

SOME CHARACTERIZATIONS OF ALMOST HERMITIAN MANIFOLDS

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ABSTRACT. We give a condition for an almost constant-type manifold to be a constant-type manifold, and holomorphic and R -invariant submanifolds of almost Hermitian manifolds are studied. Generalizations of some results in [5] are given.

1. INTRODUCTION

The theory of submanifolds of Riemannian or almost Hermitian manifolds is an important topic in differential geometry. In an almost Hermitian manifold, its complex structure J transforms a vector to another one which perpendicular to it. Perhaps this has been the natural motivation to study submanifolds of an almost Hermitian manifold, according to the behavior of its tangent bundle under the action of the almost complex structure J of the ambient manifold.

There are many classes of submanifolds in the literature. One of the classes is *holomorphic (invariant)* submanifolds. In this case the tangent space of the submanifold remains invariant under the almost complex structure J . The other one is *R -invariant submanifolds*, where R is the Riemannian curvature tensor of the ambient manifold. In which case, R_{XY} defines, at each point of the submanifold, a linear transformation on the tangent space of the submanifold at the point and the tangent space remains invariant under R_{XY} , where X and Y are elements of the tangent space. The theory of holomorphic submanifolds has been a very active area whereas the theory of R -invariant submanifolds has not been so far.

This paper is organized as follows. In Section 2, the fundamental definitions and notions of almost Hermitian manifolds are given. In Section 3, we recall some basic formulas for submanifolds of a Riemannian manifold and the definition of holomorphic submanifold of an almost Hermitian manifold. In Section 4, we study almost constant-type manifolds and give a characterization theorem. In

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section 5, after recalling the definition of R -invariant submanifold we study holomorphic, totally umbilical and R -invariant submanifolds of almost Hermitian manifolds (AH -manifolds). In particular, we shall improve Theorem 4.2 of [5] concerning Kaehlerian manifolds. In the last section, the strongly R -invariant submanifolds are considered and analogues of Theorem 3.4 and Theorem 3.5 of [5] concerning Riemannian manifolds are given. A different characterization of almost constant-type manifolds is placed in this section.

2. PRELIMINARIES

Let M be an almost Hermitian manifold, that is, its tangent bundle has an almost complex structure J and a Riemannian metric g such that $g(JX, JY) = g(X, Y)$ for all $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of C^∞ vector fields on M . Let ∇ be the Riemannian connection on M , the Riemannian curvature tensor R associated with ∇ defined by $R(X, Y) = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y]$. We denote $g(R(X, Y)Z, W)$ by $R(X, Y, Z, W)$. Curvature identities are keys to understanding the geometry of almost Hermitian manifolds. The following curvature identities are used in many studies, for example, see [1].

- (1) $R(X, Y, Z, W) = R(X, Y, JZ, JW)$,
- (2) $R(X, Y, Z, W) = R(JX, JY, Z, W) + R(JX, Y, JZ, W)$
 $+R(JX, Y, Z, JW)$,
- (3) $R(X, Y, Z, W) = R(JX, JY, JZ, JW)$.

Let AH_i denote the subclass of the class AH of almost Hermitian manifolds satisfying the curvature identity (i), $i=1,2,3$. The following inclusion relations is well known in the literature.

$$AH_1 \subset AH_2 \subset AH_3 \subset AH$$

An almost Hermitian manifold M is called *Kaehlerian* if $\nabla_X J = 0$ for all $X \in \chi(M)$. It is well known that a Kaehlerian manifold is an AH_1 -manifold ([10]). Some authors call AH_1 -manifold as a *para-Kaehlerian* and call AH_3 -manifold as an *RK-manifold* ([9]).

By a *plane section* we mean a two-dimensional linear subspace of a tangent space $T_p M$. A plane section σ is said to be *holomorphic* (resp. *anti-holomorphic* or *totally real*) if $J\sigma = \sigma$ (resp. $J\sigma \perp \sigma$). The sectional curvature K of M determined by orthonormal vector fields X and Y is given by $K(X, Y) = R(X, Y, X, Y)$. The sectional curvature of M restricted to a holomorphic (resp. an anti-holomorphic) plane σ is called *holomorphic* (resp. *anti-holomorphic*) *sectional curvature*. If the holomorphic (resp. anti-holomorphic) sectional curvature at each point $p \in M$, does not depend on σ , then M is said to be *pointwise constant holomorphic* (resp. *pointwise constant anti-holomorphic*) *sectional curvature*. A Riemannian manifold of constant curvature is called a *space form* or a *real space form*. Sometimes a space form defined as a complete connected Riemannian manifold of constant curvature is said to be *elliptic*, *hyperbolic* or *flat* (or *locally Euclidean*) according as the sectional curvature is positive, negative or zero. A Kaehlerian manifold of constant holomorphic sectional curvature is called a *complex space form* ([3, 10]).

