International Electronic Journal of Geometry
Volume 5 No. 2 pp. 10–18 (2012) ©IEJG

WEAKENED MANNHEIM CURVES IN GALILEAN 3-SPACE

MURAT KEMAL KARACAN AND YILMAZ TUNÇER

(Communicated by H. Hilmi HACISALIHOĞLU)

Abstract. In this study, Frenet-Mannheim curves and Weakened Mannheim curves are investigated in Galilean 3-space. Some characterizations for this curves are obtained.

1. Introduction

In the study of the fundamental theory and the characterizations of space curves, the corresponding relations between the curves are very interesting and important problems. The well-known Bertrand curve is characterized as a kind of such corresponding relation between the two curves. For the Bertrand curve α, it shares the normal lines with another curve β, called Bertrand mate or Bertrand partner curve of α [2].

In 1967, H.F.Lai investigated the properties of two types of similar curves (the Frenet-Bertrand curves and the Weakened Bertrand curves) under weakened conditions.

In recent works, Liu and Wang (2007, 2008) were curious about the Mannheim curves in both Euclidean and Minkowski 3-space and they obtained the necessary and sufficient conditions between the curvature and the torsion for a curve to be the Mannheim partner curves. Meanwhile, the detailed discussion concerned with the Mannheim curves can be found in literature (Wang and Liu, 2007; Liu and Wang, 2008; Orbay and Kasap, 2009) and references therein [3]. M.K.Karacan and Y.Tuncer investigated the properties of two types of similar curves (the Frenet-Mannheim curves and the Weakened Mannheim curves) under weakened conditions [5]. Also H.B.Öztekin investigated Weakened Bertrand curves under weakened conditions [4].

In this paper, our main purpose is to carry out some results which were given in [1] to Frenet-Mannheim curves and Weakened Mannheim curves in Galilean 3-space and we assume that, the angle between tangent vectors T_β and T_α is constant such that $\langle T_\alpha, T_\beta \rangle = \cos\theta \neq 0$.

2000 *Mathematics Subject Classification.* 53B30,53A35.
Key words and phrases. Mannheim Cuves, Frenet-Mannheim Curves,Weakened-Mannheim Curves, Galilean 3-Space.

2. Preliminaries

The Galilean space is a three dimensional complex projective space, P_3, in which the absolute figure $\{w, f, I_1, I_2\}$ consists of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and two complex conjugate points, $I_1, I_2 \in f$ (the absolute points).

We shall take, as a real model of the space G_3, a real projective space P_3, with the absolute $\{w, f\}$ consisting of a real plane $w \subset G_3$ and a real line $f \subset w$, on which an elliptic involution ϵ has been defined. Let ϵ be in homogeneous coordinates

$$\begin{aligned} w...x_0 &= 0, \qquad f...x_0 = x_1 = 0 \\ \epsilon &: (0:0:x_2:x_3) \to (0:0:x_3:-x_2). \end{aligned}$$

In the nonhomogeneous coordinates, the similarity group H_8 has the form

$$\begin{aligned} \overline{x} &= a_{11} + a_{12}x \\ \overline{y} &= a_{21} + a_{22}x + a_{23}\cos\theta + a_{23}\sin\theta \\ \overline{z} &= a_{31} + a_{32}x - a_{23}\sin\theta + a_{23}\cos\theta \end{aligned} \tag{2.1}$$

where a_{ij} and θ are real numbers. For $a_{11} = a_{23} = 1$, we have have the subgroup B_6 , the group of Galilean motions:

$$\begin{aligned} \overline{x} &= a_{11} + a_{12}x \\ \overline{y} &= b + cx + y\cos\theta + z\sin\theta \\ \overline{z} &= d + ex - y\sin\theta + z\cos\theta. \end{aligned}$$

In G_3, there are four classes of lines:

a) (proper) nonisotropic lines - they do not meet the absolute line f.

b) (proper) isotropic lines - lines that do not belong to the plane w but meet the absolute line f.

c) unproper nonisotropic lines - all lines of w but f.

d) the absolute line f.

