International Electronic Journal of Geometry
Volume 6 No. 1 pp. 170–180 (2013) ©IEJG

BENZ SURFACES

GROZIO STANILOV

(Communicated by Arif SALIMOV)

Abstract. We define one parametrical family of surfaces (called Benz surfaces) which are induced by a given surface in the Euclidean 3-dimensional space. Some *Maple* programs are also used to investigate these surfaces. We also consider some examples, visualizations and animations. Several theorems about Benz surfaces are also proved.

1. Introduction

Walter Benz gives the following characterization for the curvature axis in the line geometry [1]:
A ray K in the space is a curvature axis of the straight line surface $X(s) = x(s) + ev(s)$ if the following condition holds:

$$[X(s+h) - x(s)]\,.K = 0, \{h^3\}.$$

The investigations there are made by classical methods. This result gives us occasion for carrying it over the theory of surfaces. So we introduce in respect to any surface in the 3-dimensional Euclidean space a family of surfaces called Benz surfaces. Our investigations are not in classical notations because it will be very difficult and practically not possible. Because of this we applied the calculations by computer using the computer algebra and graphic of Maple.

Let the surface be determined by the equation

$$S : x = x(u, v).$$

This is a two-variable vector function and we apply the well-known Taylor's Formula:

$$x(u+h, v+k) = x(u, v) + \frac{h}{1!}xu + \frac{k}{1!}xv + \frac{h^2}{2!}xuu + \frac{2hk}{2!}xuv +$$
$$+\frac{k^2}{2!}xvv + \frac{h^3}{3!}xuuu + \frac{3h^2k}{3!}xuuv + \frac{3hk^2}{3!}xuvv + \frac{k^3}{3!}xvvv + O(h, k).$$

Date: Received: February 27, 2013 and Accepted: March 27, 2013.
2000 *Mathematics Subject Classification.* 53A05, 53A25, 53B25.
Key words and phrases. Benz surfaces, *Maple* program, Möbius strip, Cup surface, Cartesian coordinates.

Dedicated to Prof. Dr. (DHC) Walter Benz.

Now we look for a surface of the form

$$B = k_1 xu + k_2 xv + k_3 n. \tag{1.1}$$

Here k_1, k_2, k_3 are some coefficients. We take the growth $k = mh$ with some function $m = m(h)$. From the Benz condition

$$[x(u+h, v+k) - x(u,v)].B = 0,$$

and dividing by h and putting $h \mapsto 0$, we get the first equality

$$k_1(g_{11} + mg_{12}) + k_2(g_{12} + mg_{22}) = 0. \tag{1.2}$$

Here g_{11}, g_{12}, g_{22} are the coefficients of the first fundamental form for the given surface. In the same way, considering second degrees of h, we get the second equality

$$k_1 e + k_2 f + k_3 II(m) = 0. \tag{1.3}$$

For the corresponding coefficients, we have the expressions

$$e = xu.(xuu + 2mxuv + m^2 xvv),$$
$$f = xv.(xuu + 2mxuv + m^2 xvv),$$
$$II(m) = n.(xuu + 2mxuv + m^2 xvv).$$

From the equalities (2) and (3), up to homothety, we find the coefficients

$$k_1 = g_{12} + mg_{22}, \quad k_2 = -g_{11} - mg_{12}, \quad k_3 = -(k_1 e + k_2 f)/II(m) \tag{1.4}$$

if the second fundamental form for the given surface is not identically zero.

Thus, we have established the following

Theorem 1.1. *Any surface with non-zero identical second fundamental form induces one parameter family of surfaces defined by* (1) *and* (4).

These surfaces we shall call *Benz surfaces* induced by the given surface.

2. Benz surfaces induced by a given surface and parameterized by Cartesian coordinates

From the considerations we have already done, it is obvious that the parametrization of the given surface is very important. Changing the parametrization one should expect different results. Let's start with the following

Example 1. We take the hyperboloid in the non-classical parametrization:

$$x_1 = u\cos u, \ x_2 = v\cos v, \ x_3 = uv\cos u\cos v.$$

It is shown on the Fig. 1.

