure 1). First we make clear our definitions and assumptions.

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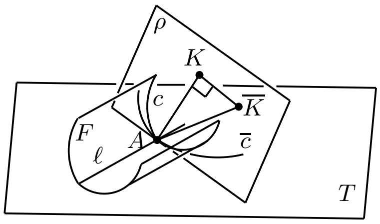


FIGURE 1. F=tame developable surface, ρ= a plane, $c = F \cap \rho$, $A \in c$, ℓ= ruling of F through A, T= tangent plane of F along ℓ, $\overline{c}$= unwrapping of c on T. If $K, \overline{K}$ = centers of curvature of $c, \overline{c}$ at A, then K = orthogonal projection of $\overline{K}$ on ρ (Proposition 1.2).

A *developable surface* is a C^2 regular ruled surface with a constant tangent plane along any ruling, and a *ruled surface* is a C^1 parametrized surface $F : x(u,v) = \alpha(u) + vw(u), u \in (a,b), v \in \mathbb{R}$ with $a(u)$ regular, $|w(u)| = 1, \forall u$, generated by a one parameter family of lines $\ell_u : y(v) = \alpha(u) + vw(u)$ called the *rulings* of F. The C^1 curve $\alpha(u)$ is called the *directrix* of F.

The given definitions allow surfaces to have self intersections and ruled surfaces to admit singularities (this means $x_u \times x_v = 0$ at some points called *singular* or non-regular) so that cones and tangent surfaces of curves are not excluded from ruled surfaces. The *tangent surface* of a regular parametrized curve $y(s)$ with no inflection points and s an arc length parameter, is the ruled surface by F : $x(s,v) = y(s) + vy'(s)$. A (generalized) *cone* is a ruled surface with all its rulings passing through a common point p called the vertex of F. Then the cone can be parametrized as $F : x(u,v) = p + v\alpha(u), u \in (a,b), v \in \mathbb{R}$. Tangent surfaces and cones are developable away from their singular points, as are also planes and (generalized) *cylinders* defined as ruled surfaces $F : x(u,v) = y(u) + vg, |g| = 1$ with rulings of a fixed direction g.

Our Proposition 1.2 deals with surfaces composed by pieces of planes, cylinders, cones and tangent surfaces of curves, which we conveniently call tame:

Definition 1.1. We call *tame* a developable surface $F : x(u,v) = \alpha(u)+vw(u), u \in I, v \in \mathbb{R}, w'(u) \neq 0, \forall u$ whenever there exists a partition of I so that the part of F corresponding to each subinterval of the partition is just a plane, a (generalized) cylinder, a (generalized) cone or the tangent surface of a curve.

As explained at the end of this section, all developable surfaces are tame provided some mild assumptions hold.

Now, tame developable surfaces have zero Gaussian curvature at all regular points ([2],[4]), so by Mindings Theorem ([2],[4]) for any C^3 tame developable surface there exists a neighborhood U of every regular point that can be mapped isometrically by an isometry f onto an appropriate open neighborhood of any point of a plane. Actually, one can guarante the above result for C^2 tame developable surfaces ([3] provides specific such isometries and our proof of Proposition 1.2 provides more). The name *development* is usually reserved for those isometries that fix some ruling inside U. Such isometries exist: For a chosen ruling ℓ intersecting U and a chosen point M of ℓ consider first an arbitrary isometry of U upon the tangent plane T of F along ℓ. Compose with a translation so that M is mapped onto itself. The ruling ℓ is a usual line, thus a geodesic of F, maped by the development onto a geodesic of T, in other words again to a line. Any point on ℓ has to be mapped on an equally distant point from M on the image line on T. So composing, if necessary, with a rotation of T the resulting isometry f fixes pointwise the ruling ℓ, implying that f is a development.

Notice that for any such development f, its composition f_{sym} with the reflection with respect to ℓ is another development of U into T fixing ℓ, but clearly only one of f, f_{sym} looks like a real life unwrapping of F on the plane. It is the one that maps all nearby points of U on the two sides of ℓ on the correct side of T with respect to ℓ. Technically this can be captured by demanding the extra property of our second definition below be satisfied for all points on the ruling:

Definition 1.2. For an open, connected subset U of a developable surface F we call *unwrapping* of U on the tangent plane T of the surface along a ruling ℓ, any development f, i.e. isometry (or isometric image if we wish) of U on T which fixes all points of ℓ inside U, and for which the tangent vector of any curve $y(t)$ on U coincides with that of its image curve $f(y(t))$ at their common points $y(t_0) = f(y(t_0))$ on ℓ. For a curve γ on U we call $f(\gamma)$ as the *unwrapping* of γ by f.

The following Proposition concerns the existence and uniqueness of unwrappings and deals with the local properties of the unwrappings needed in the sequel.

Proposition 1.1. *(a) For any regular point A on a C^2 tame developable surface F there exists a neighborhood U_A around A for which there exist exactly two developments of U_A on the tangent plane T of F fixing the ruling ℓ through A. The images of the two developments are symmetric with respect to ℓ and exactly one of them is the unique unwrapping f of U_A on T along the ruling ℓ. (b) If V is another neighborhood around A then any unwrapping of V on T along the ruling ℓ coincides with that of U_A in a neighborhood of A common to both U_A and V. (c) An isometry f from U_A to T which fixes ℓ pointwise is an unwrapping of U_A if and only the tangent vector of either parameter curve of $F : x(u, v) = \alpha(u) + vw(u)$ coincides with that of its image curve at their common points on ℓ.*

In this paper all neighborhoods are considered open and connected. Our second Proposition relates the centers of curvature of any plane section of F to those of its unwrapping at their common points on the ruling used for the unwrapping.

Proposition 1.2. *Let γ be the intersection curve of a C^2 tame developable surface F by a plane ρ, and let $\overline{\gamma}$ be the unwrapping of γ in the unique unwrapping of a neighborhood U_A of F around its regular point A of γ on the tangent plane T along the ruling ℓ through A. If A is a non-singular and non-inflection point of either curve, then the center of curvature K of γ at A is the point of the orthogonal projection on ρ of the center of curvature $\overline{K}$ of the unwrapping $\overline{\gamma}$ at A.*

Notice that if ρ is perpendicular to T and the center of curvature $\overline{K}$ of $\overline{\gamma}$ exists, then it has to be projected on ρ onto A which of course cannot be the center of curvature K of γ. We immediately get

Corollary 1.1. *With the assumptions of Proposition 1.2, whenever ρ and T are perpendicular and A a non-singular point for both $\gamma, \overline{\gamma}$, then at least one of $\gamma, \overline{\gamma}$ has an inflection point at A. Actually if U_A is part of a cylinder, then the development $\overline{\gamma}$ is part of a line.*

The proof of the last statement is a triviality.

For the completeness of presentation, let us note that as proved in [2], the building blocks of all developable surfaces are planes, cones, cylinders and tangent surfaces of curves, provided some mild assumptions hold:

Theorem 1.1. ([2]) *For a developable surface $F : x(u,v) = a(u) + vw(u), u \in I, v \in \mathbb{R}$ there exists a partition of I so that the part of F corresponding to each subinterval of the partition is just a plane, a cylinder, a cone or the tangent surface of a curve, provided the tangent vectors of $w(s)$ and of its striction line (if any) do not have a clustering of their zeros on their domain of definition. The striction line (if any) of F form the locus of all singular points of F.*

The *striction line* mentioned in the Theorem is defined on those parts of F corresponding to the subintervals of I for which $w'(u) \neq 0$ throughout, as the unique curve $\beta(u)$ of F so that $\beta' \cdot w' = 0, \forall u$ ([2], [3]). It is $\beta = \alpha - \frac{\alpha' \cdot w'}{|w'|^2} w$. In general, the striction line of F is the union of the striction lines defined in $I - \{\text{zeros of } w'\}$ which is a union of intervals. This line is not defined for planes and cylinders, it coincides with the directrix for tangent surfaces of curves, and it degenerates to the vertex for cones.

The *partition* of I in the Theorem is a collection of intervals I_i (indices in a subset of $\mathbb{Z}$) with disjoint interiors and union I, so that each I_i is closed in I and anyone of its endpoints other than that of I (if any) is also the endpoint of another interval of the partition. The partition can either be finite or infinite. A clustering of zeros at u_0 means that no left or right neighborhood of u_0 contains only zeros, but all neighborhoods of u_0 contain at least one more zero other than u_0; then u_0 is necessary a zero because of the continuity of the functions involved. The Theorem gives a necessary and sufficient condition for a developable surface to have the property that there exists some small connected neighborhood U_A around any point A of the surface such that each side of U_A with respect to the ruling ℓ through A, is part of one of the four named types of developable surfaces; the same surface on both sides whenever u_0 is not a zero of the functions of the Theorem.

