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ON THE SMOOTHABLE SCHEME RANK WITH RESPECT TO NON-DEGENERATE VARIETIES

EDOARDO BALLICO

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Abstract. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety. Set $m := \dim(X)$ and assume $n \geq m+2$. Here we prove that for each $P \in \mathbb{P}^n$ there is a zero-dimensional smooth scheme $Z \subset \mathbb{P}^n$ such that $\deg(Z) \leq n-m$ and the linear span of Z contains P. We may find Z with the additional condition that either Z is reduced or it has only two unreduced connected components, both with degree 2 or that it has a unique unreduced connected component, Z_1, and $\deg(Z_1) \in \{2,3\}$.

1. Introduction

Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate subvariety. For each $P \in \mathbb{P}^n$ the X-rank $r_X(P)$(resp. scheme-rank $z_X(P)$, resp. smoothable scheme rank $sm_X(P)$) of P is the minimal cardinality of a finite subset (resp. zero-dimensional scheme, resp. zero-dimensional smoothable scheme inside X) Z of $\mathbb{P}^n$ such that $P \in \langle Z\rangle$, where $\langle\ \rangle$ denote the linear span. We recall that a zero-dimensional scheme $Z \subset X$ is said to be *smoothable inside* X if it is a flat limit of a flat family of finite subsets of X. The key concept is the notion of X-rank, because it is the one used in the applications to tensors and symmetric tensors. If X is a Segre embedding of a multiprojective space $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_1}$, then P corresponds to a tensor of format $n_1, \dots, n_s$ and $r_X(P)$ is the tensor rank of the tensor P (here $n+1 = (m_1+1)\cdots(m_s+1)$. If X is a order d Veronese embedding of $\mathbb{P}^m$ (here $n = \binom{n+d}{n}-1$) , then P corresponds to a degree d homogeneous polynomial $f \in \mathbb{K}[x_0,\dots,n]$ and $r_X(P)$ is the minimal integer t such that $f = \sum_{i=1}^{t} L_i^d$, where each L_i is a linear form ([5]). In this note we first improve (just by 1) an upper bound for the X-rank given in [6] ([6], Proposition 5.1) and prove the following result.

Theorem 1.1. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety. Set $m := \dim(X)$ and assume $m \leq n-1$. Then $sm_X(P) \leq n-m$ for all $P \in \mathbb{P}^n$.*

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We recall that [6], Proposition 5.1, fails by 1 in a few well-described cases if $\mathrm{char}(\mathbb{K}) > 0$ ([2]).

Remark 1.1. Take X as in Theorem 1.1 and $P \in \mathbb{P}^n$. The proof of Theorem 1.1 gives the existence of a zero-dimensional smoothable scheme $Z \subset X$ such that $P \in \langle Z \rangle$, $\deg(Z) = n-m$ and either Z is reduced or it has two unreduced connected components, both of them with degree 2, or it has a unique unreduced connected component Z_1 and $\deg(Z_1) \in \{2,3\}$.

2. The proof

Remark 2.1. The scheme $Z \subset X$ is smoothable if and only if each connected component of Z is smoothable. If $Z \subset X_{reg}$, then Z is smoothable inside X if and only if it is smoothable inside $\mathbb{P}^n$. If $Z \subset X_{reg}$, then it is smoothable inside X if and only if it is smoothable inside $\mathbb{P}^n$ ([4], Proposition 2.1.5). Hence if X is smooth., then a zero-dimensional scheme $Z \subset X$ is smoothable inside X if and only if it is smoothable inside $\mathbb{P}^n$. This is the reasons why for smooth varieties X one usually write *smoothable* without specifying the ambient smooth variety allowed to do the smoothing. Notice that each degree 2 subscheme of X_{reg} is smoothable.

For each integer $k \geq 1$ let $\sigma_k(X) \subseteq \mathbb{P}^n$ denote the closure in $\mathbb{P}^n$ of the union of all linear subspaces spanned by k points of X. Each $\sigma_k(X)$ is an integral variety and $\dim(\sigma_k(X)) \leq \min\{n, k\cdot(\dim(X)+1)-1\}$. If $\dim(X) = 1$, then $\dim(\sigma_k(X)) = \min\{n, 2k-1\}$ for all $k \geq 1$ ([1], Remark 1.6). In particular $\sigma_2(X) = \mathbb{P}^3$ for each non-degenerate curve $X \subset \mathbb{P}^3$.

Remark 2.2. Fix a zero-dimensional smoothable scheme $Z \subset X$. Since Z is a flat limit of a set of $\deg(Z)$ points and $\sigma_{\deg(Z)}(X)$ is defined as a closure of all linear spans of $\deg(Z)$ distinct points, we have $\langle Z \rangle \subseteq \sigma_{\deg(Z)}(X)$.

We need the following lemma (proof of [3], Proposition 11).

Lemma 2.1. *Let $\beta'(X)$ denote the maximal integer such that $\dim(\langle Z \rangle) = \deg(Z)-1$ for each zero-dimensional and smoothable scheme of degree $\leq \beta'(X)$. Fix an integer k such that $1 \leq k \leq \beta'(X)$. Then $\sigma_k(X)$ is the union of all $\langle Z \rangle$, where Z is a smoothable subscheme of Z of degree k.*

Proof of Theorem 1.1 Fix $P \in \mathbb{P}^n$. If $P \in X$, then $r_X(P) = sm_X(P) = 1$. Hence we may assume $P \notin X$. In steps (a) and (b) we assume $\mathrm{char}(\mathbb{K}) = 0$.