We can find the family of Benz surfaces induced by the given hyperboloid and picture in the case $m = 0$ is given by Fig. 2.

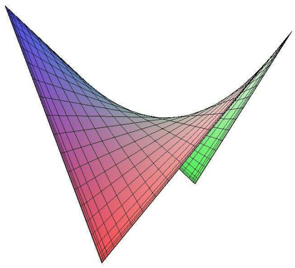
Let's take now the hyperboloid in the classical representation:

$$x_1 = u, \ x_2 = v, \ x_3 = uv.$$

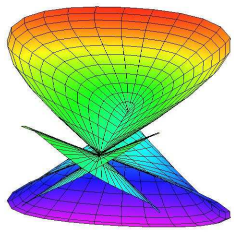
Its graphic is shown on Fig. 3 below and the graphic of the Benz surface is shown on Fig. 4. It is a segment and this is true for all values of the parameter m.

First we shall prove the following

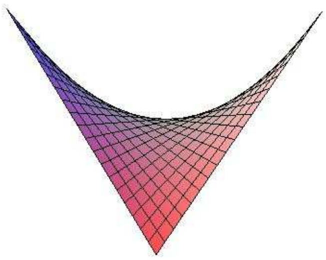
Theorem 2.1. *If the given surface is parameterized by the Cartesian coordinates then the family of the Benz surfaces consists only of segments.*



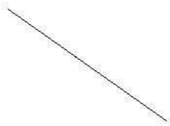
(a) Fig.1



(b) Fig.2



(c) Fig.3



(d) Fig.4

Proof. Let the surface is represented by the equation:

$$z = F(x, y)$$

or in parametrical form:

$$x = u,\ y = v,\ z = F(u, v).$$

Then the coefficients of the first fundamental form are:

$$g_{11} = 1 + \left(\frac{\partial}{\partial u}F(u,v)\right)^2,\ g_{12} = \frac{\partial}{\partial u}F(u,v)\frac{\partial}{\partial v}F(u,v),\ g_{22} = 1 + \left(\frac{\partial}{\partial u}F(u,v)\right)^2.$$

The discriminate is

$$g = 1 + (\frac{\partial}{\partial u}F(u,v))^2 + (\frac{\partial}{\partial v}F(u,v))^2.$$

Then we find:

$$e = \frac{\partial}{\partial u}F(u,v)(\frac{\partial^2}{\partial u^2}F(u,v) + 2m\frac{\partial^2}{\partial v \partial u}F(u,v) + m^2\frac{\partial^2}{\partial v^2}F(u,v))$$

$$f = \frac{\partial}{\partial v}F(u,v)(\frac{\partial^2}{\partial u^2}F(u,v) + 2m\frac{\partial^2}{\partial v \partial u}F(u,v) + m^2\frac{\partial^2}{\partial v^2}F(u,v))$$

$$II(m) = \frac{\frac{\partial^2}{\partial u^2}F(u,v) + 2m\frac{\partial^2}{\partial v \partial u}F(u,v) + m^2\frac{\partial^2}{\partial v^2}F(u,v)}{\sqrt{1 + (\frac{\partial}{\partial u}F(u,v))^2 + (\frac{\partial}{\partial v}F(u,v))^2}.}$$

$$k_1 = \frac{\partial}{\partial u}F(u,v)\frac{\partial}{\partial v}F(u,v) + m + m(\frac{\partial}{\partial v}F(u,v))^2,$$

$$k_2 = -1 - (\frac{\partial}{\partial u}F(u,v))^2 - m\frac{\partial}{\partial u}F(u,v)\frac{\partial}{\partial v}F(u,v),$$

$$k_3 = -\sqrt{1 + (\frac{\partial}{\partial u}F(u,v))^2 + (\frac{\partial}{\partial v}F(u,v))^2}.(m\frac{\partial}{\partial u}F(u,v) - \frac{\partial}{\partial v}F(u,v)).$$

Thus for the parameterization of Benz family of surfaces we get:

$$B_1 = m(1+(\frac{\partial}{\partial u}F(u,v))^2+(\frac{\partial}{\partial v}F(u,v))^2), B_2 = -1-(\frac{\partial}{\partial u}F(u,v))^2-(\frac{\partial}{\partial v}F(u,v))^2), B_3 = 0$$

The third equation shows the family consists from plane surfaces. From the first two equations we get

$$B_1 = -mB_2.$$

So the theorem is proved.

