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ON $(N(k),\xi)$-SEMI-RIEMANNIAN MANIFOLDS: PSEUDOSYMMETRIES

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Abstract. Definition of $(\mathcal{T}_a,\mathcal{T}_b)$-pseudosymmetric semi-Riemannian manifold is given. $(\mathcal{T}_a,\mathcal{T}_b)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifolds are classified. Some results for $\mathcal{T}_a$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifolds are obtained. $(\mathcal{T}_a,\mathcal{T}_b,S^{\ell})$-pseudosymmetric semi-Riemannian manifolds are also defined. $(\mathcal{T}_a,\mathcal{T}_b,S^{\ell})$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifolds are classified. Some results for $(R,\mathcal{T}_a,S^{\ell})$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifolds are obtained. In particular, some results for $(R,\mathcal{T}_a,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifolds are also obtained. After that, the definition of $(\mathcal{T}_a,S_{\mathcal{T}_b})$-pseudosymmetric semi-Riemannian manifold is given. $(\mathcal{T}_a,S_{\mathcal{T}_b})$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifolds are classified. It is proved that a $(R,S_{\mathcal{T}_a})$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold is either Einstein or $L=k$ under an algebraic condition. Some results for $(\mathcal{T}_a,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifolds are also obtained. In last, $(\mathcal{T}_a,S_{\mathcal{T}_b},S^{\ell})$-pseudosymmetric semi-Riemannian manifolds are defined and $(\mathcal{T}_a,S_{\mathcal{T}_b},S^{\ell})$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifolds are classified.

1. Introduction

Let M be an n-dimensional smooth manifold and $\mathfrak{X}(M)$ the Lie algebra of vector fields in M. We assume that $X,X_1,\ldots,X_s,Y,Z,U,V,W\in\mathfrak{X}(M)$. It is well known that every $(1,1)$ tensor field $\mathcal{A}$ on a differentiable manifold determines a derivation $\mathcal{A}\cdot$ of the tensor algebra on the manifold, commuting with contractions. For example, a $(1,1)$ tensor field $\mathcal{B}(V,U)$ induces the derivation $\mathcal{B}(V,U)\cdot$, thus given a $(0,s)$

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tensor field $\mathcal{K}$, we can associate a $(0, s+2)$ tensor $\mathcal{B}\cdot\mathcal{K}$, defined by

$$
\begin{aligned}
(\mathcal{B}\cdot\mathcal{K})(X_1,\ldots,X_s,X,Y) &= (\mathcal{B}(X,Y)\cdot\mathcal{K})(X_1,\ldots,X_s) \\
&= -\mathcal{K}(\mathcal{B}(X,Y)X_1,\ldots,X_s)-\cdots \\
&\quad -\mathcal{K}(X_1,\ldots,\mathcal{B}(X,Y)X_s).
\end{aligned} \tag{1.1}
$$

Next, for a tensor σ of type $(0,2)$, the operators $(X\wedge_\sigma Y)$ and $Q(\sigma,\mathcal{K})$are defined by

$$(X\wedge_\sigma Y)Z=\sigma(Y,Z)X-\sigma(X,Z)Y,$$

$$
\begin{aligned}
Q(\sigma,\mathcal{K})(X_1,\ldots,X_s,X,Y) &= ((X\wedge_\sigma Y)\cdot\mathcal{K})(X_1,\ldots,X_s) \\
&= -\mathcal{K}((X\wedge_\sigma Y)X_1,\ldots,X_s)-\cdots \\
&\quad -\mathcal{K}(X_1,\ldots,(X\wedge_\sigma Y)X_s).
\end{aligned} \tag{1.2}
$$

Let (M,g) be an n-dimensional semi-Riemannian manifold. Then (M,g) is said to be

(a) *pseudosymmetric* [8] if its curvature tensor R satisfies

$$R\cdot R=L_gQ(g,R),$$

where L_g is some smooth function on M,

(b) *Ricci-generalized pseudosymmetric* [5] if it satisfies

$$R\cdot R=L_SQ(S,R),$$

where L_S is some smooth function on M and S is the the Ricci tensor,

(c) *Ricci-pseudosymmetric* [4] if on the set $\mathcal{U}=\{x\in M:(S-r/n)_x\neq 0\}$,

$$R\cdot S=LQ(g,S),$$

where L is some function on $\mathcal{U}$.

A pseudosymmetric manifold is a generalization of manifold of constant curvature, symmetric manifold ($\nabla R=0$) and semisymmetric manifold ($R\cdot R=0$). Deszcz et al. [10] proved that hypersurfaces in spaces of constant curvature, with exactly two distinct principal curvatures at every point, are pseudosymmetric. Ricci-pseudosymmetric manifold is a generalization of manifold of constant curvature, Einstein manifold, Ricci symmetric manifold ($\nabla S=0$), symmetric manifold, semisymmetric manifold, pseudosymmetric manifold and Ricci-semisymmetric manifold($R\cdot S=0$). Similar to pseudosymmetry condition, Deszcz and Grycak [6, 7, 9] and Özgür [18] also studied Weyl pseudosymmetric manifolds ($R\cdot\mathcal{C}=L_gQ(g,\mathcal{C})$). In 1990, Prvanović [24] studied the condition

$$R\cdot\tilde{T}=LQ(S^\ell,\tilde{T}),\qquad \ell=0,1,2,\ldots,$$

where $\tilde{T}$ is some $(0,4)$-tensor field and L is some smooth function on M. If $\tilde{T}=R$ and $\ell=0$, this condition becomes the condition for pseudosymmetry and if $\tilde{T}=R$ and $\ell=1$, this condition becomes the condition for Ricci-generalized pseudosymmetry.

Apart from the curvature tensor and the Weyl conformal curvature tensor, quasi-conformal curvature tensor, concircular curavture tensor, conharmonic curvature tensor, pseudo-projective curvature tensor, projective curvature tensor are important curvature tensors in the semi-Riemannian point of view. The $\mathcal{T}$-curvature tensor is generalisation of quasi-conformal curvature tensor, Weyl conformal curvature tensor, conharmonic curvature tensor, concircular curavture tensor, pseudo-projective curvature tensor, projective curvature tensor, $\mathcal{W}_i$-curvature tensors ($i=0,\ldots,9$) and $\mathcal{W}_j^*$-curvature tensors ($j=0,1$). Therefore it is interesting to study the pseudosymmertic condition of $\mathcal{T}$-curvature tensor on different stuctures of manifolds. $N(k)$-contact metric manifold, (ε)-Sasakian manifold, Sasakian manifold, Kenmotsu manifold, (ε)-para Sasakian manifold and para Sasakian manifold are particular cases of $(N(k),\xi)$-semi-Riemannian manifolds. Thus it is a motivation to study the $\mathcal{T}$-curvature tensor on $(N(k),\xi)$-semi-Riemannian manifolds. In this paper, we study several derivation conditions on $(N(k),\xi)$-semi-Riemannian manifolds. The paper is organized as follows. In Section 2, we give the definition of $\mathcal{T}$-curvature tensor. In Section 3, we give examples and properties of $(N(k),\xi)$-semi-Riemannian manifolds. In Section 4, $(\mathcal{T}_a,\mathcal{T}_b)$-pseudosymmetric semi-Riemannian manifolds are defined and studied. Some results for $\mathcal{T}_a$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifolds are given. In Section 5, $(\mathcal{T}_a,\mathcal{T}_b,S^\ell)$-pseudosymmetric semi-Riemannian manifolds are defined and studied. Some results for $(R,\mathcal{T}_a,S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifolds are given. In particular, some results for $(R,\mathcal{T}_a,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifolds are obtained. In Section 6, the definition of $(\mathcal{T}_a,S_{\mathcal{T}_b})$-pseudosymmetric semi-Riemannian manifold is given. $(\mathcal{T}_a,S_{\mathcal{T}_b})$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifolds are classified. It is proved that a $(R,S_{\mathcal{T}_a})$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold is either Einstein or $L=k$ under an algebraic condition. Some results for $(\mathcal{T}_a,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifolds are obtained. In the last section, the definition of $(T_a,S_{\mathcal{T}_b},S^\ell)$-pseudosymmetric semi-Riemannian manifold is given. $(T_a,S_{\mathcal{T}_b},S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Rie-mannian manifolds are classified.

2. Preliminaries

Definition 2.1. In an n-dimensional semi-Riemannian manifold (M,g), $\mathcal{T}$*-curvature tensor* [30] is a tensor of type $(1,3)$, which is defined by

$$\begin{aligned}\mathcal{T}(X,Y)Z &= a_0 R(X,Y)Z \\ &\quad + a_1 S(Y,Z)X + a_2 S(X,Z)Y + a_3 S(X,Y)Z \\ &\quad + a_4 g(Y,Z)QX + a_5 g(X,Z)QY + a_6 g(X,Y)QZ \\ &\quad + a_7 r\,(g(Y,Z)X - g(X,Z)Y),\end{aligned} \tag{2.1}$$

where $a_0,\dots,a_7$ are real numbers; and R, S, Q and r are the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature respectively.

In particular, the $\mathcal{T}$-curvature tensor is reduced to

(1) the *curvature tensor* R if

$$a_0 = 1,\quad a_1 = \cdots = a_7 = 0,$$

(2) the *quasi-conformal curvature tensor* $\mathcal{C}_*$ [34] if

$$a_1 = -a_2 = a_4 = -a_5,\quad a_3 = a_6 = 0,\quad a_7 = -\frac{1}{n}\left(\frac{a_0}{n-1} + 2a_1\right),$$

(3) the *conformal curvature tensor* $\mathcal{C}$ [12, p. 90] if

$$a_0 = 1,\ a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{n-2},\ a_3 = a_6 = 0,\ a_7 = \frac{1}{(n-1)(n-2)},$$

(4) the *conharmonic curvature tensor* $\mathcal{L}$ [13] if

$$a_0 = 1,\quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{n-2},\quad a_3 = a_6 = 0,\quad a_7 = 0,$$

(5) the *concircular curvature tensor* $\mathcal{V}$ ([32], [33, p. 87]) if

$$a_0 = 1,\quad a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0,\quad a_7 = -\frac{1}{n(n-1)},$$

(6) the *pseudo-projective curvature tensor* $\mathcal{P}_*$ [23] if

$$a_1 = -a_2,\quad a_3 = a_4 = a_5 = a_6 = 0,\quad a_7 = -\frac{1}{n}\left(\frac{a_0}{n-1} + a_1\right),$$

(7) the *projective curvature tensor* $\mathcal{P}$ [33, p. 84] if

$$a_0 = 1, \quad a_1 = -a_2 = -\frac{1}{(n-1)}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

(8) the $\mathcal{M}$*-projective curvature tensor* [21] if

$$a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2(n-1)}, \quad a_3 = a_6 = a_7 = 0,$$

(9) the $\mathcal{W}_0$*-curvature tensor* [21, Eq. (1.4)] if

$$a_0 = 1, \quad a_1 = -a_5 = -\frac{1}{(n-1)}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

(10) the $\mathcal{W}_0^*$*-curvature tensor* [21, Eq. (2.1)] if

$$a_0 = 1, \quad a_1 = -a_5 = \frac{1}{(n-1)}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

(11) the $\mathcal{W}_1$*-curvature tensor* [21] if

$$a_0 = 1, \quad a_1 = -a_2 = \frac{1}{(n-1)}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

(12) the $\mathcal{W}_1^*$*-curvature tensor* [21] if

$$a_0 = 1, \quad a_1 = -a_2 = -\frac{1}{(n-1)}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

(13) the $\mathcal{W}_2$*-curvature tensor* [20] if

$$a_0 = 1, \quad a_4 = -a_5 = -\frac{1}{(n-1)}, \quad a_1 = a_2 = a_3 = a_6 = a_7 = 0,$$

(14) the $\mathcal{W}_3$*-curvature tensor* [21] if

$$a_0 = 1, \quad a_2 = -a_4 = -\frac{1}{(n-1)}, \quad a_1 = a_3 = a_5 = a_6 = a_7 = 0,$$

(15) the $\mathcal{W}_4$*-curvature tensor* [21] if

$$a_0 = 1, \quad a_5 = -a_6 = \frac{1}{(n-1)}, \quad a_1 = a_2 = a_3 = a_4 = a_7 = 0,$$

(16) the $\mathcal{W}_5$*-curvature tensor* [22] if

$$a_0 = 1, \quad a_2 = -a_5 = -\frac{1}{(n-1)}, \quad a_1 = a_3 = a_4 = a_6 = a_7 = 0,$$

(17) the $\mathcal{W}_6$*-curvature tensor* [22] if

$$a_0 = 1, \quad a_1 = -a_6 = -\frac{1}{(n-1)}, \quad a_2 = a_3 = a_4 = a_5 = a_7 = 0,$$

(18) the $\mathcal{W}_7$*-curvature tensor* [22] if

$$a_0 = 1, \quad a_1 = -a_4 = -\frac{1}{(n-1)}, \quad a_2 = a_3 = a_5 = a_6 = a_7 = 0,$$

(19) the $\mathcal{W}_8$*-curvature tensor* [22] if

$$a_0 = 1, \quad a_1 = -a_3 = -\frac{1}{(n-1)}, \quad a_2 = a_4 = a_5 = a_6 = a_7 = 0,$$

(20) the $\mathcal{W}_9$*-curvature tensor* [22] if

$$a_0 = 1, \quad a_3 = -a_4 = \frac{1}{(n-1)}, \quad a_1 = a_2 = a_5 = a_6 = a_7 = 0.$$

Denoting

$$\mathcal{T}(X,Y,Z,V) = g(\mathcal{T}(X,Y)Z,V),$$

we write the curvature tensor $\mathcal{T}$ in its $(0,4)$ form as follows.

$$\begin{aligned}\mathcal{T}(X,Y,Z,V) &= a_0\, R(X,Y,Z,V) \\ &+ a_1\, S(Y,Z)\, g(X,V) + a_2\, S(X,Z)\, g(Y,V) \\ &+ a_3\, S(X,Y)\, g(Z,V) + a_4\, S(X,V)\, g(Y,Z) \\ &+ a_5\, S(Y,V)\, g(X,Z) + a_6\, S(Z,V)\, g(X,Y) \\ &+ a_7\, r\,(g(Y,Z)\, g(X,V) - g(X,Z)\, g(Y,V)).\end{aligned} \tag{2.2}$$

Notation. We will call $\mathcal{T}$-curvature tensor as $\mathcal{T}_a$-curvature tensor, whenever it is necessary. If $a_0,\ldots,a_7$ are replaced by $b_0,\ldots,b_7$ in the definition of $\mathcal{T}$-curvature tensor, then we will call $\mathcal{T}$-curvature tensor as $\mathcal{T}_b$-curvature tensor.