3. SUBMANIFOLDS OF A RIEMANNIAN MANIFOLD

Let N be a submanifold of a Riemannian manifold M with a Riemannian metric g . Let $\chi(N)$ and $\chi(M)$ the Lie algebras of vector fields on N and M respectively, and $\bar{\chi}(N)$ denote the Lie algebra of restrictions to N of vector fields in $\chi(M)$. We then may write $\bar{\chi}(N) = \chi(N) \oplus \chi(N)^\perp$, where $\chi(N)^\perp$ consists of all vector fields perpendicular to N . The Gauss and Weingarten formulas are given respectively by $\nabla_X Y = \hat{\nabla}_X Y + B(X, Y)$ and $\nabla_X \xi = -A_\xi X + \nabla_X^\perp \xi$ for all $X, Y \in \chi(N)$ and $\xi \in \chi(N)^\perp$, where ∇ , $\hat{\nabla}$ and ∇^\perp are respectively the Riemannian, induced Riemannian and induced normal connection in M , N and the normal bundle $\chi(N)^\perp$ of N , and B is the second fundamental form related to shape operator A corresponding to the normal vector field ξ by $g(B(X, Y), \xi) = g(A_\xi X, Y)$. We say that N is *totally umbilical* submanifold in M if for all $X, Y \in \chi(N)$, we have

$$(3.1) \quad B(X, Y) = g(X, Y)H \quad ,$$

where $H \in \chi(N)^\perp$ is the mean curvature vector field of N in M . A vector field $\xi \in \chi(N)^\perp$ is said to be *parallel* if $\nabla_X^\perp \xi = 0$ for each $X \in \chi(N)$. The Codazzi equation is given by

$$(3.2) \quad (R(X, Y)Z)^\perp = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z)$$

for all $X, Y, Z \in \chi(N)$, where $^\perp$ denotes the normal component and the covariant derivative of B , denoted by $\nabla_X B$, is defined by

$$(3.3) \quad (\nabla_X B)(Y, Z) = \nabla_X^\perp (B(Y, Z)) - B(\hat{\nabla}_X Y, Z) - B(Y, \hat{\nabla}_X Z)$$

for all $X, Y, Z \in \chi(N)$ [3, 5, 10].

Now, let (M, J, g) (or briefly M) be an almost Hermitian manifold and N be a Riemannian submanifold of M . If $J(T_p N) = T_p N$ at each point $p \in N$, $T_p N$ being the tangent space over N in M , then N is called a *holomorphic* submanifold of M . In this case we see that $JT_p N^\perp = T_p N^\perp$. The fundamental properties and basic formulas of holomorphic submanifolds can be found in [10].

4. ALMOST CONSTANT-TYPE MANIFOLDS

The following notion of constant type, is first defined by A. Gray for nearly Kaehlerian manifolds ([2]), and then by L. Vanhecke for almost Hermitian manifolds ([9]).

Let M be an almost Hermitian manifold. Then M is said to be of *constant type* at $p \in M$ provided that for all $X \in T_p M$, we have $\lambda(X, Y) = \lambda(X, Z)$ whenever the planes $\text{span}\{X, Y\}$ and $\text{span}\{X, Z\}$ are anti-holomorphic and $g(Y, Y) = g(Z, Z)$, where the function λ is defined by $\lambda(X, Y) = R(X, Y, X, Y) - R(X, Y, JX, JY)$. If this holds for all $p \in M$, then we say that M has (*pointwise*) *constant type*. Finally, if for $X, Y \in \chi(M)$ with $g(X, Y) = g(JX, Y) = 0$, the value $\lambda(X, Y)$ is constant whenever $g(X, X) = g(Y, Y) = 1$, then we say that M has *global constant type*.

