International Electronic Journal of Geometry
Volume 5 No. 2 pp. 67–78 (2012) ©IEJG

CLASSIFICATION OF HOMOTHETIC FUNCTIONS WITH CONSTANT ELASTICITY OF SUBSTITUTION AND ITS GEOMETRIC APPLICATIONS

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(Communicated by Cihan ÖZGÜR)

Abstract. Almost all economic theories presuppose a production function, either on the firm level or the aggregate level. In this sense the production function is one of the key concepts of mainstream neoclassical theories. A homothetic function is a production function of the form $f(\mathbf{x}) = F(h(x_1, \ldots, x_n))$, where $h(x_1, \ldots, x_n)$ is a homogeneous function and F is a monotonically increasing function. The most common quantitative indices of production factor substitutability are forms of the elasticity of substitution.

In this paper we prove that a homothetic function $f = F \circ h$ satisfies the constant elasticity of substitution property if and only if the homogeneous function h is either a generalized Cobb-Douglas production function or a generalized ACMS production function. Some of its geometric applications will also be given in this paper.

1. Introduction.

In economics, a production function is a positive nonconstant function that specifies the output of a firm, an industry, or an entire economy for all combinations of inputs. Almost all economic theories presuppose a production function, either on the firm level or the aggregate level. In this sense, the production function is one of the key concepts of mainstream neoclassical theories. By assuming that the maximum output technologically possible from a given set of inputs is achieved, economists using a production function in analysis are abstracting from the engineering and managerial problems inherently associated with a particular production process.

Let $\mathbb{R}$ denote the set of real numbers. Let us put

$$\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\} \text{ and } \mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1, \ldots, x_n > 0\}.$$

2000 *Mathematics Subject Classification.* Primary: 90A11; Secondary 53C40; 91B38.
Key words and phrases. Homothetic production function, constant elasticity of substitution, generalized Cobb-Douglas production function, generalized ACMS production function, Gauss-Kronecker curvature.

In this paper, by a *production function* we mean a function $f : D \subset \mathbb{R}^n_+ \to \mathbb{R}_+$ from a domain D of $\mathbb{R}^n_+$ into $\mathbb{R}^+$ which has non-vanishing first derivatives. Throughout this paper we assume that production functions are twice differentiable.

There are two special classes of production functions that are often analyzed in microeconomics and macroeconomics; namely, homogeneous and homothetic production functions. A production function $f(x_1, \cdots, x_n)$ is said to be *homogeneous of degree* p or p*-homogeneous*, if

$$f(tx_1, \ldots, tx_n) = t^p f(x_1, \ldots, x_n) \tag{1.1}$$

holds for each $t \in \mathbb{R}$ for which (1.1) is defined. A homogeneous function of degree one is called *linearly homogeneous*.

If $p > 1$, the homogeneous function exhibits increasing returns to scale, and it exhibits decreasing returns to scale if $p < 1$. If it is homogeneous of degree one, it exhibits constant returns to scale. The presence of increasing returns means that a one percent increase in the usage levels of all inputs would result in a greater than one percent increase in output; the presence of decreasing returns means that it would result in a less than one percent increase in output. Constant returns to scale is the in-between case.

A *homothetic function* is a production function of the form:

$$Q(\mathbf{x}) = F(h(x_1, \ldots, x_n)), \tag{1.2}$$

where $h(x_1, \ldots, x_n)$ is a homogeneous function of any given degree and F is a monotonically increasing function.

In economics, an *isoquant* is a contour line drawn through the set of points at which the same quantity of output is produced while changing the quantities of two or more inputs. While an indifference curve mapping helps to solve the utility-maximizing problem of consumers, the isoquant mapping deals with the cost-minimization problem of producers. Isoquants are typically drawn on capital-labor graphs, showing the technological tradeoff between capital and labor in the production function, and the decreasing marginal returns of both inputs. A family of isoquants can be represented by an isoquant map, a graph combining a number of isoquants, each representing a different quantity of output. Isoquants are also called equal product curves.

Homothetic functions are functions whose marginal technical rate of substitution (the slope of the isoquant) is homogeneous of degree zero. Due to this, along rays coming from the origin, the slopes of the isoquants will be the same.

The most common quantitative indices of production factor substitutability are forms of the elasticity of substitution. The elasticity of substitution was originally introduced by J. R. Hicks [11] in case of two inputs for the purpose of analyzing changes in the income shares of labor and capital.