3. $(N(k),\xi)$-semi-Riemannian manifolds

Let (M,g) be an n-dimensional semi-Riemannian manifold [17] equipped with a semi-Riemannian metric g. If $\operatorname{index}(g)=1$ then g is a Lorentzian metric and (M,g) a Lorentzian manifold [1]. If g is positive definite then g is an usual Riemannian metric and (M,g) a Riemannian manifold.

The *k-nullity distribution* [28] of (M,g) for a real number k is the distribution

$$N(k) : p \mapsto N_p(k) = \{Z \in T_pM : R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\}.$$

Let ξ be a non-null unit vector field in (M,g) and η its associated 1-form. Thus

$$g(\xi,\xi) = \varepsilon,$$

where $\varepsilon = 1$ or -1 according as ξ is spacelike or timelike, and

$$\eta(X) = \varepsilon g(X,\xi), \qquad \eta(\xi) = 1. \tag{3.1}$$

Definition 3.1. An $(N(k),\xi)$*-semi-Riemannian manifold* [31] consists of a semi-Riemannian manifold (M,g), a k-nullity distribution $N(k)$ on (M,g) and a non-null unit vector field ξ in (M,g) belonging to $N(k)$.

Now, we intend to give some examples of $(N(k),\xi)$-semi-Riemannian manifolds. For this purpose we collect some definitions from the geometry of almost contact manifolds and almost paracontact manifolds as follows:

Almost contact manifolds. Let M be a smooth manifold of dimension $n = 2m+1$. Let φ, ξ and η be tensor fields of type $(1,1)$, $(1,0)$ and $(0,1)$, respectively. If φ, ξ and η satisfy the conditions

$$\varphi^2 = -I + \eta \otimes \xi, \tag{3.2}$$

$$\eta(\xi) = 1, \tag{3.3}$$

where I denotes the identity transformation, then M is said to have an almost contact structure (φ,ξ,η). A manifold M alongwith an almost contact structure is called an *almost contact manifold* [2]. Let g be a semi-Riemannian metric on M such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y), \tag{3.4}$$

where $\varepsilon = \pm 1$. Then (M,g) is an (ε)-*almost contact metric manifold* [11] equipped with an (ε)-*almost contact metric structure* $(\varphi,\xi,\eta,g,\varepsilon)$. In particular, if the metric g is positive definite, then an (ε)-almost contact metric manifold is the usual *almost contact metric manifold* [2].

From (3.4), it follows that

$$g(X,\varphi Y) = -g(\varphi X, Y) \tag{3.5}$$

and

$$g(X,\xi) = \varepsilon\eta(X). \tag{3.6}$$

From (3.3) and (3.6), we have

$$g(\xi,\xi) = \varepsilon. \tag{3.7}$$

In an (ε)-almost contact metric manifold, the fundamental 2-form Φ is defined by

$$\Phi(X,Y) = g(X,\varphi Y). \tag{3.8}$$

An (ε)-almost contact metric manifold with $\Phi = d\eta$ is an (ε)-*contact metric manifold* [27]. For $\varepsilon = 1$ and g Riemannian, M is the usual *contact metric manifold* [2]. A contact metric manifold with $\xi \in N(k)$, is called a $N(k)$-*contact metric manifold* [3].

An (ε)-almost contact metric structure $(\varphi,\xi,\eta,g,\varepsilon)$ is called an (ε)-*Sasakian structure* if

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \varepsilon\eta(Y)X,$$

where ∇ is Levi-Civita connection with respect to the metric g. A manifold endowed with an (ε)-Sasakian structure is called an (ε)-*Sasakian manifold* [27]. For $\varepsilon = 1$ and g Riemannian, M is the usual *Sasakian manifold* [2, 25].

An almost contact metric manifold is a *Kenmotsu manifold* [15] if

$$(\nabla_X\varphi)Y = g(\varphi X,Y)\xi - \eta(Y)\varphi X. \tag{3.9}$$

By (3.9), we have

$$\nabla_X\xi = X - \eta(X)\xi. \tag{3.10}$$

Almost paracontact manifolds. Let M be an n-dimensional *almost paracontact manifold* [26] equipped with an *almost paracontact structure* (φ,ξ,η), where φ, ξ and η are tensor fields of type $(1,1)$, $(1,0)$ and $(0,1)$, respectively; and satisfy the conditions

$$\varphi^2 = I - \eta\otimes\xi, \tag{3.11}$$

$$\eta(\xi) = 1. \tag{3.12}$$

Let g be a semi-Riemannian metric on M such that

$$g(\varphi X,\varphi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y), \tag{3.13}$$

where $\varepsilon = \pm 1$. Then (M,g) is an (ε)-*almost paracontact metric manifold* equipped with an (ε)-*almost paracontact metric structure* $(\varphi,\xi,\eta,g,\varepsilon)$. In particular, if index$(g) = 1$, then an (ε)-almost paracontact metric manifold is said to be a *Lorentzian almost paracontact manifold*. In particular, if the metric g is positive definite, then an (ε)-almost paracontact metric manifold is the usual *almost paracontact metric manifold* [26].

The equation (3.13) is equivalent to

$$g(X,\varphi Y) = g(\varphi X,Y) \tag{3.14}$$

along with

$$g(X,\xi) = \varepsilon\eta(X). \tag{3.15}$$

From (3.12) and (3.15), we have

$$g(\xi,\xi) = \varepsilon. \tag{3.16}$$

An (ε)-almost paracontact metric structure is called an (ε)-*para-Sasakian structure* [29] if

$$(\nabla_X\varphi)Y = -g(\varphi X,\varphi Y)\xi - \varepsilon\eta(Y)\varphi^2 X, \tag{3.17}$$

where ∇ is Levi-Civita connection with respect to the metric g. A manifold endowed with an (ε)-para-Sasakian structure is called an (ε)-*para-Sasakian manifold* [29]. For $\varepsilon = 1$ and

g Riemannian, M is the usual para-Sasakian manifold [26]. For $\varepsilon = -1$, g Lorentzian and ξ replaced by $-\xi$, M becomes a Lorentzian para-Sasakian manifold [16].

Example 3.1. [31] The following are some well known examples of $(N(k),\xi)$-semi-Riemannian manifolds:

(1) An $N(k)$-contact metric manifold [3] is an $(N(k),\xi)$-Riemannian manifold.
(2) A Sasakian manifold [25] is an $(N(1),\xi)$-Riemannian manifold.
(3) A Kenmotsu manifold [15] is an $(N(-1),\xi)$-Riemannian manifold.
(4) An (ε)-Sasakian manifold [27] an $(N(\varepsilon),\xi)$-semi-Riemannian manifold.
(5) A para-Sasakian manifold [26] is an $(N(-1),\xi)$-Riemannian manifold.
(6) An (ε)-para-Sasakian manifold [29] is an $(N(-\varepsilon),\xi)$-semi-Riemannian manifold.

In an n-dimensional $(N(k),\xi)$-semi-Riemannian manifold (M,g), it is easy to verify that

$$R(X,Y)\xi = \varepsilon k(\eta(Y)X - \eta(X)Y), \tag{3.18}$$

$$R(\xi,X)Y = \varepsilon k(\varepsilon g(X,Y)\xi - \eta(Y)X), \tag{3.19}$$

$$R(\xi,X)\xi = \varepsilon k(\eta(X)\xi - X), \tag{3.20}$$

$$R\,(X,Y,Z,\xi) = \varepsilon k(\,\eta\,(X)\,g\,(Y,Z) - \eta\,(Y)\,g\,(X,Z)), \tag{3.21}$$

$$\eta\,(R\,(X,Y)\,Z) = k(\eta\,(X)\,g\,(Y,Z) - \eta\,(Y)\,g\,(X,Z)), \tag{3.22}$$

$$S(X,\xi) = \varepsilon k(n-1)\eta(X), \tag{3.23}$$

$$Q\xi = k(n-1)\xi, \tag{3.24}$$

$$S(\xi,\xi) = \varepsilon k(n-1), \tag{3.25}$$

$$\eta(QX) = \varepsilon g(QX,\xi) = \varepsilon S(X,\xi) = k(n-1)\eta(X). \tag{3.26}$$

Moreover, define

$$S^{\ell}(X,Y) = g(Q^{\ell}X,Y) = S(Q^{\ell-1}X,Y), \tag{3.27}$$

where $\ell = 0,1,2,\dots$ and $S^0 = g$. Using (3.26) in (3.27), we get

$$S^{\ell}(X,\xi) = \varepsilon k^{\ell}(n-1)^{\ell}\eta(X). \tag{3.28}$$

Now, we give the following Lemma.

Lemma 3.1. [31] *Let M be an n-dimensional $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$\begin{aligned} \mathcal{T}_a(X,Y)\xi \;&=\; (-\varepsilon k a_0 + \varepsilon k(n-1)a_2 - \varepsilon a_7\, r)\eta(X)Y \\ &\quad + (\varepsilon k a_0 + \varepsilon k(n-1)a_1 + \varepsilon a_7\, r)\,\eta(Y)X \\ &\quad + a_3\, S(X,Y)\xi + \varepsilon a_4\,\eta(Y)QX \\ &\quad + \varepsilon a_5\eta(X)QY + k(n-1)a_6 g(X,Y)\xi, \end{aligned} \tag{3.29}$$

$$\begin{aligned} \mathcal{T}_a(\xi,X)\xi \;&=\; (-\varepsilon k a_0 + \varepsilon k(n-1)a_2 - \varepsilon a_7\, r)X + \varepsilon a_5\, QX \\ &\quad + \{\varepsilon k a_0 + \varepsilon k(n-1)a_1 + \varepsilon k(n-1)a_3 \\ &\qquad + \varepsilon k(n-1)a_4 + \varepsilon k(n-1)a_6 + \varepsilon a_7\, r\}\,\eta(X)\xi, \end{aligned} \tag{3.30}$$

$$\begin{aligned} \mathcal{T}_a(\xi,Y)Z \;&=\; (k a_0 + k(n-1)a_4 + a_7 r)g\,(Y,Z)\,\xi \\ &\quad + a_1\, S\,(Y,Z)\,\xi + \varepsilon k(n-1)a_3\eta(Y)Z \\ &\quad + \varepsilon a_5\,\eta(Z)QY + \varepsilon a_6\,\eta(Y)QZ \\ &\quad + (-\varepsilon k a_0 + \varepsilon k(n-1)a_2 - \varepsilon a_7\, r)\eta(Z)Y, \end{aligned} \tag{3.31}$$

$$\eta(\mathcal{T}_a(X,Y)\xi) = \varepsilon k(n-1)(a_1+a_2+a_4+a_5)\eta(X)\eta(Y) + a_3 S(X,Y) + k(n-1)a_6 g(X,Y), \tag{3.32}$$

$$\begin{aligned}\mathcal{T}_a(X,Y,\xi,V) &= (-\varepsilon k a_0 + \varepsilon k(n-1)a_2 - \varepsilon a_7 r)\eta(X)g(Y,V) \\ &\quad + (\varepsilon k a_0 + \varepsilon k(n-1)a_1 + \varepsilon a_7 r)\,\eta(Y)g(X,V) \\ &\quad + \varepsilon a_3 S(X,Y)\eta(V) + \varepsilon a_4\,\eta(Y)S(X,V) \\ &\quad + \varepsilon a_5\,\eta(X)S(Y,V) + \varepsilon k(n-1)a_6\, g(X,Y)\eta(V),\end{aligned} \tag{3.33}$$

$$\begin{aligned}\mathcal{T}_a(X,\xi)\xi &= \{-\varepsilon k a_0 + \varepsilon k(n-1)a_2 + \varepsilon k(n-1)a_3 \\ &\quad + \varepsilon k(n-1)a_5 + \varepsilon k(n-1)a_6 - \varepsilon a_7 r\}\,\eta(X)\xi \\ &\quad + (\varepsilon k a_0 + \varepsilon k(n-1)a_1 + \varepsilon a_7 r)\,X + \varepsilon a_4\, QX,\end{aligned} \tag{3.34}$$

$$\begin{aligned}S_{\mathcal{T}_a}(X,\xi) &= \{\varepsilon k(n-1)(a_0+na_1+a_2+a_3+a_5+a_6) \\ &\quad + \varepsilon r(a_4+(n-1)a_7)\}\,\eta(X),\end{aligned} \tag{3.35}$$

$$\begin{aligned}S_{\mathcal{T}_a}(\xi,\xi) &= \varepsilon k(n-1)(a_0+na_1+a_2+a_3+a_5+a_6) \\ &\quad + \varepsilon r(a_4+(n-1)a_7).\end{aligned} \tag{3.36}$$

Remark 3.1. The relations (3.18) – (3.36) are true for

(1) a $N(k)$-contact metric manifold [3] ($\varepsilon = 1$),
(2) a Sasakian manifold [25] ($k = 1$, $\varepsilon = 1$),
(3) a Kenmotsu manifold [15] ($k = -1$, $\varepsilon = 1$),
(4) an (ε)-Sasakian manifold [27] ($k = \varepsilon$, $\varepsilon k = 1$),
(5) a para-Sasakian manifold [26] ($k = -1$, $\varepsilon = 1$), and
(6) an (ε)-para-Sasakian manifold [29] ($k = -\varepsilon$, $\varepsilon k = -1$).

Even, all the relations and results of this paper will be true for the above six cases.