Closing this section let us note that many beautiful results about unwrappings of curves on right circular cylinders and right circular cones can be found in the very interesting paper [1] by Apostol and Mnatsakanian.

2. Proof of Proposition 1.1

For part (a): Recall that prior to Definition 1.2 we proved that for any regular point A on a tame developable surface $F : x(u,v) = \alpha(u) + vw(u)$ there exist developments around A. These are isometries which map some neighborhood U of A on the tangent plane T at A, and which fix all points of $\ell \cap U$ where ℓ is the ruling through A; all neighborhoods here are considered open and connected. Let f be such an isometry and f_{sym} be its composition with the reflection with respect to ℓ. f_{sym} is another development of U on T fixing ℓ. First we are going to show that one of f, f_{sym} is an unwrapping of U, and then that there exists a unique unwrapping of U shrank appropriately. This immediately implies that f, f_{sym} are the only two developments of U on the plane T fixing the ruling ℓ through A, settling all claims in part (a) of the Proposition.

Let us parametrize U by the restriction of the ruled structure $x(u,v) = \alpha(u) + vw(u)$ of the surface F to it. For an arbitrary point $M = x(u_0, v_0)$ in the ruling

ℓ through A, let c_1, c_2 be the u- and v-parameter curves respectively through M; $c_2 : \alpha(u_0) + vw(u_0), v \in \mathbb{R}$ is of course the ruling ℓ. Since f and f_{sym} fix $\ell = c_2$ pointwise, $f(c_2)$ and $f_{sym}(c_2)$ share with c_2 at M the same tangent vector x_v. Also, the tangent vector x_u of c_1 at M is parallel to T as are the tangent vectors u_1, u_2 of $f(c_1), f_{sym}(c_1)$ at M. Since f, f_{sym} are isometries we have $\angle(u_1, x_v) = \angle(u_2, x_v) = \angle(x_u, x_v) \neq 0$. So the nonzero vectors u_1, u_2 of T are carried by two lines through M symmetric with respect to ℓ and exactly one of u_1, u_2 forms an angle 0 with x_u whereas the other forms an angle equal to $2\angle(x_u, x_v) \neq 0$. Let $\angle(u_1, x_u) = 0$, and $\angle(u_2, x_u) = 2\angle(x_u, x_v) \neq 0$. The second relation says that f_{sym} cannot be an unwrapping of U. Now, since f is continuous and U is path-connected, the angle $\angle(u_1, x_u)$ changes continuously with v as M moves along ℓ, always taking on one of the two values 0 or $2\angle(x_u, x_v) \neq 0$, thus it always has to remain 0. But since f is an isometry, the length of u_1 is that of x_u. So $u_1 = x_u$ for all M on ℓ.

So f fixes the tangent vectors for the parameter curves for points on ℓ. This suffices to show that it also fixes the tangent vectors of any curve $c : y(t) = x(u(t), v(t))$ in U_A at points $M = y(t_0) = \overline{y}(t_0)$ on the ruling ℓ, common to c and to its image $f(c) : \overline{y}(t) = f(y(t))$ on T:

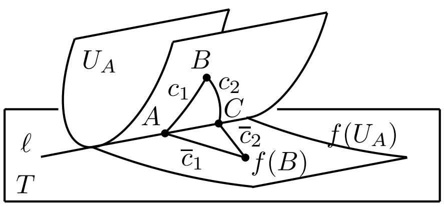
$$\frac{d\overline{y}}{dt}(t_0) = \frac{d(f \circ y)}{dt}(t_0) = \frac{d(f \circ x)}{du}(u(t_0), v(t_0))\frac{du}{dt}(t_0) + \frac{d(f \circ x)}{dv}(u(t_0), v(t_0))\frac{dv}{dt}(t_0).$$

But $\frac{d(f \circ x)}{du}(u(t_0), v(t_0))$ is the tangent vector at M of the image via f of the u- parameter line $v = v(t_0)$ through M, so it equals $\frac{dx}{du}(u(t_0), v(t_0))$. Similarly, $\frac{d(f \circ x)}{dv}(u(t_0), v(t_0)) = \frac{dx}{dv}(u(t_0), v(t_0))$ and we get

$$\frac{d\overline{y}}{dt}(t_0) = \frac{dx}{du}(u(t_0), v(t_0))\frac{du}{dt}(t_0) + \frac{dx}{dv}(u(t_0), v(t_0))\frac{dv}{dt}(t_0) = \frac{dy}{dt}(t_0)$$

So f is an unwrapping.

We now restrict U appropriately to get an almost pictorial proof for the uniqueness of the unwrapping. So let U_A be an open neighborhood of A in U small enough for A to be joined within it with any other point by a unique geodesic. Such a neighborhood U_A always exists ([4]), and of course it is path connected. Then the restriction of f in U_A is clearly an unwrapping of U_A. If g is some development of U_A on the tangent plane T along ℓ, we are going to prove $f = g$ or $f_{sym} = g$ in U_A:



$A = f(A) = g(A)$

$C = f(C) = g(C)$

$f(B) = g(B)$ or

$f(B), g(B)$ symmetric with respect to ℓ

FIGURE 2. Two developments f, g of U_A on the tangent plane T along ℓ either coincide or are symmetric with respect to ℓ.

Of course f and g agree on all points of $\ell \cap U_A$ by their definition. Now let B be a point in $U_A - \ell$ (Figure 2) and let c_1 be a geodesic arc form A to B inside U_A. Since f is an isometry, the arcs $\overline{c}_1 = f(c_1)$ and $f(\ell \cap U_A) = \ell \cap U_A$ on plane

T are also geodesic arcs, thus line segments (a fact we already know for $\ell \cap U_A$). Now $\overline{c}_1$ as a compact subset of T is covered by finitely many open disks centered at points on $\overline{c}_1$ and so that each disk is contained inside the open set $f(U_A)$). Since there exist only finitely many such disks, clearly there exists a point C on ℓ such that the line segment $Cf(B)$ lies within the union of the disks, thus within $f(U_A)$: indeed, discard disks among the given so that no one of the disks D_i left is contained in the interior of any other. Call d the minimum among the lengths of the common chords of intersecting disks and the chords of them through A or $f(B)$ that are perpendicular to $Af(B)$. Call Ax the half line of ℓ which forms with the half line $Af(B)$ (vertex at A) angle $xAf(B) \leq \frac{\pi}{2}$. Construct on T the rectangle $P = Af(B)B'A'$ with $AA' = \frac{d}{4}$ and AA', Ax on the same half plane with respect to the line $Af(B)$, and call $C = f(B)A' \cap Ax' = f(B)A' \cap \ell$. Trivially $P \subset \cup_i D_i \subset f(U_A)$ and so $Cf(B) \subset f(U_A)$ as required for point C.

So there exists an inverse image $c_2 = f^{-1}(Cf(B))$ which is an arc connecting C with the point $B = f^{-1}(f(B))$ within U_A. Denote the line segment $Cf(B) = f(c_2)$ by $\overline{c}_2$. Since $Cf(B)$ is a geodesic arc, so is also then c_2. Of course $g(A) = f(A) = A$ and $g(C) = f(C) = C$ as the points A, C lie on ℓ. Let $\overline{\overline{c}}_1 = g(c_1), \overline{\overline{c}}_2 = g(c_2)$ be the geodesic arcs on T, images by g of the geodesic arcs c_1, c_2. So $\overline{\overline{c}}_1, \overline{\overline{c}}_2$ are line segments of T connecting A to $g(B)$. Since the isometries f, g don't change lengths, and denoting by $L(\cdot)$ the lengths of segments, we conclude that $L(\overline{c}_1) = L(c_1) = L(\overline{\overline{c}}_1)$, $L(\overline{c}_2) = L(c_2) = L(\overline{\overline{c}}_2)$ thus the triangles $ACf(B), ACg(B)$ are congruent, which in turn means that either $f(B)$ and $g(B)$ coincide or else they are symmetric with respect to the line $AC = \ell$. Whichever is the case, it will hold for all points $B \in U_A - \ell$. The reason is purely topological:

Recall that U_A is open and path-connected and that the part of ℓ in U_A is closed. Thus $U_A - \ell$ consists of two disjoint path-connected open sets U_1, U_2 which are mapped under the homeomorphism f onto two disjoint path-connected open subsets $f(U_1), f(U_2)$ of T one on each half plane with respect to ℓ. If the point M is on the same U_i as B, then $f(B), f(M)$ are on the same $f(U_i)$ and similarly $g(B), g(M)$ are on the same $f(U_i)$, and since $f(B) = g(B)$ it follows that $f(M), g(M)$ are on the same $f(U_i)$. So $f(M), g(M)$ should always coincide. The same holds similarly in the remaining case when M and B are not on the same U_i.