R. G. Allen and J. R. Hicks suggested in [1] a generalization of Hicks' original two inputs elasticity concept as follows:

Let f be a production function. Put

$$H_{ij}(\mathbf{x}) = \frac{\dfrac{1}{x_i f_{x_i}} + \dfrac{1}{x_j f_{x_j}}}{-\dfrac{f_{x_i x_i}}{f_{x_i}^2} + \dfrac{2 f_{x_i x_j}}{f_{x_i} f_{x_j}} - \dfrac{f_{x_j x_j}}{f_{x_j}^2}} \tag{1.3}$$

for $\mathbf{x} \in \mathbb{R}^n_+$, $1 \le i \ne j \le n$, where the subscripts of f denote partial derivatives, that is

$$f_{x_i} = \frac{\partial f}{\partial x_i}, \quad f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

All partial derivatives are taken at the point $\mathbf{x}$ and the denominator is assumed to be different from zero.

The function H_{ij} is known as the *Hicks elasticity of substitution* of the i-th production variable (input) with respect to the j-th production variable (input). A production function f is said to satisfy the *constant elasticity of substitution* (CES) property if there is a nonzero constant $\sigma \in \mathbb{R}$ such that

$$H_{ij}(\mathbf{x}) = \sigma \;\text{ for } \mathbf{x} \in \mathbb{R}^n_+ \text{ and } 1 \le i \ne j \le n. \tag{1.4}$$

Homogeneous production functions satisfying the constant elasticity of substitution property have been completely classified recently (see [6, 12, 13] for details).

In this paper we extend this classification result by classifying all homothetic functions satisfies the CES property. More precisely, we prove that a homothetic function $f = F \circ h$ satisfies the constant elasticity of substitution property if and only if the homogeneous function h is either a generalized Cobb-Douglas production function or a generalized ACMS production function. Some of its geometric applications will also be presented in this paper.

2. Cobb-Douglas and ACMS production functions

In 1928, C. W. Cobb and P. H. Douglas introduced in [10] a famous two-input production function

$$Y = bL^k C^{1-k}, \tag{2.1}$$

where b represents the total factor productivity, Y the total production, L the labor input and C the capital input. This function is nowadays called Cobb-Douglas production function.

The Cobb-Douglas production function is widely used in economics to represent the relationship of an output to inputs. Later work in the 1940s prompted them to allow for the exponents on C and L vary, which resulting in estimates that subsequently proved to be very close to improved measure of productivity developed at that time (cf. [7, 8]).

In its generalized form the Cobb-Douglas (CD) production function may be expressed as

$$Q = \gamma x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{2.2}$$

where γ is a positive constant and $\alpha_1, \ldots, \alpha_n$ are nonzero constants. Since then a continuous search has been made for multi-input production functions with relatively simple and manageable properties.

In 1961, K. J. Arrow, H. B. Chenery, B. S. Minhas and R. M. Solow [2] introduced another two-input production function given by

$$Q = F \cdot (aK^r + (1-a)L^r)^{\frac{1}{r}}, \tag{2.3}$$

where Q is the output, F the factor productivity, a the share parameter, K and L the primary production factors, $r = (s-1)/s$, and $s = 1/(1-r)$ is the elasticity of substitution.

The generalized form of ACMS production function is given by

$$Q = \gamma \left(\sum_{i=1}^{n} a_i^{\rho} x_i^{\rho} \right)^{\frac{p}{\rho}}, \tag{2.4}$$

where a_i, p, γ, ρ are nonzero constants.

It is easy to verify that the generalized ACMS production function satisfy the CES property with

$$H_{ij}(\mathbf{x}) = \frac{1}{\rho}$$

if $\rho \neq 1$. For $\rho = 1$ the denominator of H_{ij} is zero, hence it is not defined. For this reason, the generalized ACMS production function is also known as the *generalized CES production function.*

The same functional form arises as a utility function in consumer theory. For example, if there exist n types of consumption goods x_i, then aggregate consumption C could be defined using the CES aggregator as

$$C = \left(\sum_{i=1}^{n} a_i^{\frac{1}{s}} x_i^{\frac{s-1}{s}} \right)^{\frac{s}{s-1}}, \tag{2.5}$$

where the coefficients $a_1, \ldots, a_n$ are share parameters, and s is the elasticity of substitution.