4. $(\mathcal{T}_a, \mathcal{T}_b)$-pseudosymmetry

Definition 4.1. A semi-Riemannian manifold (M, g) is said to be $(\mathcal{T}_a, \mathcal{T}_b)$-pseudosymmetric if

$$\mathcal{T}_a \cdot \mathcal{T}_b = L_g Q(g, \mathcal{T}_b), \tag{4.1}$$

where L_g is some smooth function on M. In particular, it is said to be $(R, \mathcal{T}_a)$-pseudosymmetric or, in brief, $\mathcal{T}_a$-pseudosymmetric if

$$R \cdot \mathcal{T}_a = L_g Q(g, \mathcal{T}_a) \tag{4.2}$$

holds on the set

$$\mathcal{U}_g = \left\{x \in M \,:\, (\mathcal{T}_a)_x \neq 0 \text{ at } x\right\},$$

where L_g is some smooth function on $\mathcal{U}_g$. In particular, if in (4.2), $\mathcal{T}_a$ is equal to R, $\mathcal{C}_*$, $\mathcal{C}$, $\mathcal{L}$, $\mathcal{V}$, $\mathcal{P}_*$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{W}_0$, $\mathcal{W}_0^*$, $\mathcal{W}_1$, $\mathcal{W}_1^*$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_4$, $\mathcal{W}_5$, $\mathcal{W}_6$, $\mathcal{W}_7$, $\mathcal{W}_8$, $\mathcal{W}_9$, then it becomes pseudosymmetric, quasi-conformal pseudosymmetric, Weyl pseudosymmetric, conharmonic pseudosymmetric, concircular pseudosymmetric, pseudo-projective pseudosymmetric, projective pseudosymmetric, $\mathcal{M}$-pseudosymmetric, $\mathcal{W}_0$-pseudosymmetric, $\mathcal{W}_0^*$-pseudosymmetric, $\mathcal{W}_1$-pseudosymmetric, $\mathcal{W}_1^*$-pseudosymmetric, $\mathcal{W}_2$-pseudosymmetric, $\mathcal{W}_3$-pseudosymmetric, $\mathcal{W}_4$-pseudosymmetric, $\mathcal{W}_5$-pseudosymmetric, $\mathcal{W}_6$-pseudosymmetric, $\mathcal{W}_7$-pseudosymmetric, $\mathcal{W}_8$-pseudosymmetric, $\mathcal{W}_9$-pseudosymmetric, respectively.

Theorem 4.1. *Let M be an n-dimensional $(\mathcal{T}_a,\mathcal{T}_b)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$
\begin{aligned}
&\varepsilon b_0(ka_0+\varepsilon k(n-1)a_4+a_7r)R(U,V,W,X)+\varepsilon a_1b_0S(X,R(U,V)W)\\
&-2k(n-1)a_3(kb_0+k(n-1)b_4+b_7r)\eta(X)\eta(U)g(V,W)\\
&-2k(n-1)a_3(-kb_0+k(n-1)b_5-b_7r)\eta(X)\eta(V)g(U,W)\\
&+\varepsilon a_1b_4S^2(X,U)g(V,W)+\varepsilon a_1b_5S^2(X,V)g(U,W)\\
&+\varepsilon a_1b_6S^2(X,W)g(U,V)-a_5(b_1+b_3)S^2(X,V)\eta(U)\eta(W)\\
&-a_5(b_1+b_2)S^2(X,W)\eta(U)\eta(V)-a_5(b_2+b_3)S^2(X,U)\eta(V)\eta(W)\\
&-2a_6b_1S^2(V,W)\eta(X)\eta(U)-2a_6b_2S^2(U,W)\eta(X)\eta(V)\\
&-2a_6b_3S^2(U,V)\eta(X)\eta(W)-2k^2(n-1)a_3b_6g(U,V)\eta(X)\eta(W)\\
&-2\left(k(n-1)a_3b_1+a_6(kb_0+k(n-1)b_4+b_7r)\right)\eta(X)\eta(U)S(V,W)\\
&-2\left(k(n-1)a_3b_2+a_6(-kb_0+k(n-1)b_5-b_7r)\right)\eta(X)\eta(V)S(U,W)\\
&-2k(n-1)(a_3b_3+a_6b_6)S(U,V)\eta(X)\eta(W)\\
&+\varepsilon\left(b_4(ka_0+k(n-1)a_4+a_7r)-a_1(kb_0+k(n-1)b_4)\right)S(X,U)g(V,W)\\
&+\varepsilon\left(b_5(ka_0+k(n-1)a_4+a_7r)-a_1(-kb_0+k(n-1)b_5)\right)S(X,V)g(U,W)\\
&+\varepsilon b_6(ka_0+k(n-1)(a_4-a_1)+a_7r)S(X,W)g(U,V)\\
&-\varepsilon(kb_0+k(n-1)b_4)(ka_0+k(n-1)a_4+a_7r)g(X,U)g(V,W)\\
&-\varepsilon(-kb_0+k(n-1)b_5)(ka_0+k(n-1)a_4+a_7r)g(U,W)g(X,V)\\
&-\varepsilon k(n-1)b_6(ka_0+k(n-1)a_4+a_7r)g(X,W)g(U,V)\\
&-k(n-1)\left((b_2+b_3)(ka_0+k(n-1)a_4+a_7r)\right.\\
&\left.+(a_2+a_4)(-kb_0+k(n-1)(b_5+b_6)-b_7r)\right)g(X,U)\eta(V)\eta(W)\\
&-k(n-1)\left((b_1+b_3)(ka_0+k(n-1)a_4+a_7r)\right.\\
&\left.+(a_2+a_4)(kb_0+k(n-1)(b_4+b_6)+b_7r)\right)g(X,V)\eta(U)\eta(W)\\
&-\left((b_1+b_3)(-ka_0+k(n-1)(a_1+a_2)-a_7r)\right.\\
&\left.+(a_1+a_5)(kb_0+k(n-1)(b_4+b_6)+b_7r)\right)S(X,V)\eta(U)\eta(W)\\
&-\left((b_2+b_3)(-ka_0+k(n-1)(a_1+a_2)-a_7r)\right.\\
&\left.+(a_1+a_5)(kb_0+k(n-1)(b_5+b_6)+b_7r)\right)S(X,U)\eta(V)\eta(W)\\
&-\left((b_1+b_2)(-ka_0+k(n-1)(a_1+a_2)-a_7r)\right.\\
&\left.+k(n-1)(b_4+b_5)(a_1+a_5)\right)S(X,W)\eta(U)\eta(V)\\
&-k(n-1)\left(k(n-1)(b_4+b_5)(a_2+a_4)\right.\\
&\left.+(b_1+b_2)(ka_0+k(n-1)a_4+a_7r)\right)g(X,W)\eta(U)\eta(V)\\
=\;&L_g(\varepsilon b_0R(U,V,W,X)+\varepsilon b_4S(X,U)g(V,W)+\varepsilon b_5S(X,V)g(U,W)\\
&+\varepsilon b_6S(X,W)g(U,V)-\varepsilon k(n-1)b_6g(U,V)g(X,W)\\
&-k(n-1)(b_2+b_3)g(X,U)\eta(V)\eta(W)\\
&-k(n-1)(b_1+b_3)g(X,V)\eta(U)\eta(W)\\
&-k(n-1)(b_1+b_2)g(X,W)\eta(U)\eta(V)\\
&+(b_2+b_3)S(X,U)\eta(V)\eta(W)\\
&+(b_1+b_3)S(X,V)\eta(U)\eta(W)\\
&+(b_1+b_2)S(X,W)\eta(U)\eta(V)\\
&-\varepsilon(-kb_0+k(n-1)b_5)g(U,W)g(X,V)\\
&-\varepsilon(kb_0+k(n-1)b_4)g(V,W)g(X,U)).
\end{aligned}
\tag{4.3}
$$

Proof. Let M be an n-dimensional $(\mathcal{T}_a, \mathcal{T}_b)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then

$$\mathcal{T}_a(Z, X) \cdot \mathcal{T}_b(U, V)W = L_g Q(g, \mathcal{T}_b)(U, V, W; Z, X). \tag{4.4}$$

Taking $Z = \xi$ in (4.4), we get

$$\mathcal{T}_a(\xi, X) \cdot \mathcal{T}_b(U, V)W = L_g Q(g, \mathcal{T}_b)(U, V, W; \xi, X),$$

which gives

$$\begin{aligned} & \mathcal{T}_a(\xi, X)\mathcal{T}_b(U, V)W - \mathcal{T}_b(\mathcal{T}_a(\xi, X)U, V)W \\ & - \mathcal{T}_b(U, \mathcal{T}_a(\xi, X)V)W - \mathcal{T}_b(U, V)\mathcal{T}_a(\xi, X)W \\ = \; & L_g((\xi \wedge X)\mathcal{T}_b(U, V)W - \mathcal{T}_b((\xi \wedge X)U, V)W \\ & - \mathcal{T}_b(U, (\xi \wedge X)V)W - \mathcal{T}_b(U, V)(\xi \wedge X)W), \end{aligned}$$

that is,

$$\begin{aligned} & \mathcal{T}_a(\xi, X)\mathcal{T}_b(U, V)W - \mathcal{T}_b(\mathcal{T}_a(\xi, X)U, V)W \\ & - \mathcal{T}_b(U, \mathcal{T}_a(\xi, X)V)W - \mathcal{T}_b(U, V)\mathcal{T}_a(\xi, X)W \\ = \; & L_g(\mathcal{T}_b(U, V, W, X)\xi - \mathcal{T}_b(U, V, W, \xi)X \\ & - g(X, U)\mathcal{T}_b(\xi, V)W + \varepsilon\eta(U)\mathcal{T}_b(X, V)W \\ & - g(X, V)\mathcal{T}_b(U, \xi)W + \varepsilon\eta(V)\mathcal{T}_b(U, X)W \\ & - g(X, W)\mathcal{T}_b(U, V)\xi + \varepsilon\eta(W)\mathcal{T}_b(U, V)X). \end{aligned} \tag{4.5}$$

Taking the inner product of (4.5) with ξ, we get

$$\begin{aligned} & \mathcal{T}_a(\xi, X, \mathcal{T}_b(U, V)W, \xi) - \mathcal{T}_b(\mathcal{T}_a(\xi, X)U, V, W, \xi) \\ & - \mathcal{T}_b(U, \mathcal{T}_a(\xi, X)V, W, \xi) - \mathcal{T}_b(U, V, \mathcal{T}_a(\xi, X)W, \xi) \\ = \; & L_g(\varepsilon\mathcal{T}_b(U, V, W, X) - \varepsilon\eta(X)\mathcal{T}_b(U, V, W, \xi) \\ & - g(X, U)\mathcal{T}_b(\xi, V, W, \xi) + \varepsilon\eta(U)\mathcal{T}_b(X, V, W, \xi) \\ & - g(X, V)\mathcal{T}_b(U, \xi, W, \xi) + \varepsilon\eta(V)\mathcal{T}_b(U, X, W, \xi) \\ & - g(X, W)\mathcal{T}_b(U, V, \xi, \xi) + \varepsilon\eta(W)\mathcal{T}_b(U, V, X, \xi)). \end{aligned} \tag{4.6}$$

By using (3.29),…,(3.34) in (4.6), we get (4.3). □

Theorem 4.2. *Let M be an n-dimensional $(\mathcal{T}_a, \mathcal{T}_b)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then*

$$\begin{aligned} & -\varepsilon\left\{a_5b_0 + na_5b_1 + a_5b_2 + a_5b_6 + a_5b_3 + a_5b_5\right\} S^2(V, W) \\ & + (\{(nb_1 + b_2 + b_3 + b_5 + b_6 + b_0)(\varepsilon k a_0 + \varepsilon b_7 r) \\ & \qquad - \varepsilon k(n-1)(2a_5b_6 + a_2b_3 + a_1b_6 + a_1b_3 + a_1b_5 \\ & \qquad + a_1b_1 + a_1b_2 + a_2b_2 + a_2b_6 + na_2b_1 + a_1b_0 + a_2b_0)\} \\ & - \varepsilon(n-1)a_1b_7r - \varepsilon na_5b_7r - \varepsilon b_4a_5r - \varepsilon a_1b_4r)S(V, W) \\ & + \{-\varepsilon k(n-1)(nb_1 + b_2 + b_3 + b_5 + b_6 + b_0)(a_7r + ka_0 + k(n-1)a_4) \\ & \qquad - \varepsilon k(n-1)r((n-1)b_7a_2 + (n-1)b_7a_4 + a_2b_4 + a_4b_4)\}\, g(V, W) \\ & + (a_1 + a_2 + 2a_3 + a_4 + a_5 + 2a_6)\left\{-k^2(n-1)^2(nb_1 + b_2 + b_3 + b_5 + b_6 + b_0)\right. \\ & \qquad \left. - k(n-1)^2b_7r - k(n-1)b_4r\right\} \eta(V)\eta(W) \\ = \; & L_g(\varepsilon(b_0 + b_5 + b_6)S(V, W) + \varepsilon(b_4r - k(n-1)(b_0 + nb_4 + b_5 + b_6))g(V, W) \\ & + (b_2 + b_3)(r - kn(n-1))\eta(V)\eta(W)). \end{aligned} \tag{4.7}$$

Theorem 4.3. *Let M be an n-dimensional $\mathcal{T}_a$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$\begin{aligned}
(4.8)\quad & \varepsilon a_0 kR(U,V,W,X) + \varepsilon k a_4 S(X,U)g(V,W) + \varepsilon k a_5 S(X,V)g(U,W) \\
& + \varepsilon k a_6 S(X,W)g(U,V) - \varepsilon k^2(n-1)a_6 g(X,W)g(U,V) \\
& - \varepsilon k(ka_0 + k(n-1)a_4)g(V,W)g(X,U) \\
& - \varepsilon k(-ka_0 + k(n-1)a_5)g(U,W)g(X,V) \\
& - k^2(n-1)(a_2+a_3)g(X,U)\eta(V)\eta(W) \\
& - k^2(n-1)(a_1+a_3)g(X,V)\eta(U)\eta(W) \\
& - k^2(n-1)(a_1+a_2)g(X,W)\eta(U)\eta(V) \\
& + k(a_2+a_3)S(X,U)\eta(V)\eta(W) + k(a_1+a_3)S(X,V)\eta(U)\eta(W) \\
& + k(a_1+a_2)S(X,W)\eta(U)\eta(V) \\
= \; & L_g(\varepsilon a_0 R(U,V,W,X) + \varepsilon a_4 S(X,U)g(V,W) + \varepsilon a_5 S(X,V)g(U,W) \\
& + \varepsilon a_6 S(X,W)g(U,V) - \varepsilon k(n-1)a_6 g(U,V)g(X,W) \\
& - k(n-1)(a_2+a_3)g(X,U)\eta(V)\eta(W) \\
& - k(n-1)(a_1+a_3)g(X,V)\eta(U)\eta(W) \\
& - k(n-1)(a_1+a_2)g(X,W)\eta(U)\eta(V) \\
& + (a_2+a_3)S(X,U)\eta(V)\eta(W) \\
& + (a_1+a_3)S(X,V)\eta(U)\eta(W) \\
& + (a_1+a_2)S(X,W)\eta(U)\eta(V) \\
& - \varepsilon(-ka_0 + k(n-1)a_5)g(U,W)g(X,V) \\
& - \varepsilon(ka_0 + k(n-1)a_4)g(V,W)g(X,U)).
\end{aligned}$$

Remark 4.1. Here two cases arise. First is that when $L_g = 0$. In this case, it is $\mathcal{T}_a$-semisymmetric. We exclude this case, because it is already studied in [31]. And the second case is that when $L_g \neq 0$. In the following Theorem, we consider the result for $L_g \neq 0$.

