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INTEGRATING THE DIFFERENTIAL EQUATIONS INSPIRED BY THE UMBILICITY CONDITION FOR ROTATION HYPERSURFACES IN LORENTZ-MINKOWSKI SPACE

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Abstract. U. Dursun obtained explicit parametrizations of rotation hypersurfaces in the Lorentz-Minkowski ambient space. The pointwise umbilicity condition yields a differential equation in each of the cases described by Dursun's parametrizations. In the present work we study the direct solutions of these differential equations.

1. Introduction

Recently, Uğur Dursun [11] obtained explicit parametrizations of rotation hypersurfaces in the Lorentz-Minkowski ambient space L^{n+1}; related ideas have been pursued in a later work [12]. The origin of the problem of characterizing umbilicity can be traced back to Delaunay's work [10]; for the history of the mathematical idea one can see the introduction in [11] and its references. It is natural to ask when such rotational hypersurfaces have all points umbilics. A point of a hypersurface is called an umbilic if all the principal curvatures are equal (see e.g. [16], p.72). Thus, the geometric condition we study is that all the principal curvatures are equal at every point. The pointwise umbilicity condition yields a differential equation in each case of Dursun's parametrizations obtained in [11]. The direct solutions of these differential equations are the object of the present work.

The structure of our paper naturally follows the cases discussed in [11].

Our question is motivated by the geometry of the De Sitter space-time (see [1]; for an exposition see [13]), which represents one of the particular solutions in the present study (see Corollary 3.3 in [11]).

The study of umbilics on hypersurfaces has a long history; they are extensively discussed in the classical literature (see e.g. Chapter 4 in [15]; see also the more recent monograph by Bang-Yen Chen [8]). The length of the interval where the principal curvatures lie is called the spread of the shape operator (for the study of the conformal invariant see [17], and for the algebraic foundations [14]). This geometric quantity, the spread of shape operator, represents, intuitively, how far a

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Dedicated to the memory of Edsel F. Stiel (1933-2008).

point of a submanifold is from being an umbilic. This direction of study pursues along the lines of the thorough study of the umbilical condition [2, 3, 4].

We describe first our notations, which are along the lines of notations used in [11].

Let L^{n+1} denotes the $(n+1)-$dimensional Lorentz-Minkowski space, that is, the real vector space $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric $\langle , \rangle = (dx_1)^2 + \ldots + (dx_n)^2 - (dx_{n+1})^2$, where $(x_1, ..., x_{n+1})$ are the canonical coordinates in $\mathbb{R}^{n+1}$. A vector x of L^{n+1} is said to be space-like if $\langle x, x\rangle > 0$ or $x = 0$, time-like if $\langle x, x\rangle < 0$, or light-like (or null) if $\langle x, x\rangle = 0$ and $x \neq 0$.

An immersed hypersurface M_q of L^{n+1} with index q ($q = 0, 1$) is called space-like (Riemannian) or time-like (Lorentzian) if the induced metric, which, as usual, is also denoted by $\langle , \rangle$ on M_q has the index 0 or 1, respectively. The de Sitter n-space $\mathbb{S}^n_1(x_0, c)$ centered at $x_0 \in L^{n+1}, c > 0$, is a Lorentzian hypersurface of L^{n+1} defined by

$$\mathbb{S}^n_1(x_0, c) = \{x \in L^{n+1} |\, \langle x - x_0, x - x_0\rangle = c^2\}.$$

The index T used below indicates the times axis direction.

We also use the following notation. Let $\Theta(u_1, ..., u_{n-2})$ be an orthogonal parametrization of the unit sphere $\mathbb{S}^{n-2}(1)$ in the Euclidean space $\mathbb{E}^{n-1}$ generated by $\{\eta_1, \eta_2, ...\eta_{n-1}\}$:
(1.1)

$$\Theta(u_1, ..., u_{n-2}) = \cos u_1 \eta_1 + \sin u_1 \cos u_2 \eta_2 + \cdots + \sin u_1 \cdots \sin u_{n-3} \cos u_{n-2} \eta_{n-2} +$$

$$+ \sin u_1 \cdots \sin u_{n-3} \sin u_{n-2} \eta_{n-1},$$

where $0 < u_i < \pi$, for $i = 1, ..., n - 3$, and $0 < u_{n-2} < 2\pi$.