The above prove that either $f = g$ or else $f_{sym} = g$ where f_{sym} is the composition of f with the reflection on T with respect to ℓ. But although f_{sym} is an isometry it is not an unwrapping of U. So in case g is an unwrapping of U_A it must be $f = g$ in U_A as wanted.

For part (b) of the Proposition: Let g be some unwrapping of V along ℓ. Of course the restrictions of f, g on the connected component of $U_A \cap V$ containing A are both unwrappings of it. If we restrict this component enough so that A is joined within it with any other point by a unique geodesic, then the proof of the uniqueness for the unwrappings in part (a) of the Proposition applies to it implying that $f = g$ as wanted.

For part (c) of the Proposition: If f is an unwrapping, then by definition it fixes the tangent vectors of the parameter curves at points on ℓ. Conversely, if it fixes the tangent vectors of the parameter curves, then as was proved in part (a) of the Proposition it fixes the tangent vectors of all curves in U at points on ℓ. This means that f is an unwrapping.

3. Proof of Proposition 1.2

Since we are interested in the curvature of the section curve γ at A we only need to restrict our attention locally to a connected arc of γ containing A and contained in the given neighborhood U. As mentioned in §1 we can restrict, if necessary, the neighborhood U to have it as part of a plane, generalized cylinder, generalized cone or the tangent surface of a curve. To be precise, as observed in §1 the two sides of U_A with respect to the ruling ℓ through A can be of a different kind of these surfaces. Nevertheless the arguments to be presented remain valid even if we restrict them to a single side of U_A with respect to ℓ. So there will be no harm if we consider below that the whole neighborhood U_A of A belongs entirely to a plane, cylinder, cone or the tangent surface of a curve.

If ρ is going through the ruling ℓ then γ is just the ruling itself which is a line with only inflection points and there is nothing to prove.

Similarly there is nothing to prove in case U_A is part of a plane because the intersection γ of the plane ρ with U_A is again just a line.

So from now on we consider U_A as part of a cylinder, cone or the tangent plane of a curve and $\rho : a_1x_1 + a_2x_2 + a_3x_3 = a_0$ as a plane intersecting T in a line ϵ with a unique common point A with ℓ. For later use denote $e = (a_1, a_2, a_3)$ which is a non-zero and normal vector to ρ.

We examine each case separately, but the idea in all of them is to construct explicitly the unwrapping f of U_A taking advantage of the special geometric information each time. According to Proposition 1.1 we can shrink U_A as necessary. In each case we consider U_A parametrized by the restriction to U_A of an s, v parametrization of the corresponding developable surface, where the rulings (ℓ among them) are the v-parameter curves. We use the parameter s instead of u because we intend it to be an arc length parameter of a specific curve on U_A. The details follow.

We start with a Lemma which provides equivalent statements to the conclusion of Proposition 1.2.

Lemma 3.1. *Under the assumptions of Proposition 1.2 the following are equivalent:*

(1) *the center of curvature K of γ at A is the point of the orthogonal projection on ρ of the center of curvature $\overline{K}$ of the unwrapping $\overline{\gamma}$ at A.*
(2) $k_{\gamma_A} \cdot k_{\overline{\gamma}_A} = k_{\overline{\gamma}_A} \cdot k_{\overline{\gamma}_A}$.
(3) $\kappa_{\gamma_A}(n_{\gamma_A} \cdot n_{\overline{\gamma}_A}) = \kappa_{\overline{\gamma}_A}$.

Here $k_\gamma, k_{\overline{\gamma}}$ are the curvature vectors, $n_\gamma, n_{\overline{\gamma}}$ are choices for the unit normal vectors of $\gamma, \overline{\gamma}$, and finally $\kappa_\gamma, \kappa_{\overline{\gamma}}$ are the corresponding signed curvatures.

Proof. Recall that by Proposition 1.1 the curves γ and $\overline{\gamma}$ share a common tangent vector t at A which is parallel to both planes ρ, T since γ lies on ρ and $\overline{\gamma}$ lies on T. This makes t parallel to ϵ. Since the lines $AK, A\overline{K}$ carry the normal vectors of the two curves at A, both lines are perpendicular to ϵ. This implies that $A\overline{K}$ projects on ρ onto the line AK (or just onto the point A of it), thus $\overline{K}$ projects on some point of AK. So this projection coincides with K if and only if $\overrightarrow{AK} \perp \overrightarrow{\overline{K}K}$, that is if and only if $\overrightarrow{AK} \cdot \overrightarrow{\overline{K}K} = 0$. But

$$\overrightarrow{AK} = \frac{n_{\gamma_A}}{\kappa_{\gamma_A}} = \frac{k_{\gamma_A}}{\kappa^2_{\gamma_A}} \quad \text{and} \quad \overrightarrow{\overline{K}K} = \overrightarrow{AK} - \overrightarrow{A\overline{K}} = \frac{n_{\gamma_A}}{\kappa_{\gamma_A}} - \frac{n_{\overline{\gamma}_A}}{\kappa_{\overline{\gamma}_A}} = \frac{k_{\gamma_A}}{\kappa^2_{\gamma_A}} - \frac{k_{\overline{\gamma}_A}}{\kappa^2_{\overline{\gamma}_A}} = \frac{k_{\gamma_A}}{k^2_{\gamma_A}} - \frac{k_{\overline{\gamma}_A}}{k^2_{\overline{\gamma}_A}}$$

($\kappa_{\gamma A}, \kappa_{\overline{\gamma} A}$ are non-zero as A is assumed to be a non-inflection point of $\gamma, \overline{\gamma}$). So $\overrightarrow{AK} \cdot \overrightarrow{\overline{K}K} = 0$ is equivalent to $\frac{k_{\gamma_A}}{\kappa^2_{\gamma_A}}\left(\frac{k_{\gamma_A}}{k^2_{\gamma_A}} - \frac{k_{\overline{\gamma}_A}}{k^2_{\overline{\gamma}_A}}\right) = 0$ which is clearly equivalent to $k_{\gamma_A} \cdot k_{\overline{\gamma}_A} = k_{\overline{\gamma}_A} \cdot k_{\overline{\gamma}_A}$, which in turn is equivalent to $\kappa_{\gamma_A} n_{\gamma_A} \cdot n_{\overline{\gamma}_A} = \kappa_{\overline{\gamma}_A}$, as claimed. □

The case $U_A =$ part of the tangent surface of a curve.
Let U_A be part of the tangent surface $F : x(s, v) = y(s) + vy'(s)$ (Figure 3) of the regular curve $c : y(s)$ with an arc length parameter. Denote differentiation with respect to s by primes and call respectively $t(s) = y'(s), n(s), b(s), k(s), \tau(s)$ the unit tangent vector of c, a choice for the principal unit normal vector, the corresponding binormal vector, the corresponding signed curvature, and the torsion of c. Notice that s parametrizes the section curve $\gamma = U_A \cap \rho$ as follows: if z is the point of γ on the tangent line of c at $y(s)$, then $z \in \rho : a_1x_1 + a_2x_2 + a_3x_3 = a_0$ implies $z \cdot e = a_0$ and since $z = y(s) + vt(s)$ for some $v \in \mathbb{R}$ we get $v = \frac{a_0 - y(s)\cdot e}{t(s)\cdot e}t(s)$ and so $z = z(s) = y(s) + \frac{a_0 - y(s)\cdot e}{t(s)\cdot e}t(s)$. The point A of γ and the ruling ℓ through A correspond to a value $s = s_0$. Summarizing:

$$(3.1)\quad \gamma : z(s) = y(s) + \frac{a_0 - y(s)\cdot e}{t(s)\cdot e}t(s), \quad A = z(s_0), \quad \ell : y(s_0) + vy'(s_0).$$

By the way the denominator in (3.1) is non-zero if we stay close enough to A: $t(s_0) \cdot e \neq 0$ since otherwise $t(s_0)$ would be parallel to ρ and then the ruling $\ell \parallel t(s_0)$ would be parallel to ρ too, thus ℓ and ρ wouldn't have just a single point A in common, a contradiction. Then by continuity $t(s) \cdot e \neq 0$ for a small enough interval of values of s around s_0, and if necessary, we can shrink U_A so that this relation holds for all s for which $z(s) \in U_A$.