Theorem 4.4. *Let M be an n-dimensional $\mathcal{T}_a$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold such that $a_0 + a_5 + a_6 \neq 0$.*

(1) *If $a_0 + a_2 + a_3 + na_4 + a_5 + a_6 \neq 0$, then either it is Einstein manifold or $L_g = k$.*
(2) *If $a_0 + a_2 + a_3 + na_4 + a_5 + a_6 = 0$, then either it is η-Einstein manifold or $L_g = k$.*

Proof. Let M be an n-dimensional $\mathcal{T}_a$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. On contracting (4.8), we get

$$\begin{aligned}
& k(\varepsilon(a_0+a_5+a_6)S(V,W) + \varepsilon(a_4 r - k(n-1)(a_0+na_4+a_5+a_6))g(V,W) \\
& + (a_2+a_3)(r - kn(n-1))\eta(V)\eta(W)) \\
= \; & L_g(\varepsilon(a_0+a_5+a_6)S(V,W) + \varepsilon(a_4 r - k(n-1)(a_0+na_4+a_5+a_6))g(V,W) \\
& + (a_2+a_3)(r - kn(n-1))\eta(V)\eta(W)),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
(4.9)\quad & (L_g - k)(\varepsilon(a_0+a_5+a_6)S(V,W) \\
& + \varepsilon(a_4 r - k(n-1)(a_0+na_4+a_5+a_6))g(V,W) \\
& + (a_2+a_3)(r - kn(n-1))\eta(V)\eta(W)).
\end{aligned}$$

On contracting above equation, we get

$$(L_g - k)(a_0 + a_2 + a_3 + na_4 + a_5 + a_6)(r - kn(n-1)).$$

Case 1. If $a_0 + a_2 + a_3 + na_4 + a_5 + a_6 \neq 0$, then either $L_g = k$ or $r = kn(n-1)$. If $r = kn(n-1)$, then from (4.9), we get

$$S = k(n-1)g.$$

Case 2. If $a_0 + a_2 + a_3 + na_4 + a_5 + a_6 = 0$, then by (4.9), we get either $L_g = k$ or

$$\begin{aligned}-\varepsilon(a_0 + a_5 + a_6)S(V,W) &= \varepsilon(a_4 r - k(n-1)(a_0 + na_4 + a_5 + a_6))g(V,W) \\ &\quad +(a_2 + a_3)(r - kn(n-1))\eta(V)\eta(W)).\end{aligned}$$

This proves the result. □

Corollary 4.1. *An n-dimensional $\mathcal{T}_a$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold is of the form $R \cdot \mathcal{T}_a = kQ(g, \mathcal{T}_a)$.*

In view of Theorem 4.4, we have the following

Corollary 4.2. *Let M be an n-dimensional $\mathcal{T}_a$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold such that*

$$\mathcal{T}_a \in \{R, \mathcal{V}, \mathcal{P}, \mathcal{M}, \mathcal{W}_0, \mathcal{W}_0^*, \mathcal{W}_1, \mathcal{W}_1^*, \mathcal{W}_3, \ldots, \mathcal{W}_8\}$$

Then we have the following table:

M	$L_g =$	$S =$
$N(k)$-contact metric	k	$k(n-1)g$
Sasakian	1	$(n-1)g$
Kenmotsu	-1	$-(n-1)g$
(ε)-Sasakian	ε	$\varepsilon(n-1)g$
para-Sasakian	-1	$-(n-1)g$
(ε)-para-Sasakian	$-\varepsilon$	$-\varepsilon(n-1)g$

Corollary 4.3. [19] *Let M be an n-dimensional, $n \geq 3$, Kenmotsu manifold. If M is pseudosymmetric then either it is locally isometric to the hyperbolic space $H^n(-1)$ or $L_g = -1$ holds on M.*

Corollary 4.4. [19] *Every Kenmotsu manifold M, $n \geq 3$, is a pseudosymmetric manifold of the form $R \cdot R = -Q(g, R)$.*

Corollary 4.5. *Let M be an n-dimensional quasi-conformal pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold such that $a_0 - a_1 \neq 0$ and $a_0 + (n-2)a_1 \neq 0$. Then we have the following table:*

M	$L_g =$	$S =$
$N(k)$-contact metric	k	$k(n-1)g$
Sasakian	1	$(n-1)g$
Kenmotsu	-1	$-(n-1)g$
(ε)-Sasakian	ε	$\varepsilon(n-1)g$
para-Sasakian	-1	$-(n-1)g$
(ε)-para-Sasakian	$-\varepsilon$	$-\varepsilon(n-1)g$

Corollary 4.6. *Let M be an n-dimensional pseudo-projective pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold such that $a_0 \neq 0$ and $a_0 - a_1 \neq 0$. Then we have the following table:*

M	$L_g =$	$S =$
$N(k)$-contact metric	k	$k(n-1)g$
Sasakian	1	$(n-1)g$
Kenmotsu	-1	$-(n-1)g$
(ε)-Sasakian	ε	$\varepsilon(n-1)g$
para-Sasakian	-1	$-(n-1)g$
(ε)-para-Sasakian	$-\varepsilon$	$-\varepsilon(n-1)g$

Corollary 4.7. *Let M be an n-dimensional Weyl pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$L_g =$	$S =$
$N(k)$-contact metric	k	$S = \left(\frac{r}{n-1} - k\right) g + \left(nk - \frac{r}{n-1}\right) \eta \otimes \eta$
Sasakian	1	$S = \left(\frac{r}{n-1} - 1\right) g + \left(n - \frac{r}{n-1}\right) \eta \otimes \eta$
Kenmotsu [19]	-1	$S = \left(\frac{r}{n-1} + 1\right) g - \left(n + \frac{r}{n-1}\right) \eta \otimes \eta$
(ε)-Sasakian	ε	$S = \left(\frac{r}{n-1} - \varepsilon\right) g + \varepsilon \left(n\varepsilon - \frac{r}{n-1}\right) \eta \otimes \eta$
para-Sasakian [18]	-1	$S = \left(\frac{r}{n-1} + 1\right) g - \left(n + \frac{r}{n-1}\right) \eta \otimes \eta$
(ε)-para-Sasakian	$-\varepsilon$	$S = \left(\frac{r}{n-1} + \varepsilon\right) g - \varepsilon \left(n\varepsilon + \frac{r}{n-1}\right) \eta \otimes \eta$

Corollary 4.8. *Let M be an n-dimensional conharmonic pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$L_g =$	$S =$
$N(k)$-contact metric	k	$S = \left(\frac{r}{n-1} - k\right) g + \left(nk - \frac{r}{n-1}\right) \eta \otimes \eta$
Sasakian	1	$S = \left(\frac{r}{n-1} - 1\right) g + \left(n - \frac{r}{n-1}\right) \eta \otimes \eta$
Kenmotsu	-1	$S = \left(\frac{r}{n-1} + 1\right) g - \left(n + \frac{r}{n-1}\right) \eta \otimes \eta$
(ε)-Sasakian	ε	$S = \left(\frac{r}{n-1} - \varepsilon\right) g + \varepsilon \left(n\varepsilon - \frac{r}{n-1}\right) \eta \otimes \eta$
para-Sasakian	-1	$S = \left(\frac{r}{n-1} + 1\right) g - \left(n + \frac{r}{n-1}\right) \eta \otimes \eta$
(ε)-para-Sasakian	$-\varepsilon$	$S = \left(\frac{r}{n-1} + \varepsilon\right) g - \varepsilon \left(n\varepsilon + \frac{r}{n-1}\right) \eta \otimes \eta$

Corollary 4.9. *Let M be an n-dimensional $\mathcal{W}_2$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$L_g =$	$S =$
$N(k)$-contact metric	k	$\frac{r}{n} g$
Sasakian	1	$\frac{r}{n} g$
Kenmotsu	-1	$\frac{r}{n} g$
(ε)-Sasakian	ε	$\frac{r}{n} g$
para-Sasakian	-1	$\frac{r}{n} g$
(ε)-para-Sasakian	$-\varepsilon$	$\frac{r}{n} g$

Corollary 4.10. *Let M be an n-dimensional $\mathcal{W}_9$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$L_g =$	$S =$
$N(k)$-*contact metric*	k	$S = \left(\frac{r}{n-1} - k\right) g + \left(nk - \frac{r}{n-1}\right) \eta \otimes \eta$
Sasakian	1	$S = \left(\frac{r}{n-1} - 1\right) g + \left(n - \frac{r}{n-1}\right) \eta \otimes \eta$
Kenmotsu	-1	$S = \left(\frac{r}{n-1} + 1\right) g - \left(n + \frac{r}{n-1}\right) \eta \otimes \eta$
(ε)-*Sasakian*	ε	$S = \left(\frac{r}{n-1} - \varepsilon\right) g + \varepsilon \left(n\varepsilon - \frac{r}{n-1}\right) \eta \otimes \eta$
para-Sasakian	-1	$S = \left(\frac{r}{n-1} + 1\right) g - \left(n + \frac{r}{n-1}\right) \eta \otimes \eta$
(ε)-*para-Sasakian*	$-\varepsilon$	$S = \left(\frac{r}{n-1} + \varepsilon\right) g - \varepsilon \left(n\varepsilon + \frac{r}{n-1}\right) \eta \otimes \eta$

5. $(\mathcal{T}_a, \mathcal{T}_b, S^\ell)$-PSEUDOSYMMETRY

Definition 5.1. A semi-Riemannian manifold is said to be $(\mathcal{T}_a, \mathcal{T}_b, S^\ell)$-pseudosymmetric if it satisfies

$$\mathcal{T}_a \cdot \mathcal{T}_b = L_{S^\ell} Q(S^\ell, \mathcal{T}_b), \tag{5.1}$$

where L_{S^ℓ} is some smooth function on M. In particular, it is said to be $(R, \mathcal{T}_a, S^\ell)$-pseudosymmetric if

$$R \cdot \mathcal{T}_a = L_{S^\ell} Q(S^\ell, \mathcal{T}_a) \tag{5.2}$$

holds on the set $\mathcal{U}_{S^\ell} = \left\{x \in M : Q(S^\ell, \mathcal{T}_a) \neq 0\right\}$, where L_{S^ℓ} is some smooth function on $\mathcal{U}_{S^\ell}$.

For $\ell = 1$, we can give the following definition.

Definition 5.2. A semi-Riemannian manifold is called $(\mathcal{T}_a, \mathcal{T}_b, S)$-pseudosymmetric if it satisfies

$$\mathcal{T}_a \cdot \mathcal{T}_b = L_S Q(S, \mathcal{T}_b), \tag{5.3}$$

where L_S is some smooth function on M. In particular, it is said to be $(R, \mathcal{T}_a, S)$-pseudosymmetric if

$$R \cdot \mathcal{T}_a = L_S Q(S, \mathcal{T}_a) \tag{5.4}$$

holds on the set $\mathcal{U}_S = \{x \in M : Q(S, \mathcal{T}_a) \neq 0\}$, where L_S is some smooth function on $\mathcal{U}_S$.

Remark 5.1. A semi-Riemannian manifold is said to be (R, R, S)-pseudosymmetric or in short, Ricci-generalized pseudosymmetric if

$$R \cdot R = L_S Q(S, R)$$

holds on the set $\mathcal{U}_S = \{x \in M : Q(S, R) \neq 0\}$, where L_S is some smooth function on $\mathcal{U}_S$. It is known [5] that every 3-dimensional semi-Riemannian manifold is Ricci-generalized pseudosymmetric along with $L_S = 1$, that is, $R \cdot R = Q(S, R)$.