3. Maple Program for treating Benz Surfaces

For the sake of convenience we prefer to use the following Maple Program to investigate Benz surfaces. If the given surface has the parameterization

$$x_1 = X_1(u,v), x_2 = X_2(u,v), x_3 = X_3(u,v)$$

we introduce the package

```
with(VectorCalculus): x:=<x1,x2,x3>;
```

and calculate the coefficients of its first fundamental form:

```
g11:=simplify(DotProduct(diff(x,u),diff(x,u))):
g12:=simplify(DotProduct(diff(x,u),diff(x,v))):
g22:=simplify(DotProduct(diff(x,v),diff(x,v))):
g:=simplify(g11*g22-g12^2):
```

The unit normal vector field can be calculated by:

```
n:=CrossProduct(diff(x,u),diff(x,v))/sqrt(g);
```

We calculate

```
e:=DotProduct(diff(x,u),diff(x,u,u)+2*m*diff(x,u,v)+m^2*diff(x,v,v)):
f:=DotProduct(diff(x,v),diff(x,u,u)+2*m*diff(x,u,v)+m^2*diff(x,v,v)):
II(m):=DotProduct(n,diff(x,u,u)+2*m*diff(x,u,v)+m^2*diff(x,v,v)):

 k1:=g12+m*g22:k2:=-(g11+m*g12):k3:=-(k1*e+k2*f)/II(m):
 The vector parameterization of the looking for surface is given by:

B:=k1*diff(x,u)+k2*diff(x,v)+k3*n:
```

To find a coordinate presentation of the Benz surface we introduce the usual basis:

```
e1:=<1,0,0>;e2:=<0,1,0>;e3:=<0,0,1>;
```

Then the family of Benz surface is parameterized by the formulas

```
B1:=DotProduct(B,e1):B2:=DotProduct(B,e2):B3:=DotProduct(B,e3):
```

4. Benz surfaces induced from Möbius strip

For the Möbius strip we apply the following representation

$$x_1 = \cos u + v\cos(\frac{u}{2})\cos u, x_2 = \sin u + v\cos(\frac{u}{2})\sin u, x_3 = v\sin(\frac{u}{2}).$$

By Fig. 5 and 6 are shown the positions of the rotated normal vector field along the closed curve $v = 0$at its initial point $u = 0$ and at its end point $u = 2\pi$. They are of course opposite vectors.

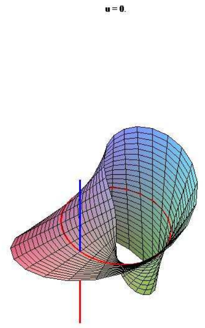
```
  with(plots):
p1:=plot3d([x1,x2,x3],u=0..2*Pi,v=-1..1):
opts:=color=red,thickness=2,numpoints=100:
p2:=animate(spacecurve,[[eval(x1+k*n1,[v=0]),eval(x2+k*n2,[v=0])
,eval(x3+k*n3,[v=0])], opts,k=0..1],u=0..2*Pi): p3:=spacecurve
([eval(x1,[v=0]),eval(x2,[v=0]), eval(x3,[v=0])],u=0..2*Pi,
color=red,thickness=3): p4:=spacecurve([eval(x1+k*n1,[v=0,u=0]),
eval(x2+k*n2,[v=0,u=0]),eval(x3+k*n3,[v=0,u=0])], k=0..1,
color=red,thickness=3): p5:=spacecurve([eval(x1+k*n1,[v=0,u=2*Pi]),
eval(x2+k*n2,[v=0,u=2*Pi]), eval(x3+k*n3,[v=0,u=2*Pi])],k=0..1,
color=blue,thickness=3):
display([p1,p2,p3,p4,p5]);
```

But what happens on the Benz surfaces? Applying the Maple program we find the Benz family of surfaces. Then we can prove the following

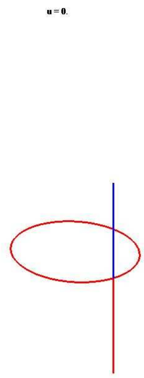
Theorem 4.1. 1. *The curve $v = 0$ is on any (for any m) Benz surface induced from Möbius strip non-closed;*
2. *There exists exactly one number namely*

$$mo = \frac{\sqrt{-18 + 12\sqrt{3}}}{6},$$

for which the moved unit normal vector field along the curve $v = 0$ at the initial point $u = 0$ and at the end point $u = 2\pi$ bring to vectors n_1,$n_2$which are opposite vectors.