2. The Differential Equation Corresponding to the Umbilicity Condition for Rotational Hypersurfaces with Time-Like Axis

Dursun proved in [11] the following result. Let $M_{q,T}$ be a rotation hypersurface of L_{n+1} with the index q and time-like axis parameterized by
(2.1)

$$f_T(u_1, \ldots, u_{n-1}, t) = \phi(t) \sin u_{n-1} \Theta(u_1, \ldots, u_{n-2}) + \phi(t) \cos u_{n-1} \eta_n + \psi(t)\eta_{n+1},$$

where $0 < u_{n-1} < \pi$. Then the directions of parameters are principal directions, and the principal curvatures along the coordinate curves u_i, $i = 1, ..., n - 1$, are all equal and given by

$$\lambda = -\frac{\dot{\psi}}{\phi\sqrt{\epsilon(\dot{\phi}^2 - \dot{\psi}^2)}}$$

with multiplicity $n - 1$, and the principal curvature along the coordinate curve t is given by

$$\mu = \frac{\dot{\psi}\ddot{\phi} - \ddot{\psi}\dot{\phi}}{(\dot{\phi}^2 - \dot{\psi}^2)\sqrt{\epsilon(\dot{\phi}^2 - \dot{\psi}^2)}}$$

where $\varepsilon = sgn(\dot{\phi}^2 - \dot{\psi}^2) = \mp 1$, and $q = 0$ if $\varepsilon = 1$, and $q = 1$ if $\varepsilon = -1$.

It should be pointed out here that Dursun obtains the umbilicity condition from the study of the constant mean curvature condition (see Theorem 3.2 in [11]) using geometric characterizations to establish his Corollary 3.3. Our study relies on a direct integration process in the differential equation (2.4).

Proposition 2.1. *Let $M_{q,T}$ be a rotation hypersurface of L_{n+1} with the index q and time-like axis parameterized by*

$$f_T(u_1,\dots,u_{n-1},t)=\phi(t)\sin u_{n-1}\Theta(u_1,\dots,u_{n-2})+\phi(t)\cos u_{n-1}\eta_n+\psi(t)\eta_{n+1}, \tag{2.2}$$

where $0<u_{n-1}<\pi$. Then this hypersurface has all points umbilics if and only if $\phi^2(t)=\psi^2(t)+2C\psi(t)+K$, at every t in the domain of definition.

Proof: The condition that the hypersurface has equal principal curvatures at every point is:

$$-\frac{\dot\psi}{\phi\sqrt{\epsilon(\dot\phi^2-\dot\psi^2)}}=\frac{\dot\psi\ddot\phi-\ddot\psi\dot\phi}{(\dot\phi^2-\dot\psi^2)\sqrt{\epsilon(\dot\phi^2-\dot\psi^2)}} \tag{2.3}$$

After simplifications, this equation becomes:

$$-\dot\psi\dot\phi^2+\dot\psi\cdot\dot\psi^2=\dot\psi\phi\ddot\phi-\ddot\psi\phi\dot\phi$$

which can be rearranged as

$$-\dot\psi(\dot\phi^2+\phi\ddot\phi)+\dot\psi^3+\ddot\psi\phi\dot\phi=0 \tag{2.4}$$

or:

$$-\dot\psi\frac{d}{dt}(\phi\dot\phi)+\dot\psi^3+\ddot\psi\phi\dot\phi=0.$$

By using the substitution $u=\phi\dot\phi$, the equation becomes:

$$-\dot\psi\frac{du}{dt}+\dot\psi^3+\ddot\psi u=0 \tag{2.5}$$

Under the restriction $\dot\psi\neq0$, the equation becomes an ordinary linear differential equation:

$$\frac{du}{dt}+\left(-\frac{\ddot\psi}{\dot\psi}\right)u=\dot\psi^2. \tag{2.6}$$

The integrating factor is: $\rho=(\dot\psi)^{-1}$, which yields the linear differential equation:

$$\frac{1}{\dot\psi}\cdot\frac{du}{dt}+\left(-\frac{\ddot\psi}{(\dot\psi)^2}\right)\cdot u=\dot\psi. \tag{2.7}$$

Completing the derivative in the left-hand side term, we get:

$$\frac{d}{dt}\left(\frac{u}{\dot\psi}\right)=\dot\psi.$$

Thus we obtain, for some real constant C :

$$\frac{u}{\dot\psi}=\psi+C.$$

Therefore $u=\psi\dot\psi+C\dot\psi$. Bearing in mind the substitution that precedes relation (2.5) we have:

$$\phi\cdot\dot\phi=\psi\cdot\dot\psi+C\dot\psi.$$

Integrating with respect to t both sides, we get, for C and K real constants:

$$\phi^2(t)=\psi^2(t)+2C\psi(t)+K. \tag{2.8}$$

The solution exists when C and K are such that $\psi^2(t)+2C\psi(t)+K\geq0$. □

As a remark, equation (2.8) can be rewritten in the form $\phi^2 = (\psi + C)^2 + K_1$.