Theorem 5.1. *Let M be an n-dimensional $(\mathcal{T}_a,\mathcal{T}_b,S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$
\begin{aligned}
(5.5)\qquad & \varepsilon b_0(ka_0+\varepsilon k(n-1)a_4+a_7r)R(U,V,W,X)+\varepsilon a_1b_0S(X,R(U,V)W)\\
&-2k(n-1)a_3(kb_0+k(n-1)b_4+b_7r)\eta(X)\eta(U)g(V,W)\\
&-2k(n-1)a_3(-kb_0+k(n-1)b_5-b_7r)\eta(X)\eta(V)g(U,W)\\
&+\varepsilon a_1b_4S^2(X,U)g(V,W)+\varepsilon a_1b_5S^2(X,V)g(U,W)\\
&+\varepsilon a_1b_6S^2(X,W)g(U,V)-a_5(b_1+b_3)S^2(X,V)\eta(U)\eta(W)\\
&-a_5(b_1+b_2)S^2(X,W)\eta(U)\eta(V)-a_5(b_2+b_3)S^2(X,U)\eta(V)\eta(W)\\
&-2a_6b_1S^2(V,W)\eta(X)\eta(U)-2a_6b_2S^2(U,W)\eta(X)\eta(V)\\
&-2a_6b_3S^2(U,V)\eta(X)\eta(W)-2k^2(n-1)a_3b_6g(U,V)\eta(X)\eta(W)\\
&-2\left(k(n-1)a_3b_1+a_6(kb_0+k(n-1)b_4+b_7r)\right)\eta(X)\eta(U)S(V,W)\\
&-2\left(k(n-1)a_3b_2+a_6(-kb_0+k(n-1)b_5-b_7r)\right)\eta(X)\eta(V)S(U,W)\\
&-2k(n-1)(a_3b_3+a_6b_6)S(U,V)\eta(X)\eta(W)\\
&+\varepsilon\left(b_4(ka_0+k(n-1)a_4+a_7r)-a_1(kb_0+k(n-1)b_4)\right)S(X,U)g(V,W)\\
&+\varepsilon\left(b_5(ka_0+k(n-1)a_4+a_7r)-a_1(-kb_0+k(n-1)b_5)\right)S(X,V)g(U,W)\\
&+\varepsilon b_6(ka_0+k(n-1)(a_4-a_1)+a_7r)S(X,W)g(U,V)\\
&-\varepsilon(kb_0+k(n-1)b_4)(ka_0+k(n-1)a_4+a_7r)g(X,U)g(V,W)\\
&-\varepsilon(-kb_0+k(n-1)b_5)(ka_0+k(n-1)a_4+a_7r)g(U,W)g(X,V)\\
&-\varepsilon k(n-1)b_6(ka_0+k(n-1)a_4+a_7r)g(X,W)g(U,V)\\
&-k(n-1)\left((b_2+b_3)(ka_0+k(n-1)a_4+a_7r)\right.\\
&\left.+(a_2+a_4)(-kb_0+k(n-1)(b_5+b_6)-b_7r)\right)g(X,U)\eta(V)\eta(W)\\
&-k(n-1)\left((b_1+b_3)(ka_0+k(n-1)a_4+a_7r)\right.\\
&\left.+(a_2+a_4)(kb_0+k(n-1)(b_4+b_6)+b_7r)\right)g(X,V)\eta(U)\eta(W)\\
&-\left((b_1+b_3)(-ka_0+k(n-1)(a_1+a_2)-a_7r)\right.\\
&\left.+(a_1+a_5)(kb_0+k(n-1)(b_4+b_6)+b_7r)\right)S(X,V)\eta(U)\eta(W)\\
&-\left((b_2+b_3)(-ka_0+k(n-1)(a_1+a_2)-a_7r)\right.\\
&\left.+(a_1+a_5)(kb_0+k(n-1)(b_5+b_6)+b_7r)\right)S(X,U)\eta(V)\eta(W)\\
&-\left((b_1+b_2)(-ka_0+k(n-1)(a_1+a_2)-a_7r)\right.\\
&\left.+k(n-1)(b_4+b_5)(a_1+a_5)\right)S(X,W)\eta(U)\eta(V)\\
&-k(n-1)\left(k(n-1)(b_4+b_5)(a_2+a_4)\right.\\
&\left.+(b_1+b_2)(ka_0+k(n-1)a_4+a_7r)\right)g(X,W)\eta(U)\eta(V)\\
=\;& L_{S^\ell}(\varepsilon b_0S^\ell(R(U,V)W,X)+\varepsilon b_4S^{\ell+1}(X,U)g(V,W)+\varepsilon b_5S^{\ell+1}(X,V)g(U,W)\\
&+\varepsilon b_6S^{\ell+1}(X,W)g(U,V)-\varepsilon k(n-1)b_6S^\ell(X,W)g(U,V)\\
&+k^\ell(n-1)^\ell(-kb_0+k(n-1)(b_5+b_6)-b_7r)g(X,U)\eta(V)\eta(W)\\
&+k^\ell(n-1)^\ell(kb_0+k(n-1)(b_4+b_6)+b_7r)g(X,V)\eta(U)\eta(W)\\
&-(-kb_0+k(n-1)(b_2+b_3+b_5+b_6)-b_7r)S^\ell(X,U)\eta(V)\eta(W)\\
&-(kb_0+k(n-1)(b_1+b_3+b_4+b_6)+b_7r)S^\ell(X,V)\eta(U)\eta(W)\\
&-k(n-1)(b_1+b_2+b_4+b_5)S(X,W)\eta(U)\eta(V)\\
&+k^{\ell+1}(n-1)^{\ell+1}(b_4+b_5)g(X,W)\eta(U)\eta(V)\\
&-\varepsilon(-kb_0+k(n-1)b_5)S^\ell(X,V)g(U,W)\\
&-\varepsilon(kb_0+k(n-1)b_4)S^\ell(X,U)g(V,W)\\
&+k^\ell(n-1)^\ell(b_1+b_3)S(X,V)\eta(U)\eta(W)\\
&+k^\ell(n-1)^\ell(b_1+b_2)S(X,W)\eta(U)\eta(V)\\
&+k^\ell(n-1)^\ell(b_2+b_3)S(X,U)\eta(V)\eta(W)).
\end{aligned}
$$

Proof. Let M be an n-dimensional $(\mathcal{T}_a, \mathcal{T}_b, S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then

$$\mathcal{T}_a(Z,X)\cdot\mathcal{T}_b(U,V)W = L_{S^\ell}Q(S^\ell,\mathcal{T}_b)(U,V,W;Z,X). \tag{5.6}$$

Taking $Z=\xi$ in (5.6), we get

$$\mathcal{T}_a(\xi,X)\cdot\mathcal{T}_b(U,V)W = L_{S^\ell}Q(S^\ell,\mathcal{T}_b)(U,V,W;\xi,X),$$

which gives

$$\begin{aligned} & \mathcal{T}_a(\xi,X)\mathcal{T}_b(U,V)W - \mathcal{T}_b(\mathcal{T}_a(\xi,X)U,V)W \\ & \quad - \mathcal{T}_b(U,\mathcal{T}_a(\xi,X)V)W - \mathcal{T}_b(U,V)\mathcal{T}_a(\xi,X)W \\ = \; & L_{S^\ell}((\xi\wedge_{S^\ell}X)\mathcal{T}_b(U,V)W - \mathcal{T}_b((\xi\wedge_{S^\ell}X)U,V)W \\ & -\mathcal{T}_b(U,(\xi\wedge_{S^\ell}X)V)W - \mathcal{T}_b(U,V)(\xi\wedge_{S^\ell}X)W), \end{aligned}$$

that is,

$$\begin{aligned} & \mathcal{T}_a(\xi,X)\mathcal{T}_b(U,V)W - \mathcal{T}_b(\mathcal{T}_a(\xi,X)U,V)W \\ & \quad - \mathcal{T}_b(U,\mathcal{T}_a(\xi,X)V)W - \mathcal{T}_b(U,V)\mathcal{T}_a(\xi,X)W \\ = \; & L_{S^\ell}(S^\ell(X,\mathcal{T}_b(U,V)W)\xi - S^\ell(\xi,\mathcal{T}_b(U,V)W)X \\ & -S^\ell(X,U)\mathcal{T}_b(\xi,V)W + S^\ell(\xi,U)\mathcal{T}_b(X,V)W \\ & -S^\ell(X,V)\mathcal{T}_b(U,\xi)W + S^\ell(\xi,V)\mathcal{T}_b(U,X)W \\ & -S^\ell(X,W)\mathcal{T}_b(U,V)\xi + S^\ell(\xi,W)\mathcal{T}_b(U,V)X). \end{aligned} \tag{5.7}$$

Taking the inner product of (5.7) with ξ, we get

$$\begin{aligned} & \mathcal{T}_a(\xi,X,\mathcal{T}_b(U,V)W,\xi) - \mathcal{T}_b(\mathcal{T}_a(\xi,X)U,V,W,\xi) \\ & \quad - \mathcal{T}_b(U,\mathcal{T}_a(\xi,X)V,W,\xi) - \mathcal{T}_b(U,V,\mathcal{T}_a(\xi,X)W,\xi) \\ = \; & L_{S^\ell}(\varepsilon S^\ell(X,\mathcal{T}_b(U,V)W) - \varepsilon\eta(X)S^\ell(\xi,\mathcal{T}_b(U,V)W) \\ & -S^\ell(X,U)\mathcal{T}_b(\xi,V,W,\xi) + S^\ell(\xi,U)\mathcal{T}_b(X,V,W,\xi) \\ & -S^\ell(X,V)\mathcal{T}_b(U,\xi,W,\xi) + S^\ell(\xi,V)\mathcal{T}_b(U,X,W,\xi) \\ & -S^\ell(X,W)\mathcal{T}_b(U,V,\xi,\xi) + S^\ell(\xi,W)\mathcal{T}_b(U,V,X,\xi)). \end{aligned} \tag{5.8}$$

By using (3.29),. . . ,(3.34) in (5.8), we get (5.5). □

Corollary 5.1. *Let M be an n-dimensional $(\mathcal{T}_a,\mathcal{T}_b,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$
\begin{aligned}
(5.9)\qquad & \varepsilon b_0(ka_0+\varepsilon k(n-1)a_4+a_7r)R(U,V,W,X)+\varepsilon a_1b_0S(X,R(U,V)W)\\
& -2k(n-1)a_3(kb_0+k(n-1)b_4+b_7r)\eta(X)\eta(U)g(V,W)\\
& -2k(n-1)a_3(-kb_0+k(n-1)b_5-b_7r)\eta(X)\eta(V)g(U,W)\\
& +\varepsilon a_1b_4S^2(X,U)g(V,W)+\varepsilon a_1b_5S^2(X,V)g(U,W)\\
& +\varepsilon a_1b_6S^2(X,W)g(U,V)-a_5(b_1+b_3)S^2(X,V)\eta(U)\eta(W)\\
& -a_5(b_1+b_2)S^2(X,W)\eta(U)\eta(V)-a_5(b_2+b_3)S^2(X,U)\eta(V)\eta(W)\\
& -2a_6b_1S^2(V,W)\eta(X)\eta(U)-2a_6b_2S^2(U,W)\eta(X)\eta(V)\\
& -2a_6b_3S^2(U,V)\eta(X)\eta(W)-2k^2(n-1)a_3b_6g(U,V)\eta(X)\eta(W)\\
& -2\left(k(n-1)a_3b_1+a_6(kb_0+k(n-1)b_4+b_7r)\right)\eta(X)\eta(U)S(V,W)\\
& -2\left(k(n-1)a_3b_2+a_6(-kb_0+k(n-1)b_5-b_7r)\right)\eta(X)\eta(V)S(U,W)\\
& -2k(n-1)(a_3b_3+a_6b_6)S(U,V)\eta(X)\eta(W)\\
& +\varepsilon\left(b_4(ka_0+k(n-1)a_4+a_7r)-a_1(kb_0+k(n-1)b_4)\right)S(X,U)g(V,W)\\
& +\varepsilon\left(b_5(ka_0+k(n-1)a_4+a_7r)-a_1(-kb_0+k(n-1)b_5)\right)S(X,V)g(U,W)\\
& +\varepsilon b_6(ka_0+k(n-1)(a_4-a_1)+a_7r)S(X,W)g(U,V)\\
& -\varepsilon(kb_0+k(n-1)b_4)(ka_0+k(n-1)a_4+a_7r)g(X,U)g(V,W)\\
& -\varepsilon(-kb_0+k(n-1)b_5)(ka_0+k(n-1)a_4+a_7r)g(U,W)g(X,V)\\
& -\varepsilon k(n-1)b_6(ka_0+k(n-1)a_4+a_7r)g(X,W)g(U,V)\\
& -k(n-1)\left((b_2+b_3)(ka_0+k(n-1)a_4+a_7r)\right.\\
& \left.+(a_2+a_4)(-kb_0+k(n-1)(b_5+b_6)-b_7r)\right)g(X,U)\eta(V)\eta(W)\\
& -k(n-1)\left((b_1+b_3)(ka_0+k(n-1)a_4+a_7r)\right.\\
& \left.+(a_2+a_4)(kb_0+k(n-1)(b_4+b_6)+b_7r)\right)g(X,V)\eta(U)\eta(W)\\
& -\left((b_1+b_3)(-ka_0+k(n-1)(a_1+a_2)-a_7r)\right.\\
& \left.+(a_1+a_5)(kb_0+k(n-1)(b_4+b_6)+b_7r)\right)S(X,V)\eta(U)\eta(W)\\
& -\left((b_2+b_3)(-ka_0+k(n-1)(a_1+a_2)-a_7r)\right.\\
& \left.+(a_1+a_5)(kb_0+k(n-1)(b_5+b_6)+b_7r)\right)S(X,U)\eta(V)\eta(W)\\
& -\left((b_1+b_2)(-ka_0+k(n-1)(a_1+a_2)-a_7r)\right.\\
& \left.+k(n-1)(b_4+b_5)(a_1+a_5)\right)S(X,W)\eta(U)\eta(V)\\
& -k(n-1)\left(k(n-1)(b_4+b_5)(a_2+a_4)\right.\\
& \left.+(b_1+b_2)(ka_0+k(n-1)a_4+a_7r)\right)g(X,W)\eta(U)\eta(V)\\
= \;& L_S(\varepsilon b_0S(R(U,V)W,X)+\varepsilon b_4S^2(X,U)g(V,W)+\varepsilon b_5S^2(X,V)g(U,W)\\
& +\varepsilon b_6S^2(X,W)g(U,V)-\varepsilon k(n-1)b_6S(X,W)g(U,V)\\
& +k(n-1)(-kb_0+k(n-1)(b_5+b_6)-b_7r)g(X,U)\eta(V)\eta(W)\\
& +k(n-1)(kb_0+k(n-1)(b_4+b_6)b_7r)g(X,V)\eta(U)\eta(W)\\
& +k^2(n-1)^2(b_4+b_5)g(X,W)\eta(U)\eta(V)\\
& -(-kb_0+k(n-1)(b_5+b_6)-b_7r)S(X,U)\eta(V)\eta(W)\\
& -(kb_0+k(n-1)(b_4+b_6)+b_7r)S(X,V)\eta(U)\eta(W)\\
& -k(n-1)(b_4+b_5)S(X,W)\eta(U)\eta(V)-\varepsilon(-kb_0+k(n-1)b_5)S(X,V)g(U,W)\\
& -\varepsilon(kb_0+k(n-1)b_4)S(X,U)g(V,W)).
\end{aligned}
$$