Planes x =constant are Euclidean and so is the plane w. Other planes are isotropic. In what follows, the coefficients a_{12} and a_{23} a will play a special role. In particular, for $a_{12} = a_{23} = 1$, (2.1) defines the group $B_6 \subset H_8$ of isometries of the Galilean space G_3.

The scalar product in Galilean space G_3 is defined by

$$\langle X, Y\rangle_{G_3} = \begin{cases} x_1y_1 & , \; if \quad x_1 \neq 0 \ \vee y_1 \neq 0 \\ x_2y_2 + x_3y_3 \, , & if \quad x_1 = 0 \ \wedge y_1 = 0 \end{cases}$$

where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ .The Galilean cross product is defined for $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ by

$$a \wedge_{G_3} b = \begin{vmatrix} 0 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

A curve $\alpha : I \subseteq R \to G_3$ of the class C^r $(r \geq 3)$ in the Galilean space G_3 is given defined by

$$\alpha(x) = (s, y(s), z(s)), \tag{2.2}$$

where s is a Galilean invariant and the arc length on α. The curvature $\kappa(s)$ and the torsion $\tau(s)$ are defined by

$$\kappa(s) = \sqrt{(y''(s))^2 + (z''(s))^2}\ , \quad \tau(x) = \frac{\det\left(\alpha'(s), \alpha''(s), \alpha'''(s)\right)}{\kappa^2(s)}\ . \tag{2.3}$$

The orthonormal frame in the sense of Galilean space G_3 is defined by

$$\begin{aligned} T &= \alpha'(s) = (1, y'(s), z'(s)) \\ N &= \frac{1}{\kappa(s)}\alpha''(s) = \frac{1}{\kappa(s)}\left(0, y''(s), z''(s)\right) \\ B &= \frac{1}{\kappa(s)}\left(0, -z''(s), y''(s)\right) \end{aligned} \tag{2.4}$$

The vectors T, N and B in (2.4) are called the vectors of the tangent, principal normal and the binormal line of α, respectively. They satisfy the following Frenet equations

$$\begin{aligned} T' &= \kappa N \\ N' &= \tau B \\ B' &= -\tau N. \end{aligned} \tag{2.5}$$

[4].

Definition 2.1. Let G_3 be the 3-dimensional Galilean space with the standard inner product $\langle , \rangle_{G_3}$. If there exists a corresponding relationship between the space curves α and β such that, at the corresponding points of the curves, the principal normal lines of β coincides with the binormal lines of α, then β is called a Mannheim curve, and α a Mannheim partner curve of β. The pair $\{\alpha, \beta\}$ is said to be a Mannheim pair [2].

Definition 2.2. A Mannheim curve $\beta(s^\star)$,$s^\star \in I$ is a C^∞ regular curve with non-zero curvature for which there exists another (different) C^∞ regular curve $\alpha(s)$ where $\alpha(s)$ is of class C^∞ and $\alpha'(s) \neq 0$ (s being the arc length of $\alpha(s)$ only), also with non-zero curvature, in bijection with it in such a manner that the principal normal to $\beta(s^\star)$ and the binormal to $\alpha(s)$ at each pair of corresponding points coincide with the line joining the corresponding points. The curve $\alpha(s)$ is called a Mannheim conjugate of $\beta(s^\star)$.

Definition 2.3. A Frenet-Mannheim curve $\beta(s^\star)$ (briefly called a FM curve) is a C^∞ Frenet curve for which there exists another C^∞ Frenet curve $\alpha(s)$, where $\alpha(s)$ is of class C^∞ and $\alpha'(s) \neq 0$, in bijection with it so that, by suitable choice of the Frenet frames the principal normal vector $N_\beta(s^\star)$ and binormal vector $B_\alpha(s)$ at corresponding points on $\beta(s^\star), \alpha(s)$, both lie on the line joining the corresponding points. The curve $\alpha(s)$ is called a FM conjugate of $\beta(s^\star)$.