(a) Assume $m = 1$. First assume $n = 3$. Since any degree 2 subscheme of X spans a line, Remark 2.1 and Lemma 2.1 give that for each $P \in \mathbb{P}^3$ there is a zero-dimensional smoothable degree 2 scheme $Z \subset X$ such that $P \in \langle Z \rangle$. Hence Theorem 1.1 is true in this case. Now assume $n \geq 4$. Fix a general $S \subset X_{reg}$ such that $\sharp(S) = n-3$. If $P \in \langle S \rangle$, then $sr_X(P) \leq r_X(P) \leq n-3$. Hence we may assume $P \notin \langle S \rangle$. For general S we may also assume that each point of $\langle S \rangle$ is a smooth point of S (in characteristic zero we may even get $\langle S \rangle \cap X = S$ as schemes). Let $\ell : \mathbb{P}^n \setminus \langle S \rangle \to \mathbb{P}^3$ denote the linear projection from $\langle S \rangle$. Since each point of $\langle S \rangle \cap X$ is a smooth point of X, $\ell|(X \setminus \langle X \rangle \cap X)$ extends to a morphism $f : X \to \mathbb{P}^3$. Notice that $f(X)$ is an integral and non-degenerate curve (in characteristic zero we may even assume that f is birational onto its image). The case $n = 3$ gives the existence of a degree two smoothable scheme $W \subset \mathbb{P}^3$ such that $\ell(P) \in \langle W \rangle$. Since $f : X \to f(D)$ is a proper and surjective morphism, there is a zero-dimensional scheme $W' \subset X$ such that $\deg(W') = 2$ and $f(W') = W$. However, in general such a scheme W' may be non-smoothable if X is singular. To find W' as above and with the additional property that W' is smoothable we use the following path. Fix an affine integral curve Δ, $o \in \Delta$ and a flat family $\{W_\lambda\}_{\lambda\in\Delta}$ of degree two schemes with $W_o = W$ and W_λ reduced for all $\lambda \neq o$. Taking a finite covering Δ' of $\Delta \setminus \{o\}$ we lift this family to a family $\mathcal{F}$ of distinct points of X. We call W' a flat limit of $\mathcal{F}$) (a flat limit exists, because X is projective and hence each connected component of $\mathrm{Hilb}(X)$ is projective). We have $\deg(W') = 2$ and $f(W') = W$. By construction W' is smoothable. First assume $W' \cap S = \emptyset$. Since $\ell(P) \in \langle W\rangle$, we get $P \in \langle S \cup W'\rangle$. Now assume $W' \cap S \neq \emptyset$. Let Z be the only scheme such that $Z_{red} = W_{red} \cup S$, it coincide with W at the points of $W'_{red} \setminus S \cap W'_{red}$ (there is at most one point, and there is such a point if W' is reduced and $\sharp(S\cap W') = 1$) and at each $Q \in S$ it has multiplicity $1 + \epsilon_Q \in \{1, 2, 3\}$, where ϵ_Q is the multiplicity of W' at Q. Since each unreduced connected component of Z is contained in $S \subset X_{reg}$, Z is smoothable. We have $P \in \langle Z\rangle$ and $\deg(Z) = n - 1$.

(b) Assume $m > 1$. Let $V \subset \mathbb{P}^n$ be a general linear subspace such that $P \in V$ and $\dim(V) = n - m + 1$. Since $P \notin X$, the linear system of all hyperplanes of $\mathbb{P}^n$ passing through P has no base points on X. Hence Bertini's theorem gives that $X \cap V$ is an integral curve. Let $H \subset \mathbb{P}^n$ be any hyperplane. Since X is integral, we have $h^1(\mathcal{I}_X) = 0$. Since X is non-degenerate, we have $h^0(\mathcal{I}_X(1)) = 0$. Hence the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_X(1) \to \mathcal{I}_{X\cap H,H}(1) \to 0$$

shows that the scheme $X \cap H$ spans H. Applying $m - 1$ times this observation we get that the curve $X \cap V$ spans V. Since $P \in V$ and $X \cap V \subset X$ we have $sm_X(P) \leq sm_{X\cap V}(P) \leq n - m$, the last inequality being true by step (a).

(c) From now on we assume $p := \mathrm{char}(\mathbb{K}) > 0$. First assume $m = 1$. The proof of the case $n = 3$ made in step (a) works verbatim. In the case $n > 3$ we were careful to use only characteristic free statements. Now assume $m > 1$. We use induction on m. Let $\alpha : \mathbb{P}^n \setminus \{P\} \to \mathbb{P}^{n-1}$ denote the linear projection from P. Since $P \notin X$, $u := \alpha|X$ is a finite morphism. First assume that u is not separable, i.e. assume that for a general $O \in X_{reg}$ we have $P \in T_OX$, where T_OX denote the Zariski tangent space to X at P. In this case the scheme $\langle\{O, P\}\rangle \cap X$ is a scheme containing O with multiplicity ≥ 2. Hence $P \in \langle E\rangle$, where E be the degree 2 subscheme of the line $\langle\{O, P\}\rangle$ with O as its support. Since $m > 0$, we have $E \subset X \cap T_OX$. Since $O \in X_{reg}$, the degree 2 scheme E is smoothable inside P (Remark 2.1). Hence $sr_X(P) \leq 2$. Now assume that u is separable, i.e. that P is not a strange point of X. In this case we may repeat the proof of the corresponding part in [2] (top of page 6) to reduced to the case $m = 1$ just proven. □

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Department of Mathematics, University of Trento, 38123 Povo (TN), Italy
E-mail address: ballico@science.unitn.it