Theorem 5.2. *Let M be an n-dimensional $(\mathcal{T}_a,\mathcal{T}_b,S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$
\begin{aligned}
&(a_1b_5 - a_5b_1 - a_5b_3)S^2(X,V)\eta(W)\\
&+(a_1b_6 - a_5b_1 - a_5b_2)S^2(X,W)\eta(V)\\
&-2a_6b_1S^2(V,W)\eta(X)\\
&+(b_5(ka_0 + k(n-1)(a_4 - a_1) + a_7r)\\
&-(b_1 + b_3)(-ka_0 + k(n-1)(a_1 + a_2) - a_7r)\\
&-(a_1 + a_5)(kb_0 + k(n-1)(b_4 + b_6) + b_7r))S(X,V)\eta(W)\\
&+(b_6(ka_0 + k(n-1)(a_4 - a_1) + a_7r)\\
&-(b_1 + b_2)(-ka_0 + k(n-1)(a_1 + a_2) - a_7r)\\
&-k(n-1)(a_1 + a_5)(b_4 + b_5))S(X,W)\eta(V)\\
&-2(k(n-1)a_3b_1 + a_6(kb_0 + k(n-1)b_4 + b_7r))S(V,W)\eta(X)\\
&-k(n-1)((b_1 + b_3 + b_5)(ka_0 + k(n-1)a_4 + a_7r)\\
&+(a_2 + a_4)(kb_0 + k(n-1)(b_4 + b_6) + b_7r))g(X,V)\eta(W)\\
&-k(n-1)((b_1 + b_2 + b_6)(ka_0 + k(n-1)a_4 + a_7r)\\
&+k(n-1)(a_2 + a_4)(b_4 + b_5))g(X,W)\eta(V)\\
&-2k(n-1)a_3(kb_0 + k(n-1)b_4 + b_7r)g(V,W)\eta(X)\\
&-\varepsilon k(n-1)((kb_0 + b_7r)(a_1 + a_2 + a_4 + a_5)\\
&+k(n-1)(b_2 + b_3 + b_5 + b_6)\times\\
&(a_1 + a_2 + 2a_3 + a_4 + a_5 + 2a_6))\eta(X)\eta(V)\eta(W)\\
&= L_{S^\ell}(-(kb_0 + k(n-1)(b_1 + b_3 + b_4 + b_5 + b_6) + b_7r)S^\ell(X,V)\eta(W)\\
&+k^\ell(n-1)^\ell(kb_0 + k(n-1)(b_4 + b_6) + b_7r)g(X,V)\eta(W)\\
&+k^{\ell+1}(n-1)^{\ell+1}(b_4 + b_5)g(X,W)\eta(V)\\
&-k(n-1)(b_1 + b_2 + b_4 + b_5 + b_6)S^\ell(X,W)\eta(V)\\
&+b_5S^{\ell+1}(X,V)\eta(W) + b_6S^{\ell+1}(X,W)\eta(V))\\
&+k^\ell(n-1)^\ell(b_1 + b_3)S(X,V)\eta(W)\\
&+k^\ell(n-1)^\ell(b_1 + b_2)S(X,W)\eta(V)).
\end{aligned}
$$

Corollary 5.2. *Let M be an n-dimensional $(\mathcal{T}_a, \mathcal{T}_b, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then*

$$
\begin{aligned}
&(a_1b_5 - a_5b_1 - a_5b_3)S^2(X,V)\eta(W)\\
&+(a_1b_6 - a_5b_1 - a_5b_2)S^2(X,W)\eta(V) - 2a_6b_1S^2(V,W)\eta(X)\\
&+(b_5(ka_0 + k(n-1)(a_4 - a_1) + a_7r)\\
&-(b_1 + b_3)(-ka_0 + k(n-1)(a_1 + a_2) - a_7r)\\
&-(a_1 + a_5)(kb_0 + k(n-1)(b_4 + b_6) + b_7r))S(X,V)\eta(W)\\
&+(b_6(ka_0 + k(n-1)(a_4 - a_1) + a_7r)\\
&-(b_1 + b_2)(-ka_0 + k(n-1)(a_1 + a_2) - a_7r)\\
&-k(n-1)(a_1 + a_5)(b_4 + b_5))S(X,W)\eta(V)\\
&-2(k(n-1)a_3b_1 + a_6(kb_0 + k(n-1)b_4 + b_7r))S(V,W)\eta(X)
\end{aligned}
\tag{5.10}
$$

$$
\begin{aligned}
&-k(n-1)((b_1+b_3+b_5)(ka_0+k(n-1)a_4+a_7r)\\
&+(a_2+a_4)(kb_0+k(n-1)(b_4+b_6)+b_7r))g(X,V)\eta(W)\\
&-k(n-1)((b_1+b_2+b_6)(ka_0+k(n-1)a_4+a_7r)\\
&+k(n-1)(a_2+a_4)(b_4+b_5))g(X,W)\eta(V)\\
&-2k(n-1)a_3(kb_0+k(n-1)b_4+b_7r)g(V,W)\eta(X)\\
&-\varepsilon k(n-1)((kb_0+b_7r)(a_1+a_2+a_4+a_5)\\
&+k(n-1)(b_2+b_3+b_5+b_6)\times\\
&(a_1+a_2+2a_3+a_4+a_5+2a_6))\eta(X)\eta(V)\eta(W)\\
\\
= \;& L_S(-(kb_0+k(n-1)(b_4+b_5+b_6)+b_7r)S(X,V)\eta(W)\\
&+k(n-1)(kb_0+k(n-1)(b_4+b_6)+b_7r)g(X,V)\eta(W)\\
&+k^2(n-1)^2(b_4+b_5)g(X,W)\eta(V)\\
&-k(n-1)(b_4+b_5+b_6)S(X,W)\eta(V)\\
&+b_5S^2(X,V)\eta(W)+b_6S^2(X,W)\eta(V)).
\end{aligned}
$$

Theorem 5.3. *Let M be an n-dimensional $(R,\mathcal{T}_a,S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$
\begin{aligned}
&k(a_1+a_3+a_5)\,(S(X,V)-\,k(n-1)g(X,V))\,\eta(W)\\
+\;&k(a_1+a_2+a_6)\,(S(X,W)-\,k(n-1)g(X,W))\,\eta(V)\\
=\;&L_{S^\ell}(-(ka_0+k(n-1)(a_1+a_3+a_4+a_5+a_6)+a_7r)S^\ell(X,V)\eta(W)\\
&+k^\ell(n-1)^\ell(ka_0+k(n-1)(a_4+a_6)+a_7r)g(X,V)\eta(W)\\
&+k^{\ell+1}(n-1)^{\ell+1}(a_4+a_5)g(X,W)\eta(V)\\
&-k(n-1)(a_1+a_2+a_4+a_5+a_6)S^\ell(X,W)\eta(V)\\
&+a_5S^{\ell+1}(X,V)\eta(W)+a_6S^{\ell+1}(X,W)\eta(V)\\
&+k^\ell(n-1)^\ell(a_1+a_3)S(X,V)\eta(W)\\
&+k^\ell(n-1)^\ell(a_1+a_2)S(X,W)\eta(V)).
\end{aligned}
\tag{5.11}
$$

Corollary 5.3. *Let M be an n-dimensional $(R,\mathcal{T}_a,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$
\begin{aligned}
&k(a_1+a_3+a_5)\,(S(X,V)-\,k(n-1)g(X,V))\,\eta(W)\\
+\;&k(a_1+a_2+a_6)\,(S(X,W)-\,k(n-1)g(X,W))\,\eta(V)\\
=\;&L_S(-(ka_0+k(n-1)(a_4+a_5+a_6)+a_7r)S(X,V)\eta(W)\\
&+k(n-1)(ka_0+k(n-1)(a_4+a_6)+a_7r)g(X,V)\eta(W)\\
&+k^2(n-1)^2(a_4+a_5)g(X,W)\eta(V)\\
&-k(n-1)(a_4+a_5+a_6)S(X,W)\eta(V)\\
&+a_5S^2(X,V)\eta(W)+a_6S^2(X,W)\eta(V)).
\end{aligned}
\tag{5.12}
$$

Corollary 5.4. *Let M be an n-dimensional (R,R,S^ℓ)-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not semisymmetric, then*

$$S^\ell = k^\ell(n-1)^\ell g$$

and $L_{S^\ell} = \dfrac{1}{k^{\ell-1}(n-1)^\ell}$. *Consequently, we have the following:*

M	$L_{S^\ell} =$	$S^\ell =$
$N(k)$-*contact metric*	$\dfrac{1}{k^{\ell-1}(n-1)^\ell}$	$k^\ell(n-1)^\ell g$
Sasakian	$\dfrac{1}{(n-1)^\ell}$	$(n-1)^\ell g$
Kenmotsu	$\dfrac{1}{(-1)^{\ell-1}(n-1)^\ell}$	$(-1)^\ell(n-1)^\ell g$
(ε)-*Sasakian*	$\dfrac{1}{(\varepsilon)^{\ell-1}(n-1)^\ell}$	$(\varepsilon)^\ell(n-1)^\ell g$
para-Sasakian	$\dfrac{1}{(-1)^{\ell-1}(n-1)^\ell}$	$(-1)^\ell(n-1)^\ell g$
(ε)-*para-Sasakian*	$\dfrac{1}{(-\varepsilon)^{\ell-1}(n-1)^\ell}$	$(-\varepsilon)^\ell(n-1)^\ell g$

Proof. Let M be an n-dimensional (R, R, S^ℓ)-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold, that is

$$R \cdot R = L_{S^\ell} Q(S^\ell, R) \tag{5.13}$$

holds on M. By putting the value for R in (5.11), we get

$$-kL_{S^\ell}\left(S^\ell(X, V) - k^\ell(n-1)^\ell g(X, V)\right)\eta(W) = 0. \tag{5.14}$$

Putting $W = \xi$ in (5.14), we get

$$-kL_{S^\ell}\left(S^\ell(X, V) - k^\ell(n-1)^\ell g(X, V)\right) = 0. \tag{5.15}$$

Since M is not semisymmetric $L_{S^\ell} \neq 0$. Therefore from (5.15), we have

$$S^\ell(X, V) = k^\ell(n-1)^\ell g(X, V).$$

So putting $S^\ell = k^\ell(n-1)^\ell g$ in (5.13), we get

$$R \cdot R = k^\ell(n-1)^\ell L_{S^\ell} Q(g, R),$$

which is the condition of pseudosymmetric manifold. By comparison with the result of pseudosymmetric manifold (Corollary 4.2), we get $L_{S^\ell} = \dfrac{1}{k^{\ell-1}(n-1)^\ell}$. This proves the result. □

Corollary 5.5. *Let M be an n-dimensional Ricci-generalized pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. If M is not semisymmetric, then M is an Einstein manifold with scalar curvature $kn(n-1)$ and $L_S = \dfrac{1}{n-1}$. Consequently, we have the following:*

M	$L_S =$	$S =$
$N(k)$-*contact metric*	$\dfrac{1}{n-1}$	$k(n-1)g$
Sasakian	$\dfrac{1}{n-1}$	$(n-1)g$
Kenmotsu [19]	$\dfrac{1}{n-1}$	$-(n-1)g$
(ε)-*Sasakian*	$\dfrac{1}{n-1}$	$\varepsilon(n-1)g$
para-Sasakian	$\dfrac{1}{n-1}$	$-(n-1)g$
(ε)-*para-Sasakian*	$\dfrac{1}{n-1}$	$-\varepsilon(n-1)g$

Corollary 5.6. *Let M be an n-dimensional $(R,\mathcal{C}_*,S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not quasi-conformal semisymmetric, then*

$$S^{\ell+1} = \left(\left(\frac{r}{n(n-1)}-k\right)\frac{a_0}{a_1}+\left(\frac{2r}{n}-k(n-1)\right)\right)\left(S^\ell-k^\ell(n-1)^\ell g\right) +k^\ell(n-1)^\ell S.$$

Consequently, we have the following:

M	$S^{\ell+1}=$
$N(k)$-contact metric	$\left(\left(\frac{r}{n(n-1)}-k\right)\frac{a_0}{a_1}+\left(\frac{2r}{n}-k(n-1)\right)\right)\left(S^\ell-k^\ell(n-1)^\ell g\right)$ $+k^\ell(n-1)^\ell S$
Sasakian	$\left(\left(\frac{r}{n(n-1)}-1\right)\frac{a_0}{a_1}+\left(\frac{2r}{n}-1(n-1)\right)\right)\left(S^\ell-(n-1)^\ell g\right)$ $+(n-1)^\ell S$
Kenmotsu	$\left(\left(\frac{r}{n(n-1)}+1\right)\frac{a_0}{a_1}+\left(\frac{2r}{n}+(n-1)\right)\right)\left(S^\ell-(-1)^\ell(n-1)^\ell g\right)$ $+(-1)^\ell(n-1)^\ell S$
(ε)-Sasakian	$\left(\left(\frac{r}{n(n-1)}-\varepsilon\right)\frac{a_0}{a_1}+\left(\frac{2r}{n}-\varepsilon(n-1)\right)\right)\left(S^\ell-(\varepsilon)^\ell(n-1)^\ell g\right)$ $+(\varepsilon)^\ell(n-1)^\ell S$
para-Sasakian	$\left(\left(\frac{r}{n(n-1)}+1\right)\frac{a_0}{a_1}+\left(\frac{2r}{n}+(n-1)\right)\right)\left(S^\ell-(-1)^\ell(n-1)^\ell g\right)$ $+(-1)^\ell(n-1)^\ell S$
(ε)-para-Sasakian	$\left(\left(\frac{r}{n(n-1)}+\varepsilon\right)\frac{a_0}{a_1}+\left(\frac{2r}{n}+\varepsilon(n-1)\right)\right)\left(S^\ell-(-\varepsilon)^\ell(n-1)^\ell g\right)$ $+(-\varepsilon)^\ell(n-1)^\ell S$

Corollary 5.7. *Let M be an n-dimensional $(R,\mathcal{C}_*,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not quasi-conformal-semisymmetric, then*

$$S^2 = \left(\left(\frac{r}{n(n-1)}-k\right)\frac{a_0}{a_1}+\frac{2r}{n}\right)S -k(n-1)\left(\left(\frac{r}{n(n-1)}-k\right)\frac{a_0}{a_1}+\left(\frac{2r}{n}-k(n-1)\right)\right)g.$$

Consequently, we have the following:

M	$S^2=$
$N(k)$-*contact metric*	$\left(\left(\frac{r}{n(n-1)}-k\right)\frac{a_0}{a_1}+\frac{2r}{n}\right)S$ $-k(n-1)\left(\left(\frac{r}{n(n-1)}-k\right)\frac{a_0}{a_1}+\left(\frac{2r}{n}-k(n-1)\right)\right)g$
Sasakian	$\left(\left(\frac{r}{n(n-1)}-1\right)\frac{a_0}{a_1}+\frac{2r}{n}\right)S$ $-(n-1)\left(\left(\frac{r}{n(n-1)}-1\right)\frac{a_0}{a_1}+\left(\frac{2r}{n}-(n-1)\right)\right)g$
Kenmotsu	$\left(\left(\frac{r}{n(n-1)}+1\right)\frac{a_0}{a_1}+\frac{2r}{n}\right)S$ $+(n-1)\left(\left(\frac{r}{n(n-1)}+1\right)\frac{a_0}{a_1}+\left(\frac{2r}{n}+(n-1)\right)\right)g$
(ε)-*Sasakian*	$\left(\left(\frac{r}{n(n-1)}-\varepsilon\right)\frac{a_0}{a_1}+\frac{2r}{n}\right)S$ $-\varepsilon(n-1)\left(\left(\frac{r}{n(n-1)}-\varepsilon\right)\frac{a_0}{a_1}+\left(\frac{2r}{n}-\varepsilon(n-1)\right)\right)g$
para-Sasakian	$\left(\left(\frac{r}{n(n-1)}+1\right)\frac{a_0}{a_1}+\frac{2r}{n}\right)S$ $+(n-1)\left(\left(\frac{r}{n(n-1)}+1\right)\frac{a_0}{a_1}+\left(\frac{2r}{n}+(n-1)\right)\right)g$
(ε)-*para-Sasakian*	$\left(\left(\frac{r}{n(n-1)}+\varepsilon\right)\frac{a_0}{a_1}+\frac{2r}{n}\right)S$ $+\varepsilon(n-1)\left(\left(\frac{r}{n(n-1)}+\varepsilon\right)\frac{a_0}{a_1}+\left(\frac{2r}{n}+\varepsilon(n-1)\right)\right)g$