Definition 2.4. A weakened Mannheim curve $\beta(s^\star)$, $s^\star \in I^\star$ (briefly called a WM curve) is a C^∞ regular curve for which there exists another C^∞ regular curve $\alpha(s)$,$s \in I$, where s is the arclength of $\alpha(s)$, and a homeomorphism $\sigma : I \to I^\star$such that (**i**) here exist two (disjoint) closed subsets Z, N of I with void interiors such that $\sigma \in C^\infty$ on $L\backslash N$, $\left(\frac{ds^\star}{ds}\right) = 0$ on Z, $\sigma^{-1} \in C^\infty$ on $\sigma\left(L\backslash Z\right)$ and $\left(\frac{ds}{ds^\star}\right) = 0$ on $\sigma(N)$.(**ii**) The line joining corresponding points s, $s^\star$ of $\alpha(s)$ and $\beta(s^\star)$ is orthogonal to $\alpha(s)$ and $\beta(s^\star)$ at the points $s, s^\star$ respectively, and is along the principal normal to $\beta(s^\star)$ or $\alpha(s)$ at the points $s, s^\star$ whenever it is well defined. The curve $\alpha(s)$ is called a WM conjugate of $\beta(s^\star)$.

Thus for a WM curve we not only drop the requirement of $\alpha(s)$ being a Frenet curve, but also allow $\left(\frac{ds^\star}{ds}\right)$ to be zero on a subset with void interior $\left(\frac{ds^\star}{ds}\right) = 0$ on an interval would destroy the injectivity of the mapping σ. Since $\left(\frac{ds^\star}{ds}\right) = 0$ implies that $\left(\frac{ds}{ds^\star}\right)$ does not exist, the apparently artificial requirements in (i) are in fact quite natural.

It is clear that a Mannheim curve is necessarily a FM curve, and a FM curve is necessarily a WM curve. It will be proved in Theorem 3 that under certain conditions a WM curve is also a FM curve.

3. Frenet-Mannheim curves

In this section we study the structure and characterization of FM curves. We begin with a lemma, by using the classical method.

Lemma 3.1. *Let $\beta(s^\star)$, $s^\star \in I^\star$ be a FM curve and $\alpha(s)$ a FM conjugate of $\beta(s^\star)$. Let*

$$\beta(s^\star) = \alpha(s) + \lambda(s)B_\alpha(s) \tag{3.1}$$

Then the distance $|\lambda|$ between corresponding points of $\alpha(s)$, $\beta(s^\star)$ is constant, and there is a constant angle θ such that $\langle T_\alpha, T_\beta\rangle = \cos\theta$ and

$$(i)\sin\theta = -\lambda\tau_\alpha\cos\theta$$
$$(ii)\sin\theta = \lambda\tau_\beta\cos\theta$$
$$(iii)\cos^2\theta = 1$$
$$(iv)\sin^2\theta = \lambda^2\tau_\alpha\tau_\beta.$$

Proof. From (3.1) it follows that

$$\lambda(s) = \langle \beta(s^\star) - \alpha(s), B_\alpha(s)\rangle$$

is of class C^∞. Differentiation of (3.1) with respect to s gives

$$T_\beta \frac{ds^\star}{ds} = T_\alpha + \lambda' B_\alpha - \lambda\tau_\alpha N_\alpha. \tag{3.2}$$

By hypothesis we have $B_\alpha = \epsilon N_\beta$ with $\epsilon = \pm 1$, scalar multiplication of (3.2) by B_α gives

$$\lambda' = 0 \Rightarrow \lambda = \text{cons}\tan\text{t}.$$

Therefore we have

$$T_\beta \frac{ds^\star}{ds} = T_\alpha - \lambda\tau_\alpha N_\alpha. \tag{3.3}$$

But by the definition of FM curve we have $\frac{ds^\star}{ds} \neq 0$, so that T_β is C^∞ function of s. Hence

$$\langle T_\alpha, T_\beta\rangle'_{G_3} = \kappa_\alpha \langle N_\alpha, T_\beta\rangle_{G_3} + \frac{ds^\star}{ds}\kappa_\beta \langle T_\alpha, N_\beta\rangle_{G_3} = 0.$$