In [7], for an *AH*-manifold with dimension $2m \geq 4$, G.B. Rizza gave an equivalent definition to the one above. For some geometrical notions and for the notations, see [6, 7].

The manifold M is said to be of *constant type* α at p if and only if we have

$$(4.1) \quad K_\sigma - \chi_{\sigma J\sigma} = \alpha \quad ,$$

where σ is any anti-holomorphic plane of T_pM and $p \in M$.

In the same paper, G.B. Rizza also generalized the notion of constant type as follows.

We say that M has *constant type* α , *in a weak sense*, at p , if and only if we have

$$(4.2) \quad K_\sigma - 2\chi_{\sigma J\sigma} + K_{J\sigma} = 2\alpha \quad ,$$

where σ is any anti-holomorphic plane of T_pM and $p \in M$.

Remark 4.1. If σ is an anti-holomorphic plane of T_pM , then $J\sigma$ is also an anti-holomorphic plane of T_pM , so (4.1) implies (4.2). In particular, if the curvature tensor R is J -invariant, that is, if R satisfies the curvature identity (3), or the manifold M belongs to class AH_3 , then we have $K_\sigma = K_{J\sigma}$. Thus (4.2) reduces to (4.1). (cf. [7]).

An AH -manifold M is said to be an *almost constant-type manifold* if and only if M has constant type, in a weak sense at any point $p \in M$. In particular, M is a *constant-type manifold* if the condition (4.1) is satisfied at any point $p \in M$, see [7].

Now, we give a condition for an almost constant-type manifold to be a constant-type manifold.

Theorem 4.1. *Let M be an AH -manifold of almost constant-type α . Then M has constant type α if and only if M belongs to class AH_3 .*

Proof. Let M be an AH -manifold of constant type α , in a weak sense, at $p \in M$. From the equation (5) in the proof of Theorem 1([7]), we have

$$(4.3) \quad \begin{aligned} & R(X, Y, X, Y) - 2R(X, Y, JX, JY) + R(JX, JY, JX, JY) \\ & = 2\alpha\{g(X, X)g(Y, Y) - (g(X, Y))^2 - (g(JX, Y))^2\} \end{aligned}$$

for any vectors $X, Y \in T_pM$. Now, suppose that M has constant type α , at $p \in M$. By a similar method given in the proof of Theorem 1([7]), we obtain

$$(4.4) \quad \begin{aligned} & R(X, Y, X, Y) - R(X, Y, JX, JY) \\ & = \alpha\{g(X, X)g(Y, Y) - (g(X, Y))^2 - (g(JX, Y))^2\} \end{aligned}$$

for any vectors $X, Y \in T_pM$. (The equation (4.4) is also known in another form, see [4]). If we multiply the equation (4.4) by 2 and then subtract it from the equation (4.3), we get

$$(4.5) \quad R(X, Y, X, Y) = R(JX, JY, JX, JY) \quad .$$

Now, let $T : (T_p(M))^4 \rightarrow \mathbb{R}$ be a four-linear mapping given by

$$T(X, Y, Z, W) = R(X, Y, Z, W) - R(JX, JY, JZ, JW)$$

for all $X, Y, Z, W \in T_p(M)$. Using the well-known properties of R and (4.5), it is not difficult to see that T satisfies the conditions of Theorem 2([1]), so it follows that $T = 0$. This means that the curvature tensor R of M satisfies the curvature identity (3); that is, M belongs to class AH_3 , in which case, by virtue of Remark

1([7]), M also belongs to the class AH_2 . On the other hand, by Remark 4.1, the converse of Theorem 4.1. is also true. \square

5. R -INVARIANT SUBMANIFOLDS

Let (M, g) be a Riemannian manifold and N be a submanifold of M . Then $R(X, Y) = R_{XY}$ defines, at each point p of N , a linear transformation on the tangent space $T_p N$ at the point, where R is the Riemannian curvature tensor of M . If for any $X, Y \in T_p N$, the relation $R_{XY}(T_p N) \subset T_p N$ is satisfied, then N is called an R -invariant submanifold of M . For the equivalent definitions of R -invariant submanifolds, see [5].