Corollary 5.8. *Let M be an n-dimensional $(R,\mathcal{C},S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not Weyl-semisymmetric, then*

$$S^{\ell+1} = \left(\frac{r}{n-1}-k\right)\left(S^\ell - k^\ell(n-1)^\ell g\right)+k^\ell(n-1)^\ell S.$$

Consequently, we have the following:

M	$S^{\ell+1}=$
$N(k)$-*contact metric*	$\left(\frac{r}{n-1}-k\right)\left(S^\ell-k^\ell(n-1)^\ell g\right)$ $+k^\ell(n-1)^\ell S$
Sasakian	$\left(\frac{r}{n-1}-1\right)\left(S^\ell-(n-1)^\ell g\right)$ $+(n-1)^\ell S$
Kenmotsu	$\left(\frac{r}{n-1}+1\right)\left(S^\ell-(-1)^\ell(n-1)^\ell g\right)$ $+(-1)^\ell(n-1)^\ell S$
(ε)-*Sasakian*	$\left(\frac{r}{n-1}-\varepsilon\right)\left(S^\ell-(\varepsilon)^\ell(n-1)^\ell g\right)$ $+(\varepsilon)^\ell(n-1)^\ell S$
para-Sasakian	$\left(\frac{r}{n-1}+1\right)\left(S^\ell-(-1)^\ell(n-1)^\ell g\right)$ $+(-1)^\ell(n-1)^\ell S$
(ε)-*para-Sasakian*	$\left(\frac{r}{n-1}+\varepsilon\right)\left(S^\ell-(-\varepsilon)^\ell(n-1)^\ell g\right)$ $+(-\varepsilon)^\ell(n-1)^\ell S$

Corollary 5.9. *Let M be an n-dimensional $(R,\mathcal{C},S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not Weyl-semisymmetric, then*

$$S^2 = \left(k(n-2)+\frac{r}{n-1}\right)S + k(n-1)\left(k-\frac{r}{n-1}\right)g.$$

Consequently, we have the following:

M	$S^2=$
$N(k)$-contact metric	$\left(k(n-2)+\frac{r}{n-1}\right)S$ $+k(n-1)\left(k-\frac{r}{n-1}\right)g$
Sasakian	$\left((n-2)+\frac{r}{n-1}\right)S$ $+(n-1)\left(1-\frac{r}{n-1}\right)g$
Kenmotsu	$\left(-(n-2)+\frac{r}{n-1}\right)S$ $+(n-1)\left(1+\frac{r}{n-1}\right)g$
(ε)-Sasakian	$\left(\varepsilon(n-2)+\frac{r}{n-1}\right)S$ $+\varepsilon(n-1)\left(\varepsilon-\frac{r}{n-1}\right)g$
para-Sasakian	$\left(-(n-2)+\frac{r}{n-1}\right)S$ $+(n-1)\left(1+\frac{r}{n-1}\right)g$
(ε)-para-Sasakian	$\left(-\varepsilon(n-2)+\frac{r}{n-1}\right)S$ $+\varepsilon(n-1)\left(\varepsilon+\frac{r}{n-1}\right)g$

Corollary 5.10. *Let M be an n-dimensional $(R,\mathcal{L},S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not conharmonic semisymmetric, then*

$$S^{\ell+1} = -kS^\ell + k^\ell(n-1)^\ell S + k^{\ell+1}(n-1)^\ell g.$$

Consequently, we have the following:

M	$S^{\ell+1}=$
$N(k)$-contact metric	$-kS^\ell + k^\ell(n-1)^\ell S + k^{\ell+1}(n-1)^\ell g$
Sasakian	$-S^\ell + (n-1)^\ell S + (n-1)^\ell g$
Kenmotsu	$S^\ell + (-1)^\ell(n-1)^\ell S + (-1)^{\ell+1}(n-1)^\ell g$
(ε)-Sasakian	$-\varepsilon S^\ell + (\varepsilon)^\ell(n-1)^\ell S + (\varepsilon)^{\ell+1}(n-1)^\ell g$
para-Sasakian	$S^\ell + (-1)^\ell(n-1)^\ell S + (-1)^{\ell+1}(n-1)^\ell g$
(ε)-para-Sasakian	$\varepsilon S^\ell + (-\varepsilon)^\ell(n-1)^\ell S + (-\varepsilon)^{\ell+1}(n-1)^\ell g$

Corollary 5.11. *Let M be an n-dimensional $(R,\mathcal{L},S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not conharmonic semisymmetric, then*

$$S^2 = k(n-2)S + k^2(n-1)g.$$

Consequently, we have the following:

M	$S^2=$
$N(k)$-contact metric	$k(n-2)S + k^2(n-1)g$
Sasakian	$(n-2)S + (n-1)g$
Kenmotsu	$-(n-2)S + (n-1)g$
(ε)-Sasakian	$\varepsilon(n-2)S + (n-1)g$
para-Sasakian	$-(n-2)S + (n-1)g$
(ε)-para-Sasakian	$-\varepsilon(n-2)S + (n-1)g$

Corollary 5.12. *Let M be an n-dimensional $(R,\mathcal{V},S^{\ell})$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not concircular semisymmetric, then M either satisfies*

$$S^{\ell}=k^{\ell}(n-1)^{\ell}g$$

or scalar curvature is $kn(n-1)$ and $L_{S^{\ell}}=\dfrac{1}{k^{\ell-1}(n-1)^{\ell}}$. Consequently, we have the following:

M	$L_{S^{\ell}}=$	***Result***
$N(k)$-*contact*	$\dfrac{1}{k^{\ell-1}(n-1)^{\ell}}$	$S^{\ell}=k^{\ell}(n-1)^{\ell}g$ or $r=kn(n-1)$
Sasakian	$\dfrac{1}{(n-1)^{\ell}}$	$S^{\ell}=(n-1)^{\ell}g$ or $r=n(n-1)$
Kenmotsu	$\dfrac{1}{(-1)^{\ell-1}(n-1)^{\ell}}$	$S^{\ell}=(-1)^{\ell}(n-1)^{\ell}g$ or $r=-n(n-1)$
(ε)-*Sasakian*	$\dfrac{1}{(\varepsilon)^{\ell-1}(n-1)^{\ell}}$	$S^{\ell}=(\varepsilon)^{\ell}(n-1)^{\ell}g$ or $r=\varepsilon n(n-1)$
para-Sasakian	$\dfrac{1}{(-1)^{\ell-1}(n-1)^{\ell}}$	$S^{\ell}=(-1)^{\ell}(n-1)^{\ell}g$ or $r=-n(n-1)$
(ε)-*para-Sasakian*	$\dfrac{1}{(-\varepsilon)^{\ell-1}(n-1)^{\ell}}$	$S^{\ell}=(-\varepsilon)^{\ell}(n-1)^{\ell}g$ or $r=-\varepsilon n(n-1)$

Corollary 5.13. *Let M be an n-dimensional $(R,\mathcal{V},S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not concircularly semisymmetric, then M is either an Einstein manifold or scalar curvature is $kn(n-1)$ and $L_S=\dfrac{1}{n-1}$. Consequently, we have the following:*

M	$L_S=$	$S=$
$N(k)$-*contact metric*	$\dfrac{1}{n-1}$	$S=k(n-1)g$ or $r=kn(n-1)$
Sasakian	$\dfrac{1}{n-1}$	$S=(n-1)g$ or $r=n(n-1)$
Kenmotsu [19]	$\dfrac{1}{n-1}$	$S=-(n-1)g$ or $r=-n(n-1)$
(ε)-*Sasakian*	$\dfrac{1}{n-1}$	$S=\varepsilon(n-1)g$ or $r=\varepsilon n(n-1)$
para-Sasakian	$\dfrac{1}{n-1}$	$S=-(n-1)g$ or $r=-n(n-1)$
(ε)-*para-Sasakian*	$\dfrac{1}{n-1}$	$S=-\varepsilon(n-1)g$ or $r=-\varepsilon n(n-1)$

Corollary 5.14. *Let M be an n-dimensional $(R,\mathcal{P}_*,S^{\ell})$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold such that $a_0+(n-1)a_1\neq 0$. If M is not pseudo-projective semisymmetric, then*

$$\begin{aligned}&\left(\left(k-\frac{r}{n(n-1)}\right)a_0+\left(k(n-1)-\frac{r}{n}\right)a_1\right)S^{\ell}\\ =\;& k^{\ell}(n-1)^{\ell}\left(\left(k-\frac{r}{n(n-1)}\right)a_0-\left(\frac{r}{n}\right)a_1\right)g\\ &+k^{\ell}(n-1)^{\ell}a_1S.\end{aligned}$$

Consequently, we have the following:

M	$S^\ell =$
$N(k)$-contact metric	$\dfrac{k^\ell (n-1)^\ell a_1}{\left(\left(k - \dfrac{r}{n(n-1)}\right) a_0 + \left(k(n-1) - \dfrac{r}{n}\right) a_1\right)} S + \dfrac{k^\ell (n-1)^\ell \left(\left(k - \dfrac{r}{n(n-1)}\right) a_0 - \left(\dfrac{r}{n}\right) a_1\right)}{\left(\left(k - \dfrac{r}{n(n-1)}\right) a_0 + \left(k(n-1) - \dfrac{r}{n}\right) a_1\right)} g$
Sasakian	$\dfrac{(n-1)^\ell a_1}{\left(\left(1 - \dfrac{r}{n(n-1)}\right) a_0 + \left((n-1) - \dfrac{r}{n}\right) a_1\right)} S + \dfrac{(n-1)^\ell \left(\left(1 - \dfrac{r}{n(n-1)}\right) a_0 - \left(\frac{r}{n}\right) a_1\right)}{\left(\left(1 - \dfrac{r}{n(n-1)}\right) a_0 + \left((n-1) - \dfrac{r}{n}\right) a_1\right)} g$
Kenmotsu	$\dfrac{(-1)^\ell (n-1)^\ell a_1}{\left(\left(-1 - \dfrac{r}{n(n-1)}\right) a_0 + \left(-(n-1) - \dfrac{r}{n}\right) a_1\right)} S + \dfrac{(-1)^\ell (n-1)^\ell \left(\left(-1 - \frac{r}{n(n-1)}\right) a_0 - \left(\dfrac{r}{n}\right) a_1\right)}{\left(\left(-1 - \dfrac{r}{n(n-1)}\right) a_0 + \left(-(n-1) - \dfrac{r}{n}\right) a_1\right)} g$
(ε)-Sasakian	$\dfrac{(\varepsilon)^\ell (n-1)^\ell a_1}{\left(\left(\varepsilon - \dfrac{r}{n(n-1)}\right) a_0 + \left(\varepsilon(n-1) - \dfrac{r}{n}\right) a_1\right)} S + \dfrac{(\varepsilon)^\ell (n-1)^\ell \left(\left(\varepsilon - \dfrac{r}{n(n-1)}\right) a_0 - \left(\dfrac{r}{n}\right) a_1\right)}{\left(\left(\varepsilon - \dfrac{r}{n(n-1)}\right) a_0 + \left(\varepsilon(n-1) - \dfrac{r}{n}\right) a_1\right)} g$
para-Sasakian	$\dfrac{(-1)^\ell (n-1)^\ell a_1}{\left(\left(-1 - \dfrac{r}{n(n-1)}\right) a_0 + \left(-(n-1) - \dfrac{r}{n}\right) a_1\right)} S + \dfrac{(-1)^\ell (n-1)^\ell \left(\left(-1 - \dfrac{r}{n(n-1)}\right) a_0 - \left(\dfrac{r}{n}\right) a_1\right)}{\left(\left(-1 - \dfrac{r}{n(n-1)}\right) a_0 + \left(-(n-1) - \dfrac{r}{n}\right) a_1\right)} g$
(ε)-para-Sasakian	$\dfrac{(-\varepsilon)^\ell (n-1)^\ell a_1}{\left(\left(-\varepsilon - \dfrac{r}{n(n-1)}\right) a_0 + \left(-\varepsilon(n-1) - \dfrac{r}{n}\right) a_1\right)} S + \dfrac{(-\varepsilon)^\ell (n-1)^\ell \left(\left(-\varepsilon - \dfrac{r}{n(n-1)}\right) a_0 - \left(\dfrac{r}{n}\right) a_1\right)}{\left(\left(-\varepsilon - \dfrac{r}{n(n-1)}\right) a_0 + \left(-\varepsilon(n-1) - \dfrac{r}{n}\right) a_1\right)} g$

Corollary 5.15. *Let M be an n-dimensional $(R,\mathcal{P}_*,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold such that $a_0+(n-1)a_1 \neq 0$. If M is not pseudo-projective semisymmetric, then either M is an Einstein manifold or $r = \dfrac{kn(n-1)a_0}{a_0+(n-1)a_1}$ and $L_S = \dfrac{1}{n-1}$.*

Consequently, we have the following:

M	$L_S =$	$S =$
$N(k)$-*contact metric*	$\dfrac{1}{n-1}$	$S = k(n-1)g$ or $r = \dfrac{kn(n-1)a_0}{a_0 + (n-1)a_1}$
Sasakian	$\dfrac{1}{n-1}$	$S = (n-1)g$ or $r = \dfrac{n(n-1)a_0}{a_0 + (n-1)a_1}$
Kenmotsu	$\dfrac{1}{n-1}$	$S = -(n-1)g$ or $r = -\dfrac{n(n-1)a_0}{a_0 + (n-1)a_1}$
(ε)-*Sasakian*	$\dfrac{1}{n-1}$	$S = \varepsilon(n-1)g$ or $r = \dfrac{\varepsilon n(n-1)a_0}{a_0 + (n-1)a_1}$
para-Sasakian	$\dfrac{1}{n-1}$	$S = -(n-1)g$ or $r = -\dfrac{n(n-1)a_0}{a_0 + (n-1)a_1}$
(ε)-*para-Sasakian*	$\dfrac{1}{n-1}$	$S = -\varepsilon(n-1)g$ or $r = -\dfrac{\varepsilon n(n-1)a_0}{a_0 + (n-1)a_1}$