Consequently $\langle T_\alpha, T_\beta\rangle$ is constant, and there exists a constant angle θ such that

$$T_\beta = T_\alpha\cos\theta + N_\alpha\sin\theta. \tag{3.4}$$

Taking the vector product of (3.3) and (3.4), we obtain

$$\sin\theta = -\lambda\tau_\alpha\cos\theta$$

which is (i). Now we can write

$$\alpha(s) = \beta(s^\star) - \epsilon\lambda(s)N_\beta(s).$$

Therefore

$$T_\alpha = \frac{ds^\star}{ds}\left[T_\beta - \lambda\epsilon\tau_\beta B_\beta\right]. \tag{3.5}$$

On the other hand, equation (3.4) gives

$$B_\beta = T_\beta \wedge_{G_3} N_\beta = -\epsilon N_\alpha\cos\theta.$$

Using (3.4) again, we get

$$T_\alpha = T_\beta\cos\theta - \epsilon B_\beta\sin\theta. \tag{3.6}$$

Taking the vector product of (3.5) and (3.6), we obtain

$$\sin\theta = \lambda\tau_\beta\cos\theta,$$

which is (ii).On the other hand, comparison of (3.3) and (3.4) gives

$$\frac{ds^\star}{ds}\cos\theta = 1, \tag{3.7}$$

$$\frac{ds^\star}{ds}\sin\theta = -\lambda\tau_\alpha. \tag{3.8}$$

Similarly (3.5), (3.6) give

$$\frac{ds^\star}{ds} = \cos\theta, \tag{3.9}$$

$$\frac{ds^\star}{ds}(\lambda\tau_\beta) = \sin\theta. \tag{3.10}$$

The properties (iii) and (iv) then easily follow from (3.7) and (3.9), (3.6) and (3.8) and (3.10). □

Theorem 3.1. *Let $\beta(s^\star)$, $s^\star \in I^\star$ be a C^∞ Frenet curve with τ_β nowhere zero and satisfying the equation for constants λ with $\lambda \neq 0$. Then $\beta(s^\star)$ is a non-planar FM curve.*

$$\sin\theta = \lambda\tau_\beta\cos\theta \tag{3.11}$$

Proof. We can write the curve $\beta(s^\star)$ with position vector

$$\beta(s^\star) = \alpha(s) + \lambda(s)B_\alpha(s)$$

Then, denoting differentiation with respect to s by a dash, we have

$$\beta'(s^\star) = T_\alpha - \lambda\tau_\alpha N_\alpha.$$

Since $\tau_\alpha \neq 0$, it follows that $\beta(s^\star)$ is a C^∞ regular curve. Then we have

$$T_\beta\frac{ds^\star}{ds} = T_\alpha - \lambda\tau_\alpha N_\alpha.$$

Hence

$$\frac{ds^\star}{ds} = \sqrt{1-\lambda^2\tau_\alpha^2}.$$

Using (3.11), we get

$$T_\beta = T_\alpha \cos\theta + N_\alpha \sin\theta,$$

notice that from (3.11) we have $\sin\theta \neq 0$. Therefore

$$\frac{T_\beta}{ds^\star}\frac{ds^\star}{ds} = \kappa_\alpha N_\alpha \cos\theta + \tau_\alpha B_\alpha \sin\theta$$

Now we write $N_\beta = \epsilon B_\alpha$,

$$\kappa_\beta = \frac{\epsilon}{\frac{ds^\star}{ds}}\tau_\alpha \sin\theta.$$

These are C^∞ functions of s (and hence of $s^\star$), and

$$\frac{T_\beta}{ds^\star} = \kappa_\beta N_\beta.$$

Further we write $B_\beta = T_\beta \wedge_{G_3} B_\alpha$ and $\tau_\beta = -\left\langle \frac{B_\beta}{ds^\star}, N_\beta \right\rangle_{G_3}$. These are also C^∞ functions on $I^\star$. It is easy to verify that with the frame $\{T_\beta, N_\beta, B_\beta\}$ and the functions κ_β, τ_β, the curve $\beta(s^\star)$ becomes a C^∞ Frenet curve. But B_α and N_β lie on the line joining corresponding points of $\alpha(s)$ and $\beta(s^\star)$. Thus $\beta(s^\star)$ is a FM curve and $\alpha(s)$ a FM conjugate of $\beta(s^\star)$. □