3. The Differential Equation Corresponding to the Umbilicity Condition for Rotational Hypersurfaces of First Kind with Space-Like Axis

In the paper [11], Dursun discusses the rotational hypersurfaces of first kind. They are obtained as follows. Suppose that the axis of rotations is the x_n-axis, that is, the vector $\eta = (0, \dots, 0, 1, 0)$ is the direction of the rotation axis, and Π is the $x_n x_{n+1}$-plane. Let $\gamma(t) = \psi(t)\eta_n + \phi(t)\eta_{n+1}$ be a parametrization of γ in the plane Π with $x_{n+1} = \phi(t) > 0$, $t \in I \subset \mathbb{R}$. in the plane Π with $x_{n+1} = \phi(t) > 0$, $t \in I \subset \mathbb{R}$. Thus Dursun gave a parametrization of a rotation hypersurface of the first kind M_{q,S_1} of L^{n+1} with space-like axis as

(3.1)

$f_{S_1}(u_1, \dots, u_{n-1}, t) = \phi(t) \sinh u_{n-1} \Theta(u_1, \dots, u_{n-2} + \psi(t)\eta_n + \psi(t) \cosh u_{n-1} \eta_{n+1}$,

where $0 < u_{n-1} < \infty$, which is also called a hyperbolic rotation hypersurface of L^{n+1} as parallels of M_{q,S_1} are hyperbolic spaces $\mathbb{H}^{n-1}(0, -\phi(t))$.

Dursun proved the following. Let M_{q,S_1} be a rotation hypersurface of the first kind of L^{n+1} with the index q and space-like axis parameterized by (3.1). Then the direction of parameters are principal directions, and the principal curvatures along the coordinate curves u_i, $i = 1, ..., n-1$, are all equal and given by

$$\lambda = -\frac{\dot{\psi}}{\phi\sqrt{\bar{\epsilon}(\dot{\psi}^2 - \dot{\phi}^2)}} \tag{3.2}$$

with multiplicity $n-1$, and the principal curvature along the coordinate curve t is given by

$$\mu = \frac{\ddot{\psi}\dot{\phi} - \dot{\psi}\ddot{\phi}}{(\dot{\psi}^2 - \dot{\phi}^2)\sqrt{\bar{\epsilon}(\dot{\psi}^2 - \dot{\phi}^2)}}, \tag{3.3}$$

where $\bar{\epsilon} = sgn(\dot{\psi}^2 - \dot{\phi}^2 = \mp 1$, and $q = 0$ if $\bar{\epsilon} = 1$, and $q = 1$ if $\bar{\epsilon} = -1$.

In this context, we prove the following.

Proposition 3.1. *Let M_{q,S_1} be a rotation hypersurface of the first kind of L^{n+1} with the index q and space-like axis parameterized by (3.1). Suppose that both ϕ and ψ are at least twice differentiable. Additionally, suppose that $\dot{\psi} \neq 0$ for the interval of definition. Then this hypersurface has all points umbilics if and only if $\phi^2(t) = \psi^2(t) + 2C\psi(t) + K$, for every t in the domain of definition.*

Proof: The condition that the hypersurface has equal principal curvatures at every point is:

$$-\dot{\psi}^3 - \dot{\psi}\dot{\phi}^2 = \dot{\psi}\ddot{\phi}\phi - \ddot{\psi}\dot{\phi}\phi.$$

This differential equation can be rewritten as:

$$\ddot{\psi}\dot{\phi}\phi - \dot{\psi}^3 - \dot{\psi}(\dot{\phi}^2 + \ddot{\phi}\phi) = 0.$$

We group the terms in the last bracket as a derivative:

$$-\dot{\psi}\left[\frac{d}{dt}(\dot{\phi}\phi)\right] - \dot{\psi}^3 - \ddot{\psi}\dot{\phi}\phi = 0.$$

We see that this integration process is the same as in the case of rotational hypersurfaces with time-like axis. We are using the same integration factor that leads to a similar prime integral for the differential equation, with a solution similar to the one found in the previous section. □

4. The Differential Equation Corresponding to the Umbilicity Condition for Rotation Hypersurfaces of Second Kind with Space-Like Axis