Corollary 5.16. *Let M be an n-dimensional $(R, \mathcal{P}, S^\ell)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. If M is not projective semisymmetric, then M is an Einstein manifold with scalar curvature $kn(n-1)$. Consequently, we have the following:*

M	$S =$
$N(k)$-*contact metric*	$k(n-1)g$
Sasakian	$(n-1)g$
Kenmotsu	$-(n-1)g$
(ε)-*Sasakian*	$\varepsilon(n-1)g$
para-Sasakian	$-(n-1)g$
(ε)-*para-Sasakian*	$-\varepsilon(n-1)g$

Corollary 5.17. *Let M be an n-dimensional $(R, \mathcal{P}, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. If M is not projective semisymmetric, then M is an Einstein manifold with scalar curvature $kn(n-1)$ and $L_S = \dfrac{1}{n-1}$. Consequently, we have the following:*

M	$L_S =$	$S =$
$N(k)$-*contact metric*	$\frac{1}{n-1}$	$k(n-1)g$
Sasakian	$\frac{1}{n-1}$	$(n-1)g$
Kenmotsu [19]	$\frac{1}{n-1}$	$-(n-1)g$
(ε)-*Sasakian*	$\frac{1}{n-1}$	$\varepsilon(n-1)g$
para-Sasakian	$\frac{1}{n-1}$	$-(n-1)g$
(ε)-*para-Sasakian*	$\frac{1}{n-1}$	$-\varepsilon(n-1)g$

Corollary 5.18. *Let M be an n-dimensional $(R, \mathcal{M}, S^\ell)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. If M is not $\mathcal{M}$-semisymmetric, then*

$$S^{\ell+1} = k(n-1)S^\ell + k^\ell (n-1)^\ell S - k^{\ell+1}(n-1)^{\ell+1} g.$$

Consequently, we have the following:

M	$S^{\ell+1} =$
$N(k)$-*contact metric*	$k(n-1)S^\ell + k^\ell(n-1)^\ell S - k^{\ell+1}(n-1)^{\ell+1}g$
Sasakian	$(n-1)S^\ell + (n-1)^\ell S - (n-1)^{\ell+1}g$
Kenmotsu	$-(n-1)S^\ell + (-1)^\ell(n-1)^\ell S - (-1)^{\ell+1}(n-1)^{\ell+1}g$
(ε)-*Sasakian*	$\varepsilon(n-1)S^\ell + (\varepsilon)^\ell(n-1)^\ell S - (\varepsilon)^{\ell+1}(n-1)^{\ell+1}g$
para-Sasakian	$-(n-1)S^\ell + (-1)^\ell(n-1)^\ell S - (-1)^{\ell+1}(n-1)^{\ell+1}g$
(ε)-*para-Sasakian*	$-\varepsilon(n-1)S^\ell + (-\varepsilon)^\ell(n-1)^\ell S - (-\varepsilon)^{\ell+1}(n-1)^{\ell+1}g$

Corollary 5.19. *Let M be an n-dimensional $(R,\mathcal{M},S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not $\mathcal{M}$-semisymmetric, then*

$$S^2 = 2(n-1)kS - k^2(n-1)^2 g.$$

Consequently, we have the following:

M	$S^2=$
$N(k)$-contact metric	$2(n-1)kS - k^2(n-1)^2 g$
Sasakian	$2(n-1)S - (n-1)^2 g$
Kenmotsu	$-2(n-1)S - (n-1)^2 g$
(ε)-Sasakian	$2(n-1)\varepsilon S - (n-1)^2 g$
para-Sasakian	$-2(n-1)S - (n-1)^2 g$
(ε)-para-Sasakian	$-2(n-1)\varepsilon S - (n-1)^2 g$

Corollary 5.20. *Let M be an n-dimensional $(R,\mathcal{W}_0,S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_0$-semisymmetric, then*

$$S^{\ell+1} = k(n-1)S^\ell + k^\ell(n-1)^\ell S - k^{\ell+1}(n-1)^{\ell+1} g.$$

Consequently, we have the following:

M	$S^{\ell+1}=$
$N(k)$-contact metric	$k(n-1)S^\ell + k^\ell(n-1)^\ell S - k^{\ell+1}(n-1)^{\ell+1} g$
Sasakian	$(n-1)S^\ell + (n-1)^\ell S - (n-1)^{\ell+1} g$
Kenmotsu	$-(n-1)S^\ell + (-1)^\ell(n-1)^\ell S - (-1)^{\ell+1}(n-1)^{\ell+1} g$
(ε)-Sasakian	$\varepsilon(n-1)S^\ell + (\varepsilon)^\ell(n-1)^\ell S - (\varepsilon)^{\ell+1}(n-1)^{\ell+1} g$
para-Sasakian	$-(n-1)S^\ell + (-1)^\ell(n-1)^\ell S - (-1)^{\ell+1}(n-1)^{\ell+1} g$
(ε)-para-Sasakian	$-\varepsilon(n-1)S^\ell + (-\varepsilon)^\ell(n-1)^\ell S - (-\varepsilon)^{\ell+1}(n-1)^{\ell+1} g$

Corollary 5.21. *Let M be an n-dimensional $(R,\mathcal{W}_0,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_0$-semisymmetric, then*

$$S^2 = 2(n-1)kS - k^2(n-1)^2 g.$$

Consequently, we have the following:

M	$S^2=$
$N(k)$-contact metric	$2(n-1)kS - k^2(n-1)^2 g$
Sasakian	$2(n-1)S - (n-1)^2 g$
Kenmotsu	$-2(n-1)S - (n-1)^2 g$
(ε)-Sasakian	$2(n-1)\varepsilon S - (n-1)^2 g$
para-Sasakian	$-2(n-1)S - (n-1)^2 g$
(ε)-para-Sasakian	$-2(n-1)\varepsilon S - (n-1)^2 g$

Corollary 5.22. *Let M be an n-dimensional $(R,\mathcal{W}_0^*,S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_0^*$-semisymmetric, then*

$$S^{\ell+1} = -k(n-1)S^\ell + k^\ell(n-1)^\ell S + k^{\ell+1}(n-1)^{\ell+1} g.$$

Consequently, we have the following:

M	$S^{\ell+1}=$
$N(k)$-contact metric	$-k(n-1)S^\ell + k^\ell(n-1)^\ell S + k^{\ell+1}(n-1)^{\ell+1} g$
Sasakian	$-(n-1)S^\ell + (n-1)^\ell S + (n-1)^{\ell+1} g$
Kenmotsu	$(n-1)S^\ell + (-1)^\ell(n-1)^\ell S + (-1)^{\ell+1}(n-1)^{\ell+1} g$
(ε)-Sasakian	$-\varepsilon(n-1)S^\ell + (\varepsilon)^\ell(n-1)^\ell S + (\varepsilon)^{\ell+1}(n-1)^{\ell+1} g$
para-Sasakian	$(n-1)S^\ell + (-1)^\ell(n-1)^\ell S + (-1)^{\ell+1}(n-1)^{\ell+1} g$
(ε)-para-Sasakian	$\varepsilon(n-1)S^\ell + (-\varepsilon)^\ell(n-1)^\ell S + (-\varepsilon)^{\ell+1}(n-1)^{\ell+1} g$

Corollary 5.23. *Let M be an n-dimensional $(R, \mathcal{W}_0^*, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_0^*$-semisymmetric, then*

$$S^2 = k^2(n-1)^2 g.$$

Consequently, we have the following:

M	$S^2 =$
$N(k)$-contact metric	$k^2(n-1)^2 g$
Sasakian	$(n-1)^2 g$
Kenmotsu	$(n-1)^2 g$
(ε)-Sasakian	$(n-1)^2 g$
para-Sasakian	$(n-1)^2 g$
(ε)-para-Sasakian	$(n-1)^2 g$

Corollary 5.24. *Let M be an n-dimensional $(R, \mathcal{W}_1, S^\ell)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_1$-semisymmetric, then*

$$2S^\ell = k^{\ell-1}(n-1)^{\ell-1} S + k^\ell (n-1)^\ell g.$$

Consequently, we have the following:

M	$2S^\ell =$
$N(k)$-contact metric	$k^{\ell-1}(n-1)^{\ell-1} S + k^\ell (n-1)^\ell g$
Sasakian	$(n-1)^{\ell-1} S + (n-1)^\ell g$
Kenmotsu	$(-1)^{\ell-1}(n-1)^{\ell-1} S + (-1)^\ell (n-1)^\ell g$
(ε)-Sasakian	$(\varepsilon)^{\ell-1}(n-1)^{\ell-1} S + (\varepsilon)^\ell (n-1)^\ell g$
para-Sasakian	$(-1)^{\ell-1}(n-1)^{\ell-1} S + (-1)^\ell (n-1)^\ell g$
(ε)-para-Sasakian	$(-\varepsilon)^{\ell-1}(n-1)^{\ell-1} S + (-\varepsilon)^\ell (n-1)^\ell g$

Corollary 5.25. *Let M be an n-dimensional $(R, \mathcal{W}_1, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_1$-semisymmetric, then M is an Einstein manifold with scalar curvature $kn(n-1)$ and $L_S = \dfrac{1}{n-1}$. Consequently, we have the following:*

M	$L_S =$	$S =$
$N(k)$-contact metric	$\frac{1}{n-1}$	$k(n-1)g$
Sasakian	$\frac{1}{n-1}$	$(n-1)g$
Kenmotsu	$\frac{1}{n-1}$	$-(n-1)g$
(ε)-Sasakian	$\frac{1}{n-1}$	$\varepsilon(n-1)g$
para-Sasakian	$\frac{1}{n-1}$	$-(n-1)g$
(ε)-para-Sasakian	$\frac{1}{n-1}$	$-\varepsilon(n-1)g$

Corollary 5.26. *Let M be an n-dimensional $(R, \mathcal{W}_1^*, S^\ell)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_1^*$-semisymmetric, then M is an Einstein manifold with scalar curvature $kn(n-1)$. Consequently, we have the following:*

M	$S =$
$N(k)$-contact metric	$k(n-1)g$
Sasakian	$(n-1)g$
Kenmotsu	$-(n-1)g$
(ε)-Sasakian	$\varepsilon(n-1)g$
para-Sasakian	$-(n-1)g$
(ε)-para-Sasakian	$-\varepsilon(n-1)g$

Corollary 5.27. *Let M be an n-dimensional $(R,\mathcal{W}_1^*,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_1^*$-semisymmetric, then M is an Einstein manifold with scalar curvature $kn(n-1)$ and $L_S = \frac{1}{n-1}$. Consequently, we have the following:*

M	$L_S =$	$S =$
$N(k)$-contact metric	$\frac{1}{n-1}$	$k(n-1)g$
Sasakian	$\frac{1}{n-1}$	$(n-1)g$
Kenmotsu	$\frac{1}{n-1}$	$-(n-1)g$
(ε)-Sasakian	$\frac{1}{n-1}$	$\varepsilon(n-1)g$
para-Sasakian	$\frac{1}{n-1}$	$-(n-1)g$
(ε)-para-Sasakian	$\frac{1}{n-1}$	$-\varepsilon(n-1)g$

Corollary 5.28. *Let M be an n-dimensional $(R,\mathcal{W}_2,S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_2$-semisymmetric, then*

$$L_{S^\ell}S^{\ell+1} = k(n-1)L_{S^\ell}S^\ell + kS - k^2(n-1)g.$$

Consequently, we have the following:

M	$L_{S^\ell}S^{\ell+1} =$
$N(k)$-contact metric	$k(n-1)L_{S^\ell}S^\ell + kS - k^2(n-1)g$
Sasakian	$(n-1)L_{S^\ell}S^\ell + S - (n-1)g$
Kenmotsu	$-(n-1)L_{S^\ell}S^\ell - S - (n-1)g$
(ε)-Sasakian	$\varepsilon(n-1)L_{S^\ell}S^\ell + \varepsilon S - (n-1)g$
para-Sasakian	$-(n-1)L_{S^\ell}S^\ell - S - (n-1)g$
(ε)-para-Sasakian	$-\varepsilon(n-1)L_{S^\ell}S^\ell - \varepsilon S - (n-1)g$

Corollary 5.29. *Let M be an n-dimensional $(R,\mathcal{W}_2,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_2$-semisymmetric, then*

$$L_S S^2 = k\left((n-1)L_S + 1\right)S - k^2(n-1)g.$$

Consequently, we have the following:

M	$L_S S^2 =$
$N(k)$-contact metric	$k\left((n-1)L_S+1\right)S - k^2(n-1)g$
Sasakian	$\left((n-1)L_S+1\right)S - (n-1)g$
Kenmotsu	$-\left((n-1)L_S+1\right)S - (n-1)g$
(ε)-Sasakian	$\varepsilon\left((n-1)L_S+1\right)S - (n-1)g$
para-Sasakian	$-\left((n-1)L_S+1\right)S - (n-1)g$
(ε)-para-Sasakian	$-\varepsilon\left((n-1)L_S+1\right)S - (n-1)g$

Corollary 5.30. *Let M be an n-dimensional $(R,\mathcal{W}_3,S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_3$-semisymmetric, then*

$$S^\ell = k^\ell(n-1)^\ell g$$

and $L_{S^\ell} = \dfrac{1}{k^{\ell-1}(n-1)^\ell}$. *Consequently, we have the following:*

M	$L_{S^\ell} =$	$S^\ell =$
$N(k)$-*contact metric*	$\dfrac{1}{k^{\ell-1}(n-1)^\ell}$	$k^\ell(n-1)^\ell g$
Sasakian	$\dfrac{1}{(n-1)^\ell}$	$(n-1)^\ell g$
Kenmotsu	$\dfrac{1}{(-1)^{\ell-1}(n-1)^\ell}$	$(-1)^\ell(n-1)^\ell g$
(ε)-*Sasakian*	$\dfrac{1}{(\varepsilon)^{\