Lemma 3.2. *For a C^∞ regular curve β to be a FM curve with a FM conjugate if and only if β should be either a line or a non-planar circular helix.*

Proof. $\Rightarrow$: Let β have a FM conjugate α which is a line. Then $\kappa_\alpha = 0$.Using Lemma 1, (iii) and (i), (ii), we have

$$\cos^2\theta = 1, \tag{3.12}$$

and then

$$\cos^2\theta \sin\theta = \lambda\tau_\beta \cos\theta, \tag{3.13}$$

$$\sin\theta = -\lambda\tau_\alpha \cos\theta. \tag{3.14}$$

From (3.14) it follows that $\cos\theta \neq 0$. Hence (3.13) is equivalent to

$$\lambda\tau_\beta = \cos\theta \sin\theta. \tag{3.15}$$

Case 1. $\sin\theta = 0$. Then $\cos\theta = \pm 1$, so that (3.12) implies that $\kappa_\beta = 0$, and β is a line. We also note that (3.15) implies that $\tau_\beta = 0$.

Case 2. $\sin\theta \neq 0$. Then $\cos\theta \neq \pm 1$, and (3.12), (3.15) imply that κ_β, τ_β are non-zero constants, and β is a non-planar circular helix.

$\Leftarrow$: If β is a non-planar circular helix

$$\beta = (as, b\cos s, b\sin s),$$

we may take

$$N_\beta = (0, -\cos s, \sin s).$$

Now put $\lambda = b$, then the curve β with

$$\beta = \alpha + \lambda B_\alpha$$

will be a line along the $x-$axis, and can be made into a FM conjugate of β if N_β is defined as equal to B_α. □

Theorem 3.2. *Let $\beta(s^\star)$ be a plane C^∞ Frenet curve with zero torsion and whose curvature is either bounded below or bounded above. Then β is a FM curve, and has FB conjugates which are plane curves.*

Proof. Let β be a curve satisfying the conditions of the hypothesis.Then there are non-zero numbers λ such that $\kappa_\beta < -\frac{1}{\lambda}$ on I or $\kappa_\beta > -\frac{1}{\lambda}$ on I. For any such λ, consider the plane curve α with position vector

$$\alpha = \beta - \lambda N_\beta.$$

Then

$$T_\alpha = T_\beta$$

It is then a straightforward matter to verify that α is a FM conjugate of β. □

4. Weakened Mannheim curves

Definition 4.1. Let D be a subset of a topological space X. A function on X into a set Y is said to be D-piecewise constant if it is constant on each component of D.

Lemma 4.1. *Let X be a proper interval on the real line and D an open subset of X. Then a necessary and sufficient condition for every continuous, D-piecewise constant real function on X to be constant is that $X \backslash D$ should have empty dense-in-itself kernel.*

We notice, however, that if D is dense in X, any C^1 and D-piecewise constant real function on X must be constant, even if D has non-empty dense-in-itself kernel.

Theorem 4.1. *A WM curve for which N and Z have empty dense-in-itself kernels is a FM curve.*

Proof. Let $\beta(s^\star), s^\star \in I^\star$ be a WM curve and $\alpha(s)$, $s \in I$ be WM conjugate of $\beta(s^\star)$. It follows from the definition that $\alpha(s)$ and $\beta(s^\star)$ each has a C^∞ family of tangent vectors $T_\beta(s^\star)$, $T_\alpha(s)$.Let

$$\beta(s) = \beta(\sigma(s)) = \alpha(s) + \lambda(s)B_\alpha(s), \tag{4.1}$$

where $B_\alpha(s)$ is some unit vector function and $\lambda(s) \geq 0$ is some scalar function. Let $D = I\backslash N$, $D^\star = I^\star\backslash\sigma(Z)$. Then $s^\star(s) \in C^\infty$on $D^\star$.