The classification of all homogeneous production functions satisfying the constant elasticity of substitution property have been completely done recently. More precisely, we have the following (see [2, 6, 12] for details).

Theorem 2.1. *Let f be a homogeneous production function with non-vanishing first partial derivatives. If f satisfies the constant elasticity of substitution property, then it is either the generalized Cobb-Douglas production function or the generalized ACMS production function.*

Remark 2.1. For $n = 2$, Theorem 2.1 is due to L. Losonczi [12]. Losonczi's result generalizes and somewhat clarifies an analogous result of [2] (cf. [12, page 122]). It was pointed out in [12, page 124] that the approach given in [12] does not work for productions function of multiple inputs. The general form can be found in [6].

3. Classification of homothetic functions with CES property.

The next theorem completely classifies homothetic functions which satisfy the constant elasticity of substitution property.

Theorem 3.1. *Let*

$$f(\mathbf{x}) = F(h(x_1, \ldots, x_n)) \tag{3.1}$$

be a homothetic production function. Then f satisfies the constant elasticity of substitution property if and only if h is either a generalized Cobb-Douglas production function or a generalized ACMS production function.

Proof. Let

$$f(\mathbf{x}) = F(h(x_1, \ldots, x_n)) \tag{3.2}$$

be a homothetic production function. Then we have

$$
\begin{aligned}
f_{x_i} &= F'(u)h_{x_i}, \\
f_{x_ix_j} &= F''(u)h_{x_i}h_{x_j} + F'(u)h_{x_ix_j}, \quad i, j = 1, \dots, n,
\end{aligned} \tag{3.3}
$$

where $u = h(x_1, \dots, x_n)$.

Assume that the homothetic function (3.1) satisfies the constant elasticity of substitution property. Then we have

$$
H_{ij}(\mathbf{x}) = \sigma \text{ for } \mathbf{x} \in \mathbb{R}^n_+ \text{ and } 1 \le i \ne j \le n \tag{3.4}
$$

for some nonzero constant σ. Therefore, after substituting (3.3) into (1.3) and after applying (3.4), we derive that

$$
\begin{aligned}
F'(u)\big\{&\sigma x_i x_j (h^2_{x_i} h_{x_jx_j} + h^2_{x_j} h_{x_ix_i} - 2h_{x_i}h_{x_j}h_{x_ix_j}) \\
&+ (x_i h_{x_i} + x_j h_{x_j}) h_{x_i} h_{x_j}\big\} = 0
\end{aligned} \tag{3.5}
$$

for $1 \le i < j \le n$. Since F is a monotonically increasing function, $F' > 0$. Hence (3.5) gives

$$
\begin{aligned}
&\sigma x_i x_j (h^2_{x_i} h_{x_jx_j} + h^2_{x_j} h_{x_ix_i} - 2h_{x_i}h_{x_j}h_{x_ix_j}) \\
&\qquad + (x_i h_{x_i} + x_j h_{x_j}) h_{x_i} h_{x_j} = 0, \quad 1 \le i < j \le n.
\end{aligned} \tag{3.6}
$$

Solving (3.6) for $h_{x_ix_j}$ yields

$$
h_{x_ix_j} = \frac{(x_i h_{x_i} + x_j h_{x_j}) h_{x_i} h_{x_j} + \sigma x_i x_j (h_{x_ix_i} h^2_{x_j} + h_{x_jx_j} h^2_{x_i})}{2\sigma x_i x_j h_{x_i} h_{x_j}} \tag{3.7}
$$

for $1 \le i < j \le n$.

Since $h(x_1, \dots, x_n)$ is assumed to be a homogeneous equation of degree, say p, it follows from the Euler Homogeneous Function Theorem that the homogeneous function h satisfies

$$
x_1 h_{x_1} + x_2 h_{x_2} + \cdots + x_n h_{x_n} = ph. \tag{3.8}
$$

If $p = 0$, then by taking the partial derivatives of (1.1), we find $th_{x_j} = h_{x_j}$ for $j = 1, \dots, n$. Thus $h_{x_1} = \cdots = h_{x_n} = 0$, which contradicts to the assumption that h is nonconstant. Hence we must have $p \ne 0$.