(e) Fig.5



(f) Fig.6

We sketch the proof. The first part follows from the results:

$$B(v=0,u=0)=(-1,m,-\frac{1+2m^2}{m})$$

$$B(v=0,u=2\pi)=(1,m,-\frac{1+2m^2}{m})$$

They show these points can be never identical. For the second part we find the unit normal vector field N on the Benz surfaces and then we calculate the vectors:

```
n1:=simplify(eval(N, [v=0,u=0]));
n2:=simplify(eval(N, [v=0,u=2*Pi]));
```

We find the expressions:

$$n_1=\frac{2(12m^2+12m^4-1)}{\sqrt{\frac{4+592m^4-11m^2+1280m^6+576m^8}{m^6}}m^3}e_1+\frac{8m^2+9}{\sqrt{\frac{4+592m^4-11m^2+1280m^6+576m^8}{m^6}}m^2}e_2$$
$$-\frac{2(-1+4m^2)}{\sqrt{\frac{4+592m^4-11m^2+1280m^6+576m^8}{m^6}}m^2}e_3,$$
$$n_2=\frac{2(12m^2+12m^4-1)}{\sqrt{\frac{4+592m^4-11m^2+1280m^6+576m^8}{m^6}}m^3}e_1-\frac{8m^2+9}{\sqrt{\frac{4+592m^4-11m^2+1280m^6+576m^8}{m^6}}m^2}e_2$$
$$+\frac{2(-1+4m^2)}{\sqrt{\frac{4+592m^4-11m^2+1280m^6+576m^8}{m^6}}m^2}e_3$$

For their sum we get

$$n_1+n_2=\frac{4(12m^2+12m^4-1)}{\sqrt{\frac{4+592m^4-11m^2+1280m^6+576m^8}{m^6}}m^3}e_1$$

The equation

$$12m^4+12m^2-1=0$$

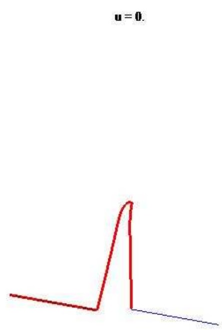
has solutions:

$$mo, -mo, \pm\frac{\sqrt{-18-12\sqrt{3}}}{6}.$$

The situation in the last theorem is shown by the Figures 7 and 8. So we got the following important situation. For the classical Möbius strip, which is a non orient table surface, the moved unit normal vector field along the closed curve at its initial point and at its end point bring to vectors which are opposite vectors in one and the same point. Now we found an extension of this result but the initial and the end points are different. In my opinion *this situation is new in the all mathematics (topology).* This was realized in our concept for introducing Benz surfaces and it was an essential motivation for introducing Benz surfaces. Here is the program for Figures 7 and 8:

```
with(plots):
l1:=plot3d([eval(B1,[m=1]),eval(B2,[m=1]),eval(B3,[m=1])],
u=0..2*Pi,v=-1..1,grid=[30,30]):
l2:=spacecurve([eval(B1,[m=1,v=0]),eval(B2,[m=1,v=0]),
eval(B3,[m=1,v=0])],u=0..2*Pi,color=red,thickness=3):
l3:=textplot3d([-1,1,-3,G1],align={LEFT},color=blue):
l4:=textplot3d([1,1,-3,G2],align={RIGHT},color=blue):
display([l1,l2,l3,l4]);
```