Theorem 5.1. *If every holomorphic submanifold of a connected AH-manifold M of dimension $2m \geq 6$ is R -invariant, then*

a) *M is a real space form or a complex space form.*

b) *M is an AH_2 -manifold.*

c) *The curvature tensor R of M has the form*

$$(5.1) \quad R(X, Y, Z, W) = \alpha R_1(X, Y, Z, W) + \beta R_2(X, Y, Z, W)$$

where α, β are constants, $R_1(X, Y, Z, W) = g(X, W)g(Y, Z) - g(X, Z)g(Y, W)$ and $R_2(X, Y, Z, W) = g(JX, W)g(JY, Z) - g(JX, Z)g(JY, W) - 2g(JX, Y)g(JZ, W)$ for all $X, Y, Z, W \in T_p M$ and $p \in M$.

Moreover, the converse of **c)** is also true.

Proof. If for any point p in M of a connected AH-manifold M of dimension $2m \geq 6$ and for any holomorphic subspace σ of $T_p M$ there exists an R -invariant holomorphic submanifold N of M through p such that $T_p N$ contains σ , then we have

$$(5.2) \quad (R(X, JX)JX)^\perp = 0$$

and

$$(5.3) \quad (R(X, Y)Y)^\perp = 0$$

for all orthonormal vectors $X, Y \in \sigma$ with $g(X, JY) = 0$ and $p \in N$. For any vector ξ normal to N at p , from (5.2) and (5.3), we get

$$(5.4) \quad R(X, JX, JX, \xi) = 0$$

and

$$(5.5) \quad R(X, Y, Y, \xi) = 0 .$$

From (5.4) and Lemma 1([3]), we conclude that M has pointwise constant holomorphic sectional curvature μ and from (5.5) and Lemma 4([3]), we find that M has pointwise constant anti-holomorphic sectional curvature ν . In this case, the assertion **a)** follows from Theorem C([3]) and assertions **b)** and **c)** follow from Theorem 3([1]).

Now we prove the converse of **c)**. Let the curvature tensor R of M satisfy (5.1), then after some calculation in (5.1), we have

$$(5.6) \quad R(X, Y)Z = \alpha\{g(Y, Z)X - g(X, Z)Y\} + \beta\{g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ\}$$

for all $X, Y, Z, W \in T_p M$ and $p \in M$. In that case, if N is any holomorphic submanifold of the manifold M , then from (5.6), we easily see that $R(X, Y)Z \in T_p N$ for all $X, Y, Z \in T_p N$ and $p \in N$, that is, N is R -invariant. \square

Remark 5.1. The assertions **a)** and **c)** of Theorem 5.1. are equivalent whenever the ambient manifold M is of class AH_3 because of Theorem 12.7([8]).

Theorem 5.2. *Let N be any connected totally umbilical submanifold of an AH -manifold M . Then N is an R -invariant submanifold if and only if it has parallel mean curvature vector field.*

Proof. Let N be any connected totally umbilical submanifold of an AH -manifold M . With the help of (3.1) and (3.3), from the Codazzi equation (3.2), we have

$$(5.7) \quad (R(X, Y)Z)^\perp = g(Y, Z)\nabla_X^\perp H - g(X, Z)\nabla_Y^\perp H$$

for all $X, Y, Z \in \chi(N)$, where H is mean curvature vector field of N in M . Let N be R -invariant, then from (5.7), we have $(R(X, Y)Z)^\perp = 0$, in which case, if we choose $Y = Z$ with $g(X, Y) = 0$ in (5.7), we obtain $\nabla_X^\perp H = 0$, which says that H is parallel. On the other hand, if H is parallel, from (5.7), we easily see that $(R(X, Y)Z)^\perp = 0$, which means that N is an R -invariant submanifold. \square

We remark that Theorem 5.1 is a generalization Theorem 4.2([5]) concerning Kaehlerian manifolds and analogue of Theorem 2.2([5]) concerning Riemannian manifolds.

6. STRONGLY R -INVARIANT SUBMANIFOLDS

Let (M, g) be a Riemannian manifold and N be a submanifold of M . If for any $X, Y \in T_p M$, the relation $R_{XY}(T_p N) \subset T_p N$ is satisfied, then N is called a *strongly R -invariant* submanifold of M . It is easy to see that a strongly R -invariant submanifold is an R -invariant submanifold. For the equivalent definitions of strongly R -invariant submanifolds, see [5].