By taking the partial derivatives of (3.8) with respect to $x_1, \dots, x_n$, respectively, we find

$$
\begin{aligned}
x_1 h_{x_1x_1} + x_2 h_{x_1x_2} + \cdots + x_n h_{x_1x_n} &= (p-1)h_{x_1}, \\
x_1 h_{x_1x_2} + x_2 h_{x_2x_2} + \cdots + x_n h_{x_2x_n} &= (p-1)h_{x_2}, \\
&\vdots \\
x_1 h_{x_1x_n} + x_2 h_{x_2x_n} + \cdots + x_n h_{x_nx_n} &= (p-1)h_{x_n}.
\end{aligned} \tag{3.9}
$$

Now, by substituting (3.7) into (3.9) and applying (3.8), we obtain

$$
h_{x_ix_i} = \left(\frac{1 + (p-1)\sigma}{p\sigma} \cdot \frac{h_{x_i}}{h} - \frac{1}{\sigma x_i} \right) h_{x_i}, \quad i = 1, \dots, n. \tag{3.10}
$$

Hence we find from (3.7) and (3.10) that

$$
h_{x_ix_j} = \left(\frac{1 + (p-1)\sigma}{p\sigma} \right) \frac{h_{x_i} h_{x_j}}{h}, \quad 1 \le i < j \le n. \tag{3.11}
$$

Case (a): $\sigma = 1$. In this case, system (3.10) and (3.11) reduce to

$$h_{x_i x_i} = h_{x_i}\left(\frac{h_{x_i}}{h} - \frac{1}{x_i}\right), \;\; i = 1, \ldots, n, \tag{3.12}$$

$$h_{x_i x_j} = \frac{h_{x_i} h_{x_j}}{h}, \;\; 1 \leq i \neq j \leq n. \tag{3.13}$$

After solving the system (3.12)-(3.13) we obtain

$$h = \alpha x_1^{d_1} \cdots x_n^{d_n} \tag{3.14}$$

for some positive number α and nonzero constants $d_i, \ldots, d_n$, with $\sum_{i=1}^{n} d_i = p$. Hence h is a generalized Cobb-Douglas production function.

Case (b): $\sigma \neq 1$. In this case, after solving (3.12) for $i = 1$, we have

$$h = u(x_2, \ldots, x_n)\big(x_1^{\frac{\sigma-1}{\sigma}} + v(x_2, \ldots, x_n)\big)^{\frac{p\sigma}{\sigma-1}} \tag{3.15}$$

for some functions $u(x_2, \ldots, x_n)$ and $v(x_2, \ldots, x_n)$.

After substituting equation (3.15) into (3.13) with $i = 1$ and $j \in \{2, \ldots, n\}$, we derive that

$$u_{x_2} = \cdots = u_{x_n} = 0.$$

Thus u is a constant. Hence we may put $u = \alpha c_1^{\frac{\sigma-1}{\sigma}}$ for some positive numbers α and c_1. Therefore (3.15) becomes

$$f = \alpha\big(c_1^{\frac{\sigma-1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \tilde{v}(x_2, \ldots, x_n)\big)^{\frac{p\sigma}{\sigma-1}}. \tag{3.16}$$

Next, by substituting (3.16) into (3.13) with $2 \leq i < j \leq n$, we find $\tilde{v}_{x_i x_j} = 0$, which imply that

$$\tilde{v} = v^{(2)}(x_2) + \cdots + v^{(n)}(x_n) \tag{3.17}$$

for some non-constant functions $v^{(2)}(x_2), \ldots, v^{(n)}(x_n)$. Now, by combining (3.16) and (3.17) we get

$$h = \alpha\big(a_1^{\frac{\sigma-1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + v^{(2)}(x_2) + \cdots + v^{(n)}(x_n)\big)^{\frac{h\sigma}{\sigma-1}}.$$

After substituting this into (3.13) with $i = 2, \ldots, n$, we obtain

$$v^{(i)}(x_i) = c_i^{\frac{\sigma-1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}}$$

for some positive numbers $c_2, \ldots, c_n$. Therefore h is a generalized ACMS function.

The converse can be verify by direct computation. □

Remark 3.1. Theorem 3.1 is a natural extension of Theorem 2.1.