Step 1. To prove λ =constant.

Since $\lambda = \|\beta(s) - \alpha(s)\|$, it is continuous on I and is of class C^∞ on every interval of D on which it is nowhere zero. Let $P = \{s \in I : \lambda(s) \neq 0\}$ and X any component of P. Then P, and hence also X, is open in I. Let L be any component interval of $X \cap D$. Then on L, $\lambda(s)$ and $B_\alpha(s)$ are of class C^∞, and from (4.1) we have

$$\beta'(s) = \alpha'(s) + \lambda'(s)B_\alpha(s) + \lambda(s)B'_\alpha(s).$$

Now by definition of a WM curve we have $\langle\alpha'(s), B_\alpha(s)\rangle_{G_3} = 0 = \langle\beta'(s^\star), B_\alpha(s)\rangle_{G_3}$. Hence, using the identity $\langle B'_\alpha(s), B_\alpha(s)\rangle_{G_3} = 0$, we have

$$0 = \lambda'(s)\left\langle B_\alpha(s), B_\alpha(s)\right\rangle_{G_3}.$$

Therefore λ =constant on L.

Hence λ is constant on each interval of the set $X \cap D$. But by hypothesis $X\backslash D$ has empty dense-in-itself kernel. It follows from Lemma 2 that λ is constant (and non-zero) on X. Since λ is continuous on I, X must be closed in I. But X is also open in I. Therefore by connectedness we must have $X = I$, that is, λ is constant on I.

Step 2. To prove the existence of two frames

$$\{T_\alpha(s), N_\alpha(s), B_\alpha(s)\}, \{T_\beta(s^\star), N_\beta(s^\star), B_\beta(s^\star)\}$$

which are Frenet frames for $\alpha(s)$, $\beta(s^\star)$ on D, $D^\star$ respectively.

Since λ is a non-zero constant, it follows from (4.1) that $B_\alpha(s)$ is continuous on I and C^∞ on D, and is always orthogonal to $T_\alpha(s)$. Now we write $B_\alpha(s) = T_\alpha(s) \wedge_{G_3} N_\alpha(s)$. Then $\{T_\alpha(s), N_\alpha(s), B_\alpha(s)\}$ forms a right-handed orthonormal frame for $\alpha(s)$ which is continuous on I and C^∞ on D.

Now from the definition of WM curve we see that there exists a scalar function $\kappa_\beta(s^\star)$ such that $T'_\beta(s^\star) = \kappa_\beta(s^\star) N_\beta(s^\star)$ on $I^\star$. Hence $\kappa_\beta(s^\star) = \left\langle T'_\beta(s^\star), N_\beta(s^\star)\right\rangle_{G_3}$ is continuous on $I^\star$ and C^∞ on $D^\star$. Thus the first Frenet formula holds on $D^\star$. It is then straightforward to show that there exists a C^∞ function $\tau_\alpha(s)$ on D such that the Frenet formulas hold. Thus $\{T_\alpha(s), N_\alpha(s), B_\alpha(s)\}$ is a Frenet frame for $\alpha(s)$ on D.

Similarly there exists a right-handed orthonormal frame $\{T_\beta(s^\star), N_\beta(s^\star), B_\beta(s^\star)\}$ for $\beta(s^\star)$ which is continuous on $I^\star$ and is a Frenet frame for $\beta(s^\star)$ on $D^\star$. Moreover, we can choose

$$B_\alpha(s) = N_\beta(\sigma(s))$$

Step 3. To prove that $N = \varnothing$, $Z = \varnothing$, we first notice that on D we have

$$\langle T_\beta, T_\alpha\rangle'_{G_3} = \left\langle \kappa_\beta N_\beta \frac{ds^\star}{ds}, T_\alpha\right\rangle_{G_3} + \langle T_\beta, \kappa_\alpha N_\alpha\rangle_{G_3} = 0,$$