Theorem 6.1. *There exists no holomorphic strongly R -invariant submanifold of an AH -manifold M with pointwise non-zero constant holomorphic sectional curvature μ .*

Proof. In [6], G.B. Rizza, for an AH -manifold M with pointwise holomorphic constant sectional curvature μ , proved the following fundamental identity:

$$(6.1) \quad \begin{aligned} & 3\{R(X, Y, Z, W) + R(X, Y, JZ, JW) + R(JX, JY, Z, W) \\ & + R(JX, JY, JZ, JW)\} - 2\{R(X, W, JY, JZ) + R(JX, JW, Y, Z) \\ & - R(X, Z, JY, JW) - R(JX, JZ, Y, W)\} - \{R(X, JY, JZ, W) \\ & + R(JX, Y, Z, JW) + R(X, JY, Z, JW) + R(JX, Y, JZ, W)\} \\ & = 4\mu\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z) + g(X, JZ)g(Y, JW) \\ & - g(X, JW)g(Y, JZ) + 2g(X, JY)g(Z, JW)\} \end{aligned}$$

for all $X, Y, Z, W \in T_p M$ and $p \in M$. After some calculations in (6.1), we conclude that

$$(6.2) \quad \begin{aligned} & 3\{R(X, Y)Z - J(R(X, Y)JZ) + R(JX, JY)Z \\ & - J(R(JX, JY)JZ)\} - 2\{R(JY, JZ)X - J(R(Y, Z)JX) \\ & + J(R(X, Z)JY) - R(JX, JZ)Y\} - \{R(X, JY)JZ \\ & - J(R(JX, Y)Z) - J(R(X, JY)Z) + R(JX, Y)JZ\} \\ & = 4\mu\{g(X, Z)Y - g(Y, Z)X - g(X, JZ)JY \\ & + g(Y, JZ)JX - 2g(X, JY)JZ\} \end{aligned}$$

for all $X, Y, Z \in T_p M$ and $p \in M$.

Now, assume that N is a holomorphic strongly R -invariant submanifold of an AH -manifold M with pointwise non-zero constant holomorphic sectional curvature μ . Then from (6.2) we have

$$(6.3) \quad \begin{aligned} & 3R(X, \xi)X + 5R(JX, J\xi)X - R(X, J\xi)JX - R(JX, \xi)JX \\ & - 5J(R(X, \xi)JX) - 3J(R(JX, J\xi)X) + J(R(X, J\xi)X) + J(R(JX, \xi)X) \\ & = 4\mu g(X, X)\xi \end{aligned}$$

for all $X \in T_p N, p \in N$ and $\xi \in T_p N^\perp$. From Proposition 3.3([5]) and (6.3), we find $\mu = 0$. \square

Since a strongly R -invariant submanifold is necessarily R -invariant submanifold, from Theorem 5.1 and Theorem 6.1, we have the following result.

Corollary 6.1. *If every holomorphic submanifold of a connected AH -manifold M of dimension $2m \geq 6$, is a strongly R -invariant submanifold, then M is flat.*

We note that Theorem 6.1 and Corollary 6.1 may be considered as analogues of Theorem 3.4([5]) and Theorem 3.5([5]) concerning Riemannian manifolds respectively, and Theorem 6.1 is a generalizations of Theorem 4.3([5]) concerning Kaehlerian manifolds. We end this paper, giving a different characterization of almost constant-type AH -manifolds.

Theorem 6.2. *There exists no holomorphic strongly R -invariant submanifold of an AH -manifold M of non-zero constant type α , in a weak sense, at $p \in M$.*

Proof. Assume that N is a holomorphic strongly R -invariant submanifold of an AH -manifold M of non-zero constant type α , in a weak sense, at $p \in M$. Then from (4.3), for $X \in T_p N$ and $\xi \in T_p N^\perp$ with $g(X, X) = g(\xi, \xi) = 1$, we have

$$(6.4) \quad R(X, \xi, X, \xi) - 2R(X, \xi, JX, J\xi) + R(JX, J\xi, JX, J\xi) = 2\alpha.$$

From Proposition 1.4([5]) and (6.4), it follows that $\alpha = 0$. This is a contradiction. \square

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