4. Curvature of production functions.

Each production function $f(\mathbf{x})$ can be identified with its *graph*, which is the non-parametric hypersurface of a Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$ given by (cf. [6, 15, 16])

$$L(\mathbf{x}) = (x_1, \ldots, x_n, f(x_1, \ldots, x_n)). \tag{4.1}$$

For a hypersurface M of a Euclidean $(n+1)$-space, the *Gauss map*

$$\nu : M \to S^{n+1}$$

maps M to the unit hypersphere S^n of $\mathbb{E}^{n+1}$. The Gauss map is a continuous map such that $\nu(p)$ is a unit normal vector $\xi(p)$ of M at p. The Gauss map can always be defined locally, i.e., on a small piece of the hypersurface. It can be defined globally if the hypersurface is orientable.

The differential $d\nu$ of the Gauss map ν can be used to define a type of extrinsic curvature, known as the *shape operator* or Weingarten map. Since at each point $p \in M$, the tangent space T_pM is an inner product space, the shape operator S_p can be defined as a linear operator on this space by the formula:

$$g(S_p v, w) = g(d\nu(v), w) \tag{4.2}$$

for $v, w \in T_pM$, where g is the metric tensor on M induced from the Euclidean metric on $\mathbb{E}^{n+1}$.

The second fundamental form σ is related with the shape operator S by

$$g(\sigma(v, w), \xi(p)) = g(S_p(v), w) \tag{4.3}$$

for tangent vectors v, w of M at p. The eigenvalues of the shape operator S_p are called the principal curvatures.

The determinant of the shape operator S_p is called the *Gauss-Kronecker curvature*, which is denoted by $G(p)$. Thus the Gauss-Kronecker curvature $G(p)$ is nothing but the product of the principal curvature at p. When $n = 2$, the Gauss-Kronecker curvature is simply called the *Gauss curvature*, which is intrinsic due to Gauss' theorema egregium.

For an n-input production function f, we put

$$w = \sqrt{1 + \sum\nolimits_{i=1}^{n} f_{x_i}^2}. \tag{4.4}$$

We recall the following lemma (see, e.g. [5, 4, 14]).

Lemma 4.1. *For the production hypersurface of* $\mathbb{E}^{n+1}$ *defined by*

$$L(x_1, \dots, x_n) = (x_1, \dots, x_n, f(x_1, \dots, x_n)),$$

we have:

(1) *The coefficient* $g_{ij} = g(\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j})$ *of the metric tensor is*

$$g_{ij} = \delta_{ij} + f_{x_i} f_{x_j}, \quad \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 1, & \text{if } i \neq j; \end{cases} \tag{4.5}$$

(2) *The inverse matrix* (g^{ij}) *of* (g_{ij}) *is*

$$g^{ij} = \delta_{ij} - \frac{f_{x_i} f_{x_j}}{w^2}; \tag{4.6}$$

(3) *The matrix of the second fundamental form* σ *is*

$$\sigma_{ij} = \frac{f_{ij}}{w}; \tag{4.7}$$

(4) *The mean curvature* H *is*

$$H = \frac{1}{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{f_{x_j}}{w} \right); \tag{4.8}$$

(5) *The Gauss-Kronecker curvature* G *is*

$$G = \frac{\det(f_{x_i x_j})}{w^{n+2}}; \tag{4.9}$$

(6) *The sectional curvature* K_{ij} *of the plane section spanned by* $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}$ *is*

$$K_{ij} = \frac{f_{x_i x_i} f_{x_j x_j} - f^2_{x_i x_j}}{w^2(1 + f^2_{x_i} + f^2_{x_j})}; \tag{4.10}$$

(7) *The Riemann curvature tensor* R *satisfies*

$$g\left(R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell}\right) = \frac{f_{x_i x_\ell} f_{x_j x_k} - f_{x_i x_k} f_{x_j x_\ell}}{w^4}. \tag{4.11}$$

5. Graphs with null Gauss-Kronecker curvature

As an application of Theorem 3.1 we give the following simple characterization of linearly homogeneous generalized ACMS production functions in terms of Gauss-Kronecker curvature.

Theorem 5.1. *Let* $f(\mathbf{x}) = F(h(x_1, \ldots, x_n))$ *be a homothetic function satisfying the constant elasticity of substitution property. Then the graph of* f *has vanishing Gauss-Kronecker curvature if and only if* f *is either*

(a) *a linearly homogeneous generalized Cobb-Douglas production function or*

(b) *a linearly homogeneous generalized ACMS production function.*

Proof. Let $f(\mathbf{x}) = F(h(x_1, \ldots, x_n))$ be a homothetic function satisfying the constant elasticity of substitution property. Then h is a generalized Cobb-Douglas function or a generalized ACMS function according to Theorem 3.1.