so that $\langle T_\beta, T_\alpha\rangle$ is constant on each component of D and hence on I by Lemma 2. Consequently there exists a angle θ such that

$$T_\beta = T_\alpha \cos\theta + N_\alpha \sin\theta.$$

Further,

$$B_\alpha(s) = N_\beta(\sigma(s))$$

and so

$$B_\beta(s^\star) = -T_\alpha \sin\theta + N_\alpha \cos\theta.$$

Thus $\{T_\beta(s^\star), N_\beta(s^\star), B_\alpha(s)\}$ are also of class C^∞ on D. On the other hand $\{T_\beta(s^\star), N_\beta(s^\star), B_\beta(s^\star)\}$ are of class C^∞ with respect to $s^\star$ on $D^\star$. Writing (4.1) in the form

$$\alpha = \beta - \lambda N_\beta.$$

and differentiating with respect to s on $D \cap \sigma^{-1}(D^\star)$, we have

$$T_\alpha = \frac{ds^\star}{ds}\left[T_\beta - \lambda\tau_\beta B_\beta\right].$$

But we have

$$T_\alpha = T_\beta \cos\theta - B_\beta \sin\theta.$$

Hence we get

$$\frac{ds^\star}{ds} = \cos\theta \text{ and } \lambda\tau_\beta = \sin\theta. \tag{4.2}$$

Since $\kappa_\beta(s^\star) = \left\langle T'_\beta, N_\beta\right\rangle_{G_3}$ is defined and continuous on $I^\star$ and $\sigma^{-1}(D^\star)$ is dense, it follows by continuity that (4.2) holds throughout D.

Case 1. $\cos\theta \neq 0$. Then (4.2) implies that $\frac{ds^\star}{ds} \neq 0$ on D. Hence $N = \varnothing$. Similarly $Z = \varnothing$.

Case 2. $\cos\theta = 0$. Then we have

$$T_\beta = \pm N_\alpha. \tag{4.3}$$

Differentiation of (4.1) with respect to s in D gives

$$T_\beta \frac{ds^\star}{ds} = T_\alpha - \lambda\tau_\alpha N_\alpha.$$

Hence using (4.3) we have

$$\frac{ds^\star}{ds} = \mp\lambda\tau_\alpha.$$

Therefore

$$\tau_\alpha = \mp\frac{1}{\lambda}\frac{ds^\star}{ds},$$

and so also on I, by Lemma 2. It follows that τ_α is nowhere zero on I. Consequently $\beta(s^\star) = \alpha(s) + \lambda(s)B_\alpha(s)$ is of class C^∞ on $I^\star$. Hence $N = \varnothing$. Similarly $Z = \varnothing$. □

References

[1] Lai,H.F. , Weakened Bertrand curves, Tohoku Math. Journ., Vol: 19, No. 2, pp:141-155, 1967.

[2] Liu, H., Wang, F., Mannheim partner curves in 3-space, Journal of Geometry, Vol. 88, No. 1-2, pp:120-126, 2008.

[3] Orbay, K., Kasap, E., On Mannheim partner curves in E^3,International Journal of Physical Sciences Vol. 4 (5), pp: 261-264, May, 2009.

[4] Oztekin, H.B., Weakened Bertrand curves in The Galilean Space G_3, J. Adv. Math. Studies,Vol. 2, No. 2, pp:69-76, 2009.

[5] Karacan, M.K., Tuncer, Y., Weakened Mannheim curves, Int. Journal of the Physical Sciences,Vol. 6(20), pp. 4700-4705, 23 September, 2011.

[6] Tuncer,Y., Karacan, M.K., Weakened Mannheim curves in Minkowski 3-Space, (submitted).

Department of Mathematics, University of Uşak-TURKEY
E-mail address: murat.karacan@usak.edu.tr

Department of Mathematics, University of Uşak-TURKEY
E-mail address: yilmaz.tuncer@usak.edu.tr