Observe also that the point $M = y(s_0)$ of c lies on the ruling ℓ on the plane T, so the tangent vector $t(s_0) = y'(s_0)$ is parallel to T. At $s = s_0$ the vectors $x_s = t(s_0) + vt'(s_0), x_v = t(s_0)$ are parallel to T for all v, so subtracting them we get $vt'(s_0)$ parallel to T. But s is an arc length parameter of c, thus $t'(s_0)$ is parallel to the normal vector $n(s_0)$ of c, so $n(s_0)$ is parallel to T. Of course moreover $n(s_0) \perp t(s_0)$, a fact we shall use shortly. Summarizing:

$$(3.2)\qquad t(s_0), n(s_0) \parallel T, \quad n(s_0) \perp t(s_0), \quad |t(s_0)| = 1.$$

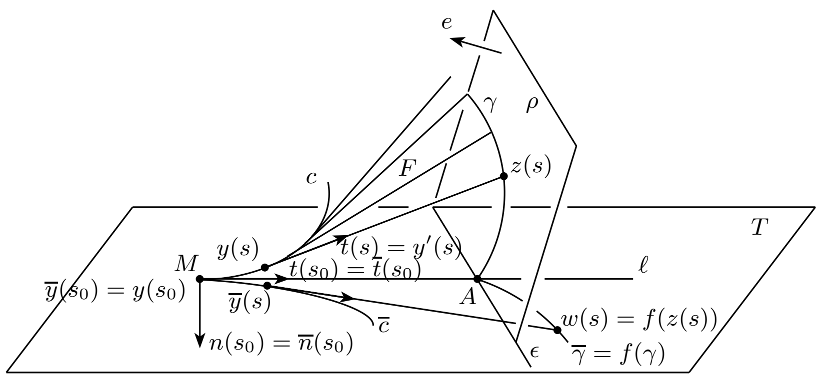


FIGURE 3. The case $U_A =$ part of the tangent surface F of a curve c.

Lemma 3.2. *Consider the curve $\overline{c} : \overline{y}(s)$ on T with natural parameter s and curvature*

$$\overline{k}(s) = k(s), \tag{3.3}$$

for which

$$\overline{y}(s_0) = y(s_0), \quad \overline{t}(s_0) = t(s_0) \quad \text{and} \quad \overline{n}(s_0) = n(s_0), \tag{3.4}$$

where $\overline{t}(s) = \overline{y}'(s), \overline{n}(s)$ are respectively the unit tangent vector and the unit normal vector corresponding to the signed curvature $\overline{k}(s)$. Then there exists a neighborhood of A in U_A (which we still call U_A) whose unique unwrapping f on T along ℓ is

$$f(x(s,v)) = f\left(y(s) + vt(s)\right) = \overline{y}(s) + v\overline{t}(s) \overset{call\ it}{=} \overline{x}(s,v).$$

Proof. Note first that the curve $\overline{c}$ of the Lemma exists: the Existence Theorem for curves guaranties the existence of a curve $\overline{c}$ on T satisfying (3.3). Since $|\overline{t}(s_0)| = 1,\ \overline{t}(s_0) \parallel T$, and since by (3.2) $|t(s_0)| = 1,\ t(s_0) \parallel T$, we can translate and rotate, if necessary, $\overline{c}$ on T so that relations $\overline{y}(s_0) = y(s_0),\ \overline{t}(s_0) = t(s_0)$ of (3.4) are satisfied too. Now the curve $\overline{c}$ lies on T, thus its unit normal vector $\overline{n}(s_0)$ is necessary parallel to T and normal to the tangent vector $\overline{t}(s_0) = t(s_0)$, exactly as $n(s_0)$ is by (3.2). So $\overline{n}(s_0), n(s_0)$ either coincide or are opposite. We can assume they coincide, that is we can assume $\overline{c}$ satisfies $\overline{n}(s_0) = n(s_0)$ of (3.4) as well, otherwise we can replace $\overline{c}$ with its symmetric with respect to ℓ which is another curve on T and which satisfy both relations (3.3), (3.4) as wanted.

To prove that f is an unwrapping we have to show by Definition 1.2 and Proposition 1.1 that (a) it fixes all points of ℓ, (b) it fixes the tangent vectors of the parameter curves at their points on ℓ, (c) it is one-to-one in a small enough neighborhood of A which we rename as U_A, and (d) it preserves the first fundamental form at corresponding points. (c) and (d) make f to an isometry and (a), (b) to an unwrapping. The details follow. Recall that throughout the calculations s is an arc length parameter of c and $\overline{c}$.

For (a):

$$f(x(s_0,v)) = f\left(y(s_0) + \frac{a_0 - y(s_0)\cdot e}{t(s_0)\cdot e} t(s_0)\right) = \overline{y}(s_0) + \frac{a_0 - y(s_0)\cdot e}{t(s-0)\cdot e}\ \overline{t}(s_0) \overset{(3.4)}{=} x(s_0,v).$$

For (b): The v-parameter curve c_1 of U_A is a ruling for which $c_1 \cap \ell = \emptyset$ (for $s \neq s_0$; shrinking U_A further if necessary) or else $c_1 = \ell$ (for $s = s_0$). In the second case it is $f(c_1) = f(\ell) = \ell = c_1$, and clearly $c_1, f(c_1)$ share a common tangent vector at any one of their common points. The s-parameter curve $c_2 : x(s,v_0) = y(s) + v_0 t(s)$ has a unique common point $x(s_0,v_0)$ with ℓ, and f sends it to the curve $f(c_2) : \overline{y}(s) + v_0\overline{t}(s)$. The tangent vectors w_1, w_2 of c_2 and $f(c_2)$ at their common point $x(s_0,v_0)$ are the same as wanted:

$$w_1 = t(s_0) + v_0 t'(s_0) = t(s_0) + v_0 k(s_0) n(s_0)$$

$$w_2 = \overline{t}(s_0) + v_0\overline{t}'(s_0) = \overline{t}(s_0) + v_0\overline{k}(s_0)\overline{n}(s_0) \overset{(3.3),(3.4)}{=} t(s_0) + v_0 k(s_0) n(s_0).$$

For (c): f is one-to-one in a neighborhood of $A = z(s_0) = y(s_0) + \frac{a_0 - y(s)\cdot e}{t(s)\cdot e} t(s) = x(s_0,v_0)$ (where $v_0 = \frac{a_0 - y(s)\cdot e}{t(s)\cdot e} \neq 0$ as A does not belong to y), because $f \circ x$ is so in a neighborhood of (s_0,v_0): $(f\circ x)(s,v) = \overline{y}(s) + v\overline{t}(s) \overset{let}{=} (\overline{y}_1(s), \overline{y}_2(s)) + v(\overline{t}_1(s), \overline{t}_2(s))$

and for its Jacobian we compute

$$J = \begin{vmatrix} \overline{y}_1' + v\overline{t}_1' & \overline{y}_2' + v\overline{t}_2' \\ \overline{t}_1 & \overline{t}_2 \end{vmatrix} \overset{\overline{y}_i' = \overline{t}_i}{=} v(\overline{t}_1'\overline{t}_2 - \overline{t}_1\overline{t}_2') = v(\overline{t}_1, \overline{t}_2) \cdot (-\overline{t}_2', \overline{t}_1').$$

But $(-\overline{t}_2', \overline{t}_1')$ is a unit vector perpendicular to $(\overline{t}_1', \overline{t}_2') = \overline{n}$, therefore equal to $\pm\overline{t}$ and for $s = s_0$ we get $J(s_0, v) = \pm v\overline{t}(s_0)^2 \neq 0$ for $v \neq 0$.