First, let us assume that h is a generalized Cobb-Douglas production function given by

$$h = \gamma x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{5.1}$$

where γ is a positive constant and $\alpha_1, \ldots, \alpha_n$ are nonzero constants.

Suppose that the graph of f has vanishing Gauss-Kronecker curvature. Then it follows from $f = F \circ h$, (5.1) and Lemma 4.1(5) that

$$(p-1)F'(u) = puF''(u), \tag{5.2}$$

where $p = \sum_{i=1}^n \alpha_i$ and $u = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. By solving (5.2) we get

$$F(u) = \alpha u^{\frac{1}{p}} + \gamma \tag{5.3}$$

for some constants α, γ with $\alpha \neq 0$. Consequently, after a suitable translation, we obtain

$$f(\mathbf{x}) = \alpha \left(x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right)^{\frac{1}{p}}, \quad p = \sum_{i=1}^n \alpha_i. \tag{5.4}$$

Therefore, the homothetic function f is a generalized Cobb-Douglas production function with the degree of homogeneity equal to one.

Next, assume that h is a generalized ACMS production function. Then, without loss of generality, we may assume that f takes the form:

$$f(\mathbf{x}) = F\Big(c_1 x_1^{\frac{\sigma-1}{\sigma}} + \cdots + c_n x_n^{\frac{\sigma-1}{\sigma}}\Big) \tag{5.5}$$

for some nonzero constants $\sigma, c_1, \ldots, c_n$ with $\sigma \neq 0, 1$

If the graph of f has vanishing Gauss-Kronecker curvature, then it follows from $f = F \circ h$, (5.5) and Lemma 4.1(5) that

$$F'(u) = (\sigma - 1)uF''(u) \tag{5.6}$$

with

$$u = c_1 x_1^{\frac{\sigma-1}{\sigma}} + \cdots + c_n x_n^{\frac{\sigma-1}{\sigma}}. \tag{5.7}$$

After solving (5.6) and applying a suitable translation on u, we get

$$F(u) = \alpha u^{\frac{\sigma}{\sigma-1}} \tag{5.8}$$

for some nonzero constant α. After combining (5.7) and (5.8) we obtain

$$F(u) = \alpha \left(c_1 x_1^{\frac{\sigma-1}{\sigma}} + \cdots + c_n x_n^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}. \tag{5.9}$$

Therefore, we conclude that the homothetic function f is a linearly homogeneous generalized ACMS production function.

The converse can be verify by direct computation. □

Another application of Theorem 3.1 is the following.

Theorem 5.2. *Let $f(\mathbf{x}) = F(h(x_1, \ldots, x_n))$ be a homothetic function satisfying the constant elasticity of substitution property. Then the graph of f is a flat space if and only if f is either*

(1) *a linearly homogeneous generalized Cobb-Douglas production function or*

(2) *a linearly homogeneous generalized ACMS production function.*

Proof. It is straightforward to verify that the graphs of all linearly homogeneous generalized Cobb-Douglas production functions and of all linearly homogeneous generalized ACMS production functions are flat spaces. By combining this with Theorem 5.1 we obtain Theorem 5.2. □

6. Minimal graphs

In economics, goods that are completely substitutable with each other are called perfect substitutes, e.g., margarine and butter, tea and coffee. They may be characterized as goods having a constant marginal rate of substitution. Mathematically, a production function is a *perfect substitute* if it is of the form:

$$f(\mathbf{x}) = \sum_{i=1}^{n} a_i x_i \tag{6.1}$$

for some nonzero constants $a_1, \ldots, a_n$.

The following result was obtained by the author in [4].

Theorem 6.1. *A two-input homogeneous production function is a perfect substitute if and only if the graph of the production function is a minimal surface.*

The next result classifies all homogeneous functions with CES property which have minimal graphs.

Theorem 6.2. *Let $f(\mathbf{x}) = h(x_1, \ldots, x_n)$ be a p-homogeneous production function satisfying the constant elasticity of substitution property. Then f has minimal graph if and only if f is one of the following:*

(i) *a perfect substitute;*

(ii) *a three-input production function of the form* $f(x,y,z)=xy/z$.