In section 5 of the work [11], Dursun studied rotation hypersurfaces of the second kind M_{1,S_2} with space-like axis and constant mean curvature. Let $\gamma(t) = \phi(t)\eta_{n-1} + \psi(t)\eta_n$ be a parametrization of γ in the plane Π considered to be $x_{n-1}x_n$. Let $x_{n-1} = \phi(t) > 0$, $t \in I \subset \mathbb{R}$. For this plane, the curve γ is space-like. In this content, Dursun gave a parametrization of a hypersurface of the second kind M_{q,S_2} of the Lorentz space L^{n+1} with space like-axis as:

(4.1)

$$f_(S_2)(u_1, \ldots, u_{n-1}, t) = \phi(t) \cosh u_{n-1} \Theta(u_1, \ldots, u_{n-2} + \psi(t)\eta_n + \phi(t) \sinh u_{n-1}\eta_{n+1},$$

where the coordinate u_{n-1} can take any real values. The image of this function is called a pseudospherical rotation hypersurface of L^{n+1}, as parallels of M_{q,S_2} are actually pseudospheres $\mathbb{S}_1^{n-1}(0, \phi(t))$ when $n > 2$. In [11] it is shown that the index $q = 1$, thus it is legitimate to talk about M_{1,S_2}.

In [11] it is proved the following result. Let M_{1,S_2} be a rotation hypersurface of the second kind of L^{n+1} with space-like axis parameterized by (4.1). Then the direction of parameters are principal directions, and the principal curvatures along the coordinate curves u_i, $i = 1, ..., n-1$, are all equal and given by

$$\lambda = -\frac{\dot\psi}{\phi\sqrt{(\dot\psi^2 + \dot\phi^2)}}, \tag{4.2}$$

with multiplicity $n - 1$, and the principal curvature along the coordinate curve t is given by

$$\mu = \frac{\dot\psi\ddot\phi - \ddot\psi\dot\phi}{(\dot\psi^2 + \dot\phi^2)\sqrt{(\dot\psi^2 + \phi^2)}}. \tag{4.3}$$

In this context, we prove the following.

Proposition 4.1. *Let M_{1,S_2} be a rotation hypersurface of the second kind of L^{n+1} with space-like axis parameterized by (4.1). Suppose that both ϕ and ψ are at least twice differentiable. Additionally, suppose that $\dot\psi \neq 0$ for the interval of definition. Then this hypersurface has all points umbilics if and only if $\phi^2(t) = -\psi^2(t) + 2C\psi(t) + K$, for every t in the domain of definition.*

Proof: The geometric condition we study is that all principal curvatures are equal. This is expressed by $\lambda(t) = \mu(t)$, at every t in the domain of definition. This condition generates the differential equation:

$$\frac{-\dot\psi}{\phi} = \frac{\dot\psi\ddot\phi - \ddot\psi\dot\phi}{(\dot\psi^2 + \dot\phi^2)}.$$

Cross-multiplying, we get:

$$\dot\psi\phi\ddot\phi + \dot\psi\dot\phi^2 + \dot\psi^3 - \ddot\psi\phi\dot\phi = 0.$$

By grouping

$$\dot{\psi}(\phi\ddot{\phi}+\dot{\phi}^2)+\dot{\psi}^3-\ddot{\psi}\phi\dot{\phi}=0,$$

we naturally see that the substitution $u=\phi\dot{\phi}$ yields the linear equation

$$\dot{\psi}\frac{du}{dt}-\ddot{\psi}u+\dot{\psi}^3=0.$$

Under the condition $\dot{\psi}\neq 0$ on the domain of definition, we get:

$$\frac{du}{dt}-\frac{\ddot{\psi}}{\dot{\psi}}u+\dot{\psi}^2=0.$$

Using the integrating factor $\rho(t)=\dot{\psi}^{-1}$, we obtain the linear ordinary differential equation:

$$\frac{1}{\dot{\psi}}\ \frac{du}{dt}-\frac{\ddot{\psi}}{\dot{\psi}^2}\ u=-\dot{\psi}$$

which can be integrated, starting from

$$\frac{d}{dt}\left(\frac{1}{\dot{\psi}}\ u\right)=-\dot{\psi}$$

and obtaining:

$$\frac{1}{\dot{\psi}}\phi\dot{\phi}=-\int\dot{\psi}+C$$

which yields

$$\phi\dot{\phi}=-\psi\dot{\psi}+C\dot{\psi}.$$

By integrating one more time this relation, we obtained the claimed result. □

5. The Differential Equation Corresponding to the Umbilicity Condition for Rotational Hypersurfaces with Light-Like Axis

Finally, we discuss in this section the last case, when the hypersurface has a light-like axis. Dursun obtains that the rotation hypersurface $M_{q,L}$ of L^{n+1} with light-like axis is defined as (see equation (2.8) in [11]):