For (d): for $x(s,v)$ and its corresponding point $f(x(s,v))$ it is $x_s = y' + vt' = t + vkn, x_v = t$ and $\overline{x}_s = \overline{y}' + v\overline{t}' = \overline{t} + v\overline{k}\overline{n}, \overline{x}_v = \overline{t}$ and so

$$E = x_s \cdot x_s = 1 + v^2k^2,\ F = x_s \cdot x_v = t^2 = 1,\ G = x_v \cdot x_v = 1$$
$$\overline{E} = \overline{x}_s \cdot \overline{x}_s = 1 + v^2\overline{k}^2 \overset{(3.3)}{=} 1 + v^2k^2,\ \overline{F} = \overline{x}_s \cdot \overline{x}_v = \overline{t}^2 = 1,\ \overline{G} = \overline{x}_v \cdot \overline{x}_v = 1$$

as wanted, finishing the proof of Lemma 3.2. □

Proof of Proposition 1.2 in the case U_A is part of the tangent surface of a curve.
Since the unwrapping of U_A is given by f of Lemma 3.2, the unwrapping $\overline{\gamma}$ of γ is $f(\gamma)$. In other words

$$\overline{\gamma} : w(s) = f(z(s)) = f\left(y(s) + \frac{a_0 - y(s)\cdot e}{t(s)\cdot e}t(s)\right) = \overline{y}(s) + \frac{a_0 - y(s)\cdot e}{t(s)\cdot e}\overline{t}(s)$$

and we are now prepared to prove $k_{\gamma_A} \cdot k_{\overline{\gamma}_A} = k_{\overline{\gamma}_A} \cdot k_{\overline{\gamma}_A}$, and so by Lemma 3.1 to prove the Proposition in this case.

Differentiate the first equation in (3.1) with respect to s and use $y' = t,\ t' = kn$ to compute

$$\begin{aligned} z' &= \left(y + \frac{a_0 - y\cdot e}{t\cdot e}t\right)' = t + \frac{(-t\cdot e)(t\cdot e) - (a_0 - y\cdot e)(t'\cdot e)}{(t\cdot e)^2}t + \frac{a_0 - y\cdot e}{t\cdot e}t' \\ &= \frac{(a_0 - y\cdot e)(kn\cdot e)}{(t\cdot e)^2}t + \frac{k(a_0 - y\cdot e)}{t\cdot e}n. \end{aligned}$$

Call f_1, f_2 the coefficients of t, n and use $t' = kn,\ n' = -kt + \tau b$ to get $|z'| = (f_1^2 + f_2^2)^{\frac{1}{2}}$ and

$$\begin{aligned} k_\gamma &= \frac{1}{|z'|}\left(\frac{z'}{|z'|}\right)' = \frac{1}{(f_1^2+f_2^2)^{\frac{1}{2}}}\left(\frac{f_1t + f_2n}{(f_1^2+f_2^2)^{\frac{1}{2}}}\right)' \\ &= \frac{1}{(f_1^2+f_2^2)^{\frac{1}{2}}}\left[\left(\frac{f_1}{(f_1^2+f_2^2)^{\frac{1}{2}}}\right)' t + \frac{f_1}{(f_1^2+f_2^2)^{\frac{1}{2}}}t' + \left(\frac{f_2}{(f_1^2+f_2^2)^{\frac{1}{2}}}\right)' n + \frac{f_2}{(f_1^2+f_2^2)^{\frac{1}{2}}}n'\right] \\ &= \frac{1}{(f_1^2+f_2^2)^{\frac{1}{2}}}\left[\left(\left(\frac{f_1}{(f_1^2+f_2^2)^{\frac{1}{2}}}\right)' - \frac{kf_2}{(f_1^2+f_2^2)^{\frac{1}{2}}}\right)t + \left(\frac{kf_1}{(f_1^2+f_2^2)^{\frac{1}{2}}} + \left(\frac{f_2}{(f_1^2+f_2^2)^{\frac{1}{2}}}\right)'\right)n \right. \\ &\quad \left. + \frac{f_2\tau}{(f_1^2+f_2^2)^{\frac{1}{2}}}b\right]. \end{aligned}$$

Use $w' = \overline{t},\ \overline{t}' = k\overline{n}$ to get

$$\begin{aligned} w' &= \left(\overline{y} + \frac{a_0 - y\cdot e}{t\cdot e}\overline{t}\right)' = \overline{t} + \frac{(-t\cdot e)(t\cdot e) - (a_0 - y\cdot e)(t'\cdot e)}{(t\cdot e)^2}\overline{t} + \frac{a_0 - y\cdot e}{t\cdot e}\overline{t}' \\ &= \frac{(a_0 - y\cdot e)(kn\cdot e)}{(t\cdot e)^2}\overline{t} + \frac{k(a_0 - y\cdot e)}{t\cdot e}\overline{n} = f_1\overline{t} + f_2\overline{n}. \end{aligned}$$

It follows $|w'| = (f_1^2 + f_2^2)^{\frac{1}{2}}$. Use $\overline{t}' = \overline{k}\overline{n} = k\overline{n}$, $\overline{n}' = -\overline{k}\,\overline{t}$ (since $\overline{c}$ is a plane curve) to find

$$\begin{aligned} k_{\overline{\gamma}} &= \frac{1}{|w'|}\left(\frac{w'}{|w'|}\right)' = \frac{1}{(f_1^2+f_2^2)^{\frac{1}{2}}}\left(\frac{f_1\overline{t}+f_2\overline{n}}{(f_1^2+f_2^2)^{\frac{1}{2}}}\right)' \\ &= \frac{1}{(f_1^2+f_2^2)^{\frac{1}{2}}}\left[\left(\frac{f_1}{(f_1^2+f_2^2)^{\frac{1}{2}}}\right)'\overline{t} + \frac{f_1}{(f_1^2+f_2^2)^{\frac{1}{2}}}\overline{t}' + \left(\frac{f_2}{(f_1^2+f_2^2)^{\frac{1}{2}}}\right)'\overline{n} + \frac{f_2}{(f_1^2+f_2^2)^{\frac{1}{2}}}\overline{n}'\right] \\ &= \frac{1}{(f_1^2+f_2^2)^{\frac{1}{2}}}\left[\left(\left(\frac{f_1}{(f_1^2+f_2^2)^{\frac{1}{2}}}\right)' - \frac{kf_2}{(f_1^2+f_2^2)^{\frac{1}{2}}}\right)t + \left(\frac{kf_1}{(f_1^2+f_2^2)^{\frac{1}{2}}} + \left(\frac{f_2}{(f_1^2+f_2^2)^{\frac{1}{2}}}\right)'\right)n\right]. \end{aligned}$$

So

$$k_\gamma = Pt + Qn + Rb \quad \text{and} \quad k_{\overline{\gamma}} = P\overline{t} + Q\overline{n}$$

for some functions P, Q, R, thus

$$\begin{aligned} k_{\gamma_A} = k_\gamma(s_0) &= P(s_0)t(s_0) + Q(s_0)n(s_0) + R(s_0)b(s_0) \\ k_{\overline{\gamma}_A} = k_{\overline{\gamma}}(s_0) &= P(s_0)\overline{t}(s_0) + Q(s_0)\overline{n}(s_0) \overset{(3.4)}{=} P(s_0)t(s_0) + Q(s_0)n(s_0) \end{aligned}$$

and

$$k_{\gamma_A} \cdot k_{\overline{\gamma}_A} = (P(s_0))^2 + (Q(s_0))^2 = k_{\overline{\gamma}_A} \cdot k_{\overline{\gamma}_A}$$

as wanted, finishing the proof of Proposition 1.2 in the case U_A is part of the tangent surface of a curve. □

The case $U_A =$ part of a generalized cylinder.

Let U_A be part of a (generalized) cylinder $F : x(u,v) = y(u) + vg, |g| = 1$. We can assume that the section curve $\gamma = U_A \cap \rho$ is given by $\gamma : y(u)$. Indeed, it is $\gamma : z(u) = y(u) + \frac{a_0 - y(u)\cdot e}{g \cdot e} g$ (recall $\rho : a_1x_1 + a_2x_2 + a_3x_3 = a_0$) and U_A is part of the cylinder $F_0 : x(u,v) = z(u) + vg$. By the way note that $g \cdot e \neq 0$ for if not, g would be parallel to ρ, thus the intersection of ρ and $\ell \parallel g$ would not be just a point A, a contradiction. F_0 is clearly a cylinder as defined in §1, and moreover clearly again its image coincides with that of F. Since A is a non-singular point of γ we have $\frac{dy}{du} \neq 0$ at A and by continuity $\frac{dy}{du} \neq 0$ in some arc of γ containing A. Restricting attention to this arc we can consider u to be an arc length parameter s of γ. So from now on U_A will be considered part of the cylinder $F : x(s,v) = y(s) + vg$ (Figure 4) and

$$\gamma : y(s), \quad s \text{ an arc length parameter of } \gamma, \quad A = y(s_0), \quad |g| = 1. \tag{3.5}$$

Denote differentiation with respect to s by primes and call $t_\gamma = y'$ the unit tangent vector of γ.