Proof. Let $f=h(x_1,\ldots,x_n)$ be a p-homogeneous production function satisfying the constant elasticity of substitution property. Then, according to Theorem 2.1, f is either a generalized Cobb-Douglas production function or a generalized ACMS production function.

Case (a): *f is a generalized Cobb-Douglas production function.* Without loss of generality, we may assume that

$$f(\mathbf{x})=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}, \tag{6.2}$$

where $\alpha_1,\ldots,\alpha_n$ are nonzero numbers. It follows from Lemma 4.1(4) that the mean curvature of the graph of f satisfies

$$\begin{aligned}H=&\frac{u}{n(1+u^2\sum_{i=1}^n\alpha_i^2x_i^{-2})^{3/2}}\times\\ &\left\{\sum_{j=1}^n\frac{\alpha_j(\alpha_j-1)}{x_j^2}-u^2\sum_{1\le i<j\le n}\frac{\alpha_i\alpha_j(\alpha_i+\alpha_j)}{x_i^2x_j^2}\right\}\end{aligned} \tag{6.3}$$

with $u=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. Therefore, if the graph of f is minimal, then

$$\sum_{j=1}^n\frac{\alpha_j(\alpha_j-1)}{x_j^2}=u^2\sum_{i<j}\frac{\alpha_i\alpha_j(\alpha_i+\alpha_j)}{x_i^2x_j^2}. \tag{6.4}$$

By taking the partial derivative of (6.4) with respect to x_k, we obtain

$$\frac{2(1-\alpha_k)}{x_k^2}=2u^2\sum_{i<j}\frac{\alpha_i\alpha_j(\alpha_i+\alpha_j)}{x_i^2x_j^2}-\frac{2u^2}{x_k^2}\sum_{j\ne k}\frac{\alpha_j(\alpha_k+\alpha_j)}{x_j^2}, \tag{6.5}$$

which implies that, for $1\le k\ne\ell\le n$, we have

$$\begin{aligned}&\frac{1-\alpha_k}{x_k^2}+\frac{u^2}{x_k^2}\sum_{j\ne k}\frac{\alpha_j(\alpha_k+\alpha_j)}{x_j^2}\\ &\qquad=\frac{1-\alpha_\ell}{x_\ell^2}+\frac{u^2}{x_\ell^2}\sum_{j\ne \ell}\frac{\alpha_j(\alpha_\ell+\alpha_j)}{x_j^2}.\end{aligned} \tag{6.6}$$

By multiplying $x_k^2x_\ell^2$ to (6.6), we find

$$\begin{aligned}&(1-\alpha_k)x_\ell^2+x_\ell^2\sum_{j\ne k}\alpha_j(\alpha_k+\alpha_j)x_1^{2\alpha_1}\cdots x_{j-1}^{2\alpha_{j-1}}x_j^{2\alpha_j-2}x_{j+1}^{2\alpha_{j+1}}\cdots x_n^{2\alpha_n}\\ &=(1-\alpha_\ell)x_k^2+x_k^2\sum_{j\ne \ell}\alpha_j(\alpha_\ell+\alpha_j)x_1^{2\alpha_1}\cdots x_{j-1}^{2\alpha_{j-1}}x_j^{2\alpha_j-2}x_{j+1}^{2\alpha_{j+1}}\cdots x_n^{2\alpha_n}.\end{aligned} \tag{6.7}$$

In particular, for $k=1,\ell=2$, (6.7) gives

$$\begin{aligned}&(1-\alpha_1)x_2^2+x_2^2\sum_{j\ne 1}\alpha_j(\alpha_1+\alpha_j)x_1^{2\alpha_1}\cdots x_{j-1}^{2\alpha_{j-1}}x_j^{2\alpha_j-2}x_{j+1}^{2\alpha_{j+1}}\cdots x_n^{2\alpha_n}\\ &=(1-\alpha_2)x_1^2+x_1^2\sum_{j\ne 2}\alpha_j(\alpha_2+\alpha_j)x_1^{2\alpha_1}\cdots x_{j-1}^{2\alpha_{j-1}}x_j^{2\alpha_j-2}x_{j+1}^{2\alpha_{j+1}}\cdots x_n^{2\alpha_n}.\end{aligned} \tag{6.8}$$

Case (a.1) $n=2$. In this case (6.8) reduces to

$$\begin{aligned}&(1-\alpha_1)x_2^2+\alpha_2(\alpha_1+\alpha_2)x_1^{2\alpha_1}x_2^{2\alpha_2}\\ &\qquad=(1-\alpha_2)x_1^2+\alpha_1(\alpha_1+\alpha_2)x_1^{2\alpha_1}x_2^{2\alpha_2}.\end{aligned} \tag{6.9}$$

Since α_1, α_2 are nonzero numbers, it follows from (6.9) that

$$1 - \alpha_1 = 1 - \alpha_2 = \alpha_1(\alpha_1 + \alpha_2) = \alpha_2(\alpha_1 + \alpha_2) = 0,$$

which is impossible.