(g) Fig.7



(h) Fig.8

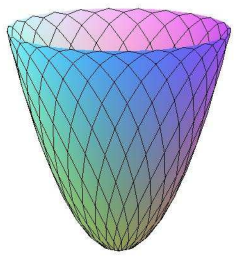
Benz surfaces over a cup

We define the following surface

$$x_1 = \cos u + \sin v, x_2 = \cos v + \sin u, x_3 = (\cos u + \sin v)^2 + (\cos v + \sin u)^2.$$

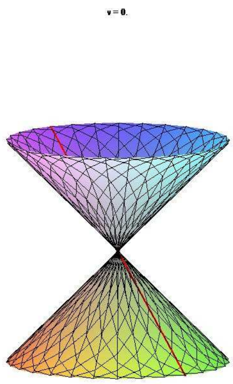
Evidently it is a part of a rotational paraboloid; we call it **cup**. It is shown by Fig. 9:

```
with(plots):
l1:=plot3d([x1,x2,x3],u=0..2*Pi,v=0..2*Pi,grid=[25,25]):
l2:=spacecurve([eval(x1,[u=Pi]),eval(x2,[u=Pi]),
eval(x3,[u=Pi])],v=0..2*Pi,color=red,thickness=3):
display([l1]);
```

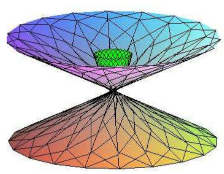


(i) Fig.9

As in the previews cases we find the family $B(u, v; m)$ of Benz surfaces induced from our cup. We show some of them. By Fig. 10 is shown the case $m = 0$ - it is a part of a cone. By Fig.11, the same figure together with the given cup.

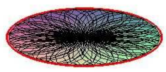


(j) Fig.10

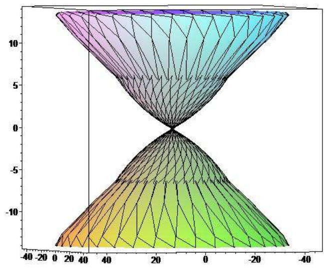


(k) Fig.11

By Fig. 12 is shown the case $m = 1$ and by Fig. 13 -the case $m = 2$.
We give the following classification theorem for Benz surfaces over the cup.



(l) Fig.12



(m) Fig.13

Theorem 4.2. *The Benz family of surfaces induced from the cup can be classified in the following way:* 1. *If* $m = 0$ *the Benz surface is really a part of a cone;*
2. *If* $m = 1$ *the Benz surface is a circle of radius*

$$r \approx 13.15378318.$$

Sketch for the proof. We find the coefficients of the second fundamental of any Benz surface induced from the cup. In the case 1 we establish that

$$h_{11} = h_{12} = 0, h_{22} \neq 0.$$

So the surface is flat but not a plane surface. It must be a cone surface. In the case 2 we find that

$$h_{11} = h_{12} = h_{22} = 0,$$
$$B_3(m = 0) = 0.$$

These equalities show the corresponding surface is a plane piece. To see that it is a circle (more precisely surround from circle) we calculate

$$b_1 = B_1(m = 1, v = u), b_2 = B_2(m = 1, v = u), b_3 = B_3(m = 1, v = u).$$

and find:

$$b_1 = -32\cos^5 u - 32 \sin u \cos^4 u + 30\cos^3 u + 34 \sin u \cos^2 u - 7 \cos u - 9 \sin u,$$
$$b_2 = 32\cos^5 u + 32 \sin u \cos^4 u - 30\cos^3 u - 34 \sin u \cos^2 u + 7 \cos u + 9 \sin u,$$
$$b_3 = 0.$$

Defining the function

$$R = \sqrt{b_1^2 + b_2^2}$$

we get that

$$R = \sqrt{2}((-1 + 2\cos^2 u)^2 (512 \sin u \cos^5 u + 320\cos^4 u - 512\cos^3 u \sin u - 320\cos^2 u + 126 \sin u \cos u + 81))^{1/2}$$

and solving the equation

$$\frac{dR}{du} = 0$$

we find, for example, as an solution

$$u_2 = \arctan\left(\frac{5}{2} - \frac{\sqrt{2830 + 70\sqrt{2345}}}{32} + \frac{\sqrt{2345}}{14} - \frac{\sqrt{2830 + 70\sqrt{2345}}\sqrt{2345}}{1120}\right).$$