It will be convenient to introduce a Cartesian coordinate system Ax_1x_2 on T with origin A and unit vectors e_1, e_2 along the positive semi-axis Ax_1, Ax_2 respectively, with $e_1 = g$.

Lemma 3.3. *Let $\phi(s) = \angle(g, t_\gamma(s))$, and $\overline{c}$ be the curve on T whose equation in Ax_1x_2 is*

$$\overline{c} : \overline{y}(s) = \int_{s_0}^{s} (\cos\phi(\sigma)e_1 + \sin\phi(\sigma)e_2)d\sigma + y(s_0). \tag{3.6}$$

Denote the unit tangent vector of $\overline{c}$ by $\overline{t}$ and choose e_2 so that

$$\overline{t}(s_0) = t_\gamma(s_0). \tag{3.7}$$

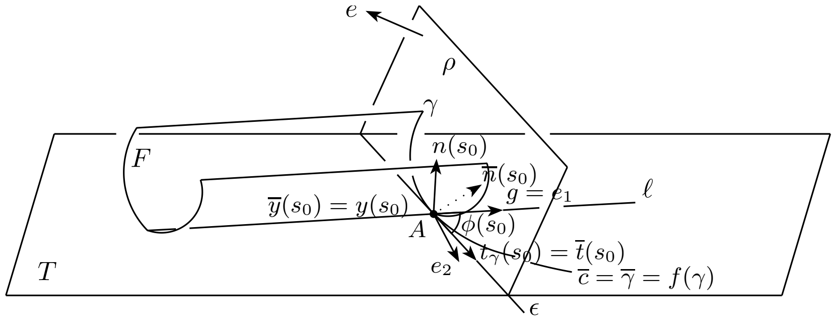


FIGURE 4. The case U_A = part of a generalized cylinder F.

Then there exists a neighborhood of A in U_A (which we still call U_A) whose unique unwrapping f on T along ℓ is

$$f(x(s,v)) = f\left(y(s) + vg\right) = \overline{y}(s) + vg \overset{call\ it}{=} \overline{x}(s,v) \in T.$$

Proof. First note that e_2 can indeed be chosen so that relation (3.7) is satisfied: s is an arc length parameter of $\overline{c}$ since $\overline{y}'(s) = \cos\phi(s)e_1 + \sin\phi(s)e_2$ (thus clearly $|\overline{y}'(s)| = 1$), and $\overline{y}(s_0) = y(s_0) = A$. So the unit tangent vector $\overline{t}(s_0)$ of $\overline{c}$ at A is $\overline{y}'(s) = \cos\phi(s_0)e_1 + \sin\phi(s_0)e_2$ and it forms with $g = e_1$ an angle $\angle(g, \overline{t}(s_0)) = \phi(s_0) = \angle(g, t_\gamma(s_0))$. Depending on the choice of e_2, either $\overline{t}(s_0)$ or its symmetric with respect to the line $\ell \parallel g$ coincides with $t_\gamma(s_0)$. If our original choice of e_2 was not the right one, we choose its opposite and (3.7) is satisfied as claimed.

To prove that f is an unwrapping, we have to show by Definition 1.2 and Proposition 1.1 that (a) it fixes all points of ℓ, (b) it fixes the tangent vectors of the parameter curves at their points on ℓ, (c) it is one-to-one in a small enough neighborhood of A which we rename as U_A, and (d) it preserves the first fundamental form at corresponding points. (c),(d) make f to an isometry and (a), (b) to an unwrapping. The details follow. Recall that throughout the calculations s is an arc length parameter of c and $\overline{c}$.

For (a): $f(x(s_0, v)) = f(y(s_0)+vg) = \overline{y}(s_0)+vg \overset{\overline{y}(s_0)=y(s_0)}{=} y(s_0)+vg = x(s_0, v)$.

For (b): The v-parameter curve c_1 of U_A is a ruling for which $c_1 \cap \ell = \emptyset$ (for $s \neq s_0$) or else $c_1 = \ell$ (for $s = s_0$). In the second case it is $f(c_1) = f(\ell) = \ell = c_1$, and clearly $c_1, f(c_1)$ share a common tangent vector at any one of their common points. The s-parameter curve $c_2 : x(s, v_0) = y(s) + v_0 g$ has a unique common point $x(s_0, v_0)$ with ℓ, and f sends c_2 to the curve $f(c_2) : \overline{y}(s) + v_0 g$. The tangent vectors w_1, w_2 of c_2 and $f(c_2)$ at their common point $x(s_0, v_0)$ are the same:

$$w_1 = t_\gamma(s_0) + v_0 g$$
$$w_2 = \overline{t}(s_0) + v_0 g \overset{(3.7)}{=} t_\gamma(s_0) + v_0 g.$$

For (c): f is one-to-one in a neighborhood of $A = y(s_0) = x(s_0, 0)$, because $f \circ x$ is so in a neighborhood of $(s_0, 0)$: $(f \circ x)(s, v) = \overline{y}(s)+vg \overset{let}{=} (\overline{y}_1(s), \overline{y}_2(s))+v(g_1, g_2)$ and for its Jacobian we compute

$$J = \begin{vmatrix} \overline{y}_1' & \overline{y}_2' \\ g_1 & g_2 \end{vmatrix} \overset{\overline{y}_i'=\overline{t}_i}{=} \overline{t}_1 g_2 - g_1 \overline{t}_2 = (g_1, g_2) \cdot (-\overline{t}_2, \overline{t}_1).$$

But $(-\overline{t}_2, \overline{t}_1)$ is a unit vector perpendicular to $(\overline{t}_1, \overline{t}_2) = \overline{t}$, therefore equal to $\pm\overline{n}$, and for $s = s_0$ we get $J(s_0, 0) = \pm g \cdot \overline{n}(s_0) \neq 0$ since $g, \overline{n}(s_0)$ are not perpendicular (if they were, it would be $g||t(s_0)$, that is $\ell||\epsilon$ which cannot be).

For (d): for $x(s, v)$ and its corresponding point $f(x(s, v))$ it is $x_s = y' = t_\gamma, x_v = g$ and $\overline{x}_s = \overline{y}' = \overline{t}, \overline{x}_v = g$ and so

$$E = x_s \cdot x_s = t^2 = 1,\ F = x_s \cdot x_v = t_\gamma \cdot g,\ G = x_v \cdot x_v = g^2$$
$$\overline{E} = \overline{x}_s \cdot \overline{x}_s = \overline{t}^2,\ \overline{F} = \overline{x}_s \cdot \overline{x}_v = \overline{t} \cdot g,\ \overline{G} = \overline{x}_v \cdot \overline{x}_v = g^2.$$

Therefore for the claimed invariance of the first fundamental form it must be

$$t_\gamma \cdot g = \overline{t} \cdot g \tag{3.8}$$

which is immediate as

$$t_\gamma \cdot g = |t_\gamma||g| \cos(t, g) = \cos(\phi(s))$$
$$\overline{t} \cdot g = (\cos\phi(s)e_1 + \sin\phi(s)e_2)e_1 = \cos(\phi(s)),$$

and Lemma 3.3 is proved. □

Proof of Proposition 1.2 in the case U_A is part of a generalized cylinder.

Since the unwrapping of U_A is given by f of Lemma 3.3, the unwrapping $\overline{\gamma}$ of γ is $f(\gamma)$. In other words

$$\overline{\gamma} = f(\gamma) : f(y(s)) \overset{(3.5)}{=} f(x(s, 0)) = \overline{y}(s) + 0g = \overline{y}(s),$$

and because of (3.6) we have $\overline{\gamma} = \overline{c}$. We are now prepared to prove $\kappa_\gamma(n_\gamma \cdot n_{\overline{\gamma}}) = \kappa_{\overline{\gamma}}$ at A, and so by Lemma 3.1 to prove the Proposition in this case.

Differentiate (3.8) to get $t'_\gamma \cdot g = \overline{t}' \cdot g$, that is $k_\gamma \cdot g = k_{\overline{\gamma}} \cdot g$ from which

$$\kappa_\gamma(n_\gamma \cdot g) = \kappa_{\overline{\gamma}}(n_{\overline{\gamma}} \cdot g) \tag{3.9}$$

which for $s = s_0$ relates the normal vectors and signed curvatures of $\gamma, \overline{\gamma}$ at A. All calculations below are done at A.