(5.1)

$$f_L(u_1,...,u_{n-1},t)=2\phi(t)u_{n-1}\Theta(u_1,...,u_{n-2})+\sqrt{2}\phi\hat{\eta}_n+\sqrt{2}(\psi(t)-\phi(t)u_{n-1}^2)\hat{\eta}_{n+1},$$

with $u_{n-1}\neq 0$. Here, the basis has been chosen such that $\hat{\eta}_1=(1,0,...,0)$, $\hat{\eta}_2=(0,1,0,...,0)$,... $\hat{\eta}_{n-1}=(0,0,...,0,1,0,0)$, $\hat{\eta}_n=\frac{1}{\sqrt{2}}(0,0,...,0,1,-1)$, and $\hat{\eta}_{n+1}=\frac{1}{\sqrt{2}}(0,0,...,0,1,1)$. Dursun proved the following. Let $M_{q,L}$ be a rotation hypersurface of L_{n+1} with the index q and time-like axis parameterized by (5.1). Then the direction of parameters are principal directions, and the principal curvatures along the coordinate curves u_i, for $i=1,...,n-1$ are equal and given by

$$\lambda=-\frac{\dot{\phi}}{2\phi\sqrt{\hat{\varepsilon}\dot{\phi}\dot{\psi}}}$$

with multiplicity $n-1$, and the principal curvature along the coordinate curve t is given by

$$\mu=\frac{\dot{\phi}\ddot{\psi}-\dot{\psi}\ddot{\phi}}{4\dot{\phi}\dot{\psi}\sqrt{\hat{\varepsilon}\dot{\phi}\dot{\psi}}}$$

where $\hat{\varepsilon}=sgn(\dot{\phi}\dot{\psi})=\mp 1$, and $q=0$ if $\hat{\varepsilon}=1$, and $q=1$ if $\hat{\varepsilon}=-1$.

We integrate the differential equation obtained from the umbilicity condition, under the restriction $\dot{\phi} \neq 0$.

Proposition 5.1. *Let $M_{q,L}$ be a rotation hypersurface of L_{n+1} with the index q and time-like axis parameterized by (5.1). Suppose $\dot{\phi} \neq 0$. Suppose also that $\phi > 0$.*

Then this hypersurface has all points umbilics if and only if there exists the constants C and K such that

$$-\frac{1}{\phi} = C\psi + K.$$

Proof: The differential equation we need to integrate is:

$$-\frac{\dot{\phi}}{2\phi} = \frac{\dot{\phi}\ddot{\psi} - \dot{\psi}\ddot{\phi}}{4\dot{\phi}\dot{\psi}}.$$

By cross-multiplying, we can write (with two identical terms among them):

$$\dot{\psi}\phi\ddot{\phi} - (\dot{\phi})^2\dot{\psi} - (\dot{\phi})^2\dot{\psi} - \phi\dot{\phi}\ddot{\psi} = 0.$$

This relation can be rewritten as:

$$\dot{\psi} \cdot \phi^2 \frac{d}{dt}\left(\frac{\dot{\phi}}{\phi}\right) = \dot{\phi} \cdot [\dot{\phi}\dot{\psi} + \phi\ddot{\psi}].$$

We regroup the terms as follows:

$$\dot{\psi}\phi\frac{d}{dt}\left(\frac{\dot{\phi}}{\phi}\right) = \left(\frac{\dot{\phi}}{\phi}\right)\frac{d}{dt}(\dot{\psi}\phi).$$

This can be rewritten as:

$$\frac{\frac{d}{dt}(\dot{\phi}/\phi)}{(\dot{\phi}/\phi)} = \frac{\frac{d}{dt}(\dot{\psi}\phi)}{\dot{\psi}\phi}.$$

When we integrate both sides with respect to t, we get:

$$\frac{\dot{\phi}}{\phi} = C\phi\dot{\psi}.$$

Separating the two functions we have:

$$\frac{\dot{\phi}}{\phi^2} = C\dot{\psi}.$$

This differential equation integrates as

$$-\frac{1}{\phi} = C\psi + K.$$

□

This last case completes the discussion of the direct integration question of the differential equations obtained from the pointwise umbilicity condition on the rotational hypersurfaces in all the cases in Dursun's classification from [11].

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