Now we calculate the maximum for

$$R:$$

$$r = R(u = u_2)$$

and find

$$r = \frac{8(572651\sqrt{2830+70\sqrt{5}\sqrt{469}}\sqrt{5}-45161640\sqrt{5}+59125\sqrt{469}\sqrt{2830+70\sqrt{5}\sqrt{469}}-4663080\sqrt{469})}{(25\sqrt{80}-\sqrt{2830+70\sqrt{5}\sqrt{469}}(51\sqrt{2830+70\sqrt{5}\sqrt{469}}\sqrt{5}\sqrt{469}-4024\sqrt{5}\sqrt{469}+2473\sqrt{2830+70\sqrt{5}\sqrt{469}}-194984)}$$

which approximation is

$$r \approx 13.15378318.$$

5. Sufficient conditions the Benz surface to be segment or plane piece

We prove here two theorems valid for the Benz surfaces in the case $m = 0$.

Theorem 5.1. *If one of the coordinates of the given surface is function only of parameter v, the Benz surface in the case $m = 0$ is a segment.*

Proof. We calculate for $m = 0$:

```
eval(B1,[X3(u,v)=X3(v)])=0;
eval(B2,[X3(u,v)=X3(v)])=0;
eval(B3,[X3(u,v)=X3(v)])=/0: ,

eval(B1,[X2(u,v)=X2(v)])=0;
eval(B2,[X2(u,v)=X2(v)]))=/0:
eval(B3,[X2(u,v)=X2(v)])=0; ,


eval(B1,[X1(u,v)=X1(v)])=/0:
eval(B2,[X1(u,v)=X1(v)])=0;
eval(B3,[X1(u,v)=X1(v)])=0; .
```

Theorem 5.2. *If two of the coordinates of the given surface are functions of the following art:*

$$f(u)g(v) + i(v), f(u)h(v) + j(v)$$

where f, g, h, i, j are arbitrary functions, the Benz surface in the case $m = 0$ is a plane piece.

Proof. We calculate for $m = 0$:

```
eval(B1,[X2(u,v)=f(u)*g(v)+j2(v),X3(u,v)=f(u)*h(v)+j3(v)])=0;
eval(B2,[X2(u,v)=f(u)*g(v)+j2(v),X3(u,v)=f(u)*h(v)+j3(v)])=/0:
eval(B3,[X2(u,v)=f(u)*g(v)+j2(v),X3(u,v)=f(u)*h(v)+j3(v)])=/0:,

eval(B1,[X1(u,v)=f(u)*g(v)+j1(v),X3(u,v)=f(u)*h(v)+j3(v)])=/0:
eval(B2,[X1(u,v)=f(u)*g(v)+j1(v),X3(u,v)=f(u)*h(v)+j3(v)])=0;
eval(B3,[X1(u,v)=f(u)*g(v)+j1(v),X3(u,v)=f(u)*h(v)+j3(v)])=/0:;

eval(B1,[X1(u,v)=f(u)*g(v)+j1(v),X2(u,v)=f(u)*h(v)+j2(v)])=/0 :
eval(B2,[X1(u,v)=f(u)*g(v)+j1(v),X2(u,v)=f(u)*h(v)+j2(v)])=/0  :
eval(B3,[X1(u,v)=f(u)*g(v)+j1(v),X2(u,v)=f(u)*h(v)+j2(v)])=0; .
```

Remarks: Some parts of this paper were plenary talks on the following international meetings:
1. VIII. Geometry Symposium in Antalia, Turkey, 29.04-02.05. 2010;
2. Fest Colloquium zum 80. Geburtstag von Prof. Dr. (DHC) Walter Benz am 13.05.2011 in Hamburg.

References

[1] W. Benz. Eine gemeinsame Kennzeichnung der Krüummungsachse bei Regelächen und Kurven. Beiträge zur Algebra und Geometrie, 41:1-6, 2000.

Sofia University "St. Kl. Ohridski", Blvd. J. Bourchier 5, 1164 Sofia, Bulgaria
E-mail address: stanilov@fmi.uni-sofia.bg