Case (a.2) $n \geq 3$. It follows from (6.8) that

$$\begin{aligned}
&(1-\alpha_2)x_1^2 - (1-\alpha_1)x_2^2 + (\alpha_1^2 - \alpha_2^2)x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} \\
&= \sum_{j=3}^{n} (\alpha_1 + \alpha_j)\alpha_j x_1^{2\alpha_1} x_2^{2\alpha_2+2} x_3^{2\alpha_3} \cdots x_{j-1}^{2\alpha_{j-1}} x_j^{2\alpha_j - 2} x_{j+1}^{2\alpha_{j+1}} \cdots x_n^{2\alpha_n} \\
&\quad - \sum_{j=3}^{n} (\alpha_2 + \alpha_j)\alpha_j x_1^{2\alpha_1+2} x_2^{2\alpha_2} \cdots x_{j-1}^{2\alpha_{j-1}} x_j^{2\alpha_j - 2} x_{j+1}^{2\alpha_{j+1}} \cdots x_n^{2\alpha_n}.
\end{aligned} \tag{6.10}$$

If $n = 3$, (6.10) yields

$$\begin{aligned}
&(1-\alpha_2)x_1^2 - (1-\alpha_1)x_2^2 + (\alpha_1^2 - \alpha_2^2)x_1^{2\alpha_1} x_2^{2\alpha_2} x_3^{2\alpha_3} \\
&= (\alpha_1 + \alpha_3)\alpha_3 x_1^{2\alpha_1} x_2^{2\alpha_2+2} x_3^{2\alpha_3-2} \\
&\quad - (\alpha_2 + \alpha_3)\alpha_3 x_1^{2\alpha_1+2} x_2^{2\alpha_2} x_3^{2\alpha_3-2},
\end{aligned} \tag{6.11}$$

which implies that $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = -1$. Therefore, the production function is of the form:

$$f = \frac{x_1 x_2}{x_3}.$$

If $n \geq 4$, then it follows from (6.10) that $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = \cdots = \alpha_n = -1$. By substituting these into (6.4) we find

$$\sum_{j=3}^{n} \frac{1}{x_j^2} + \sum_{3 \leq i < j \leq n} \frac{u^2}{x_i^2 x_j^2} = \frac{u^2}{x_1^2 x_2^2}, \tag{6.12}$$

which is impossible. Consequently, if the homogeneous function f is a generalized Cobb-Douglas production function with minimal graph, then f is a three-input production function of the form $f = xy/z$.

Case (b): *f is a generalized ACMS production function.* In this case, without loss of generality, we may assume that

$$f(\mathbf{x}) = \left(\sum_{i=1}^{n} a_i x_i^{\rho} \right)^{\frac{p}{\rho}}, \tag{6.13}$$

where a_i, p, γ, ρ are constants with $a_i, \gamma, p \neq 0$ and $\rho \neq 0, 1$.

It follows from (6.13) and Lemma 4.1(4) that the graph of f is minimal if and only if we have

$$\begin{aligned}
0 &= p^2(\rho - 1) \left(\sum_{i=1}^{n} a_i x_i^{\rho} \right)^{\frac{2p}{\rho} - 1} \sum_{1 \leq i \neq j \leq n} c_i c_j^2 x_i^{\rho-2} x_j^{2\rho-2} \\
&\quad + (\rho - 1) \sum_{1 \leq i \neq j \leq n} c_i c_j x_i^{\rho-2} x_j^{\rho} + (p-1) \sum_{i=1}^{n} c_i^2 x_i^{2\rho-2}.
\end{aligned} \tag{6.14}$$

Clearly, if (6.14) holds, then we obtain $p = \rho = 1$. Consequently, the production function f is a perfect substitute.

The converse is easy to verify. □

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