Observe that $n_{\overline{\gamma}} \cdot g \neq 0$, otherwise $n_{\overline{\gamma}}$ would be normal to g forcing g to be parallel to t, that is ℓ to be a line of ρ, a contradiction. So then $\kappa_\gamma \frac{n_\gamma \cdot g}{n_{\overline{\gamma}} \cdot g} = \kappa_{\overline{\gamma}}$ and the required relation $\kappa_\gamma(n_\gamma \cdot n_{\overline{\gamma}}) = \kappa_{\overline{\gamma}}$ is equivalent to

$$\kappa_\gamma(n_\gamma \cdot n_{\overline{\gamma}})(n_{\overline{\gamma}} \cdot g) = \kappa_\gamma(n_\gamma \cdot g). \tag{3.10}$$

To prove it write the vector $n_{\overline{\gamma}}$ of T as a linear combination $n_{\overline{\gamma}} = at_\gamma + bg$ of the linearly independent vectors t_γ, g. Recall that $t_\gamma \cdot n_\gamma = 0 = t_\gamma \cdot n_{\overline{\gamma}}$, $|t_\gamma| = |g| = 1$ and consider the inner product of both sides of $n_{\overline{\gamma}} = at_\gamma + bg$ with n_γ, g, t_γ to get respectively

$$n_\gamma \cdot n_{\overline{\gamma}} = b(n_\gamma \cdot g), \quad n_{\overline{\gamma}} \cdot g = a(t_\gamma \cdot g) + b, \quad 0 = a + b(t_\gamma \cdot g).$$

So (3.10) is equivalent to

$$\kappa_\gamma b(n_\gamma \cdot g)\left(a\frac{-a}{b} + b\right) = \kappa_\gamma(n_\gamma \cdot g),$$

and this will hold provided $b\left(a\frac{-a}{b} + b\right) = 1$. Note first that it cannot be $b = 0$, since this would imply $n_{\overline{\gamma}} = at_\gamma$ making $n_{\overline{\gamma}}$ and $t_{\overline{\gamma}} = t_\gamma$ dependent, a contradiction. Now, $b\left(a\frac{-a}{b} + b\right) = -a^2 + b^2$. And observe that $|n_{\overline{\gamma}}| = 1$ gives $|at_\gamma + bg| = 1$, so $a^2 + b^2 + 2ab(t_\gamma \cdot g) = 1$, in other words $a^2 + b^2 + 2ab\frac{-a}{b} = 1$ and equivalently $-a^2 + b^2 = 1$, as wanted finishing the proof of Proposition 1.2 in the case $U_A =$ part of a generalized cylinder. □

The case $U_A =$ part of a generalized cone.

Let U_A be part of a (generalized) cone F. Without any loss of generality we work in a Cartesian coordinate system $Ox_1x_2x_3$ with origin at the vertex O of the cone, and let $F : x(u, v) = vy(u)$, where $y(u)$ is the directrix. Actually we can consider that the section curve $\gamma = U_A \cap \rho$ is this directrix. Indeed, it is $\gamma : z(u) = \frac{a_0}{y(u)\cdot e} y(u)$ (recall that $\rho : a_1x_1 + a_2x_2 + a_3x_3 = a_0$) and U_A is part of the cone $F_0 : x(u, v) = vz(u)$. It must be $y(u) \cdot e \neq 0$ for if not, $y(u_0)$ corresponding to A would be parallel to ρ and then the ruling ℓ would be parallel to ρ, thus ℓ and ρ wouldn't have just a single point A in common, a contradiction. Then by continuity $y(u) \cdot e \neq 0$ for a small enough interval of values of u around u_0, and if necessary we shrink U_A so that this relation holds for all u for which $z(u) \in U_A$. F_0 is clearly a cone as defined in §1, and moreover clearly again its image coincides with that of F. As in the case of U_A being part of a cylinder, we can consider u to be an arc length parameter s of γ. So from now on U_A is part of the cone $F : x(s, v) = vy(s)$ (Figure 5) and

$$\gamma : y(s), s \in I, \quad s \text{ an arc length parameter of } \gamma, \quad A = y(s_0). \tag{3.11}$$

Denote differentiation with respect to s by primes and call $t_\gamma = y'$ the unit tangent vector of γ.

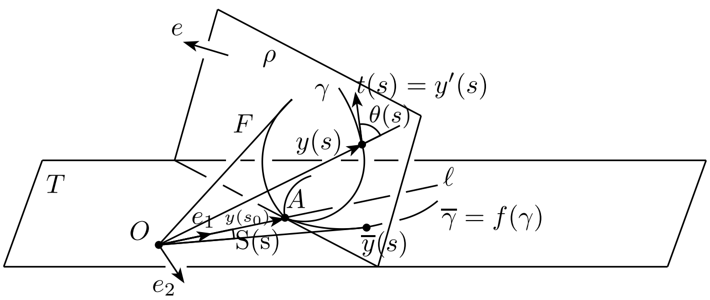


Figure 5. The case $U_A =$ part of a generalized cone F.

Since the vertex O does not lie on γ, the points $y(s)$ correspond to values of v which are all positive or all negative, let without any loss of generality $v > 0$. Consider the part F_γ of the cone corresponding to $(s, v) \in I \times (0, +\infty)$. It is convenient to consider as unit vectors e_1, e_2 along the positive semi-axis Ox_1, Ox_2 of the Cartesian system, two vectors of the plane T and especially to consider e_1 as a vector with the direction of $\overrightarrow{OA} = y(s_0)$.

Lemma 3.4. *Let $\theta(s) = \angle(y(s), t_\gamma(s))$ and let us choose e_2 so that*

$$y'(s_0) = (\cos(\theta(s_0)), \sin(\theta(s_0)), 0). \tag{3.12}$$

Then there exists a neighborhood of A in U_A (which we still call U_A) whose unique unwrapping f on T along ℓ is

$$f(x(s, v)) = v|y(s)| \left(\cos\left(\int_{s_0}^{s} \frac{\sin(\theta)}{|y(s)|} d\sigma \right), \sin\left(\int_{s_0}^{s} \frac{\sin(\theta)}{|y(s)|} d\sigma \right), 0 \right) \overset{\text{call it}}{=} \overline{x}(s, v) \in T.$$

Proof. First note that e_2 can indeed be chosen so that relation (3.12) is satisfied: since $\angle(e_1, y'(s_0)) = \angle(y(s_0), y'(s_0)) = \angle(y(s_0), t_\gamma(s_0)) = \theta(s_0)$ and since

$y'(s_0)$ is a vector of T of unit length, we have either $y'(s_0) = \cos(\theta(s_0))e_1 + \sin(\theta(s_0))e_2 = (\cos(\theta(s_0)), \sin(\theta(s_0)), 0)$ or $y'(s_0) = \cos(\theta(s_0))e_1 - \sin(\theta(s_0))e_2 = (\cos(\theta(s_0)), -\sin(\theta(s_0)), 0)$ depending on the choice of e_2. So e_2 can be chosen appropriately.

To prove that f is an unwrapping we show below that (a) it is one to one, (b) it preserves the first fundamental form at corresponding points in a small enough neighborhood of A which we rename as U_A ((a), (b) make f to an isometry), (c) it fixes all points of ℓ and (d) it fixes the tangent vectors of the parameter lines at their points on ℓ ((c), (d) make f to an unwrapping). Here are the details:

For (a): First we shrink I (and U_A) so that

$$\text{length of } I < 1, \text{ and } \frac{\sin(\theta(s))}{|y(s)|} \in I_0 \text{ with length of } I_0 < \pi. \tag{3.13}$$

Trivially, this can be achieved: Consider s close enough to s_0 so that by continuity $0 \neq \frac{|y(s_0)|}{2} < |y(s)|$ and restrict s further if necessary, and close to s_0 so that for the corresponding values $\theta(s)$ which are close to $0 = \theta(s_0)$ because of continuity, it holds $|\sin(\theta(s))| < \frac{3\pi|y(s_0)|}{2}$. Then $\frac{|\sin(\theta(s))|}{|y(s)|} < \frac{\pi}{3}$, so $\frac{\sin(\theta(s))}{|y(s)|} \in I_0 = (-\frac{\pi}{3}, \frac{\pi}{3})$ of length less than π as wanted.

Now if f were not one to one, it would be $f(s_1, v_1) = f(s_2, v_2)$ for some $(s_1, v_1) \neq (s_2, v_2)$, so $|f(s_1, v_1)| = |f(s_2, v_2)|$ thus $v_1|y(s_1)| = v_2|y(s_2)|$. This immediately gives $s_1 \neq s_2$ for if not the equality would also imply $v_1 = v_2$, a contradiction. But $s_1 \neq s_2$ also leads to a contradiction as the equality $f(s_1, v_1) = f(s_2, v_2)$ together with $v_1|y(s_1)| = v_2|y(s_2)|$ imply $\left(\cos\left(\int_{s_0}^{s_1} \frac{\sin(\theta)}{|y(s)|} d\sigma\right), \sin\left(\int_{s_0}^{s_1} \frac{\sin(\theta)}{|y(s)|} d\sigma\right)\right) = \left(\cos\left(\int_{s_0}^{s_2} \frac{\sin(\theta)}{|y(s)|} d\sigma\right), \sin\left(\int_{s_0}^{s_2} \frac{\sin(\theta)}{|y(s)|} d\sigma\right)\right)$ which gives $\int_{s_0}^{s_1} \frac{\sin(\theta)}{|y(s)|} d\sigma = \int_{s_0}^{s_2} \frac{\sin(\theta)}{|y(s)|} d\sigma$ because relation (3.13) makes both integrals lie in $[0, \pi)$. But this equality of integrals simply cannot hold as their common integrand is a positive function and $s_1 \neq s_2$, thus the claimed contradiction.

For (b): $x_s = vy', x_u = y$ from which

$$E = v^2(y')^2 = v^2, \quad F = vyy', \quad G = y^2.$$

and calling $S(s) = \int_{s_0}^{s} \frac{\sin(\theta)}{|y(s)|} d\sigma$ we get

$$\overline{x}_s \;=\; v|y|'(\cos S, \sin S, 0) + v|y(s)|(-\sin S, \cos S, 0), \overline{x}_v = |y|(\cos S, \sin S, 0)$$

from which

$$\overline{F} = \overline{x}_s\overline{x}_v = v|y||y|' = v|y|\frac{y \cdot y'}{|y|} = vyy' = F$$
$$\overline{G} = \overline{x}_v\overline{x}_v = |y|^2 = G$$

and

$$\begin{aligned}\overline{E} &= \overline{x}_s\overline{x}_s = (v|y|')^2 + (v|y|)^2\left(\frac{\sin(\theta)}{|y|}\right)^2 = v^2\left[(|y|')^2 + \sin^2(\theta)\right] \\ &= v^2\left(\cos^2(\theta)^2 + \sin^2(\theta)\right) = v^2 = E\end{aligned}$$

where we used the relation

$$|y|' = \left((|y|^2)^{\frac{1}{2}}\right)' = \frac{(|y|^2)'}{2((|y|^2)^{\frac{1}{2}}} = \frac{(y^2)'}{2|y|} = \frac{2y \cdot y'}{2|y|} = \frac{y \cdot y'}{|y|} = \cos(\theta). \tag{3.14}$$

For (c): Recall the notation $S(s) = \int_{s_0}^{s} \frac{\sin(\theta(s))}{|y(s)|} d\sigma$ to compute

$$S(s_0) = \int_{s_0}^{s_0} \frac{\sin(\theta(s))}{|y(s)|} d\sigma = 0 \tag{3.15}$$

and

$$\begin{aligned} f(x(s_0, v)) = f(vy(s_0)) = v|y(s_0)| \left(\cos(S(s_0)), \sin(S(s_0)), 0\right) &\overset{(3.15)}{=} v|y(s_0)|(\cos(0, \sin(0)), 0) \\ = v|y(s_0)|(1, 0, 0) = v|y(s_0)|e_1 = v|y(s_0)|\frac{y(s_0)}{|y(s_0)|} &= vy(s_0) = x(s_0, v). \end{aligned}$$

For (d): The v-parameter curve c_1 of U_A is a ruling for which $c_1 \cap \ell = \emptyset$ (for $s \neq s_0$) or else $c_1 = \ell$ (for $s = s_0$). In the second case it is $f(c_1) = f(\ell) = \ell = c_1$, and clearly $c_1, f(c_1)$ share a common tangent vector at any one of their common points. The s-parameter curve $c_2 : x(s, v_0) = v_0 y(s)$ has a unique common point $x(s_0, v_0)$ with ℓ, and f sends c_2 to the curve $f(c_2) : v_0|y(s)| \left(\cos(S(s)), \sin(S(s)), 0\right)$. The tangent vectors w_1, w_2 of c_2 and $f(c_2)$ at their common point $x(s_0, v_0)$ are the same:

$$w_1 = v_0 y'(s_0)$$

and

$$\begin{aligned} w_2 &= v_0|y(s)|'_{|s_0}(\cos S(s_0), \sin S(s_0), 0) + v_0|y(s_0)|\frac{\sin(\theta(s_0))}{|y(s_0)|}(-\sin S(s_0), \cos S(s_0), 0) \\ &\overset{(3.14),(3.15)}{=} v_0 \cos(\theta(s_0))(\cos(0), \sin(0), 0) + v_0 \sin(\theta(s_0))(-\sin(0), \cos(0), 0) \\ &= v_0 \cos(\theta(s_0))(1, 0, 0) + v_0 \sin(\theta(s_0))(0, 1, 0) = v_0(\cos(\theta(s_0)), \sin(\theta(s_0)), 0) \\ &\overset{(3.12)}{=} v_0 y'(s_0) = w_1, \end{aligned}$$

finishing the proof of Lemma 3.4. □

Proof of Proposition 1.2 in the case U_A is part of a generalized cone.

Since the unwrapping of U_A is given by f of Lemma 3.4, the unwrapping $\overline{\gamma}$ of γ is $f(\gamma)$. In other words $\overline{\gamma} = f(\gamma) \overset{(3.11)}{:} f(y(s)) = f(x(s, 1))$, so

$$\overline{\gamma}: |y(s)| \left(\cos\left(\int_{s_0}^{s} \frac{sin(\theta)}{|y(s)|} d\sigma\right), \sin\left(\int_{s_0}^{s} \frac{sin(\theta)}{|y(s)|} d\sigma\right), 0\right) \overset{\text{call it}}{=} \overline{y}(s).$$

Observe that s is an arc length parameter of $\overline{\gamma}$:

$$\begin{aligned} \overline{y}' &= |y|'(\cos(S), \sin(S), 0) + |y|\frac{\sin(\theta)}{|y|}(-\sin(S), \cos(S), 0) \\ &= |y|'(\cos(S), \sin(S), 0) + \sin(\theta)(-\sin(S), \cos(S), 0) \end{aligned}$$

and

$$|\overline{y}'|^2 = \left(|y|'\right)^2 + (\sin(\theta))^2 \overset{(3.14)}{=} (\cos(\theta))^2 + (\sin(\theta))^2 = 1.$$

The above expressions for $\overline{y}$ and $\overline{y}'$ give

$$\overline{y} \cdot \overline{y}' = |y||y|' = |y|\frac{y \cdot y'}{|y|} = y \cdot y'$$

which by differentiation implies $k_\gamma \cdot y + y'^2 = k_{\overline{\gamma}} \cdot \overline{y} + \overline{y}'^2$, thus $k \cdot y = k_{\overline{\gamma}} \cdot \overline{y}$ after noting that $y'^2 = \overline{y}'^2 = 1$. For $s = s_0$ it follows trivially that $y(s_0) = \overline{y}(s_0) = A$. Call $g = y(s_0) = \overline{y}(s_0)$ to get $k_\gamma(s_0) \cdot g = k_{\overline{\gamma}}(s_0) \cdot g$, also written as

$$\kappa_\gamma(s_0)(n_\gamma(s_0) \cdot g) = \kappa_{\overline{\gamma}}(s_0)(n_{\overline{\gamma}}(s_0) \cdot g)$$

which is the same as relation (3.9) for $s = s_0$ in the case of U_A being part of a cylinder. It was proved there that relation (3.9) for $s = s_0$ implies the required relation (3) of Lemma 3.1, i.e. $\kappa_\gamma(n_\gamma \cdot n_{\overline{\gamma}}) = \kappa_{\overline{\gamma}}$ at $s = s_0$. But this proof can be repeated word by word here and so by Lemma 3.1 we have finished with the proof of Proposition 1.2 in the case $U_A =$ part of a generalized cone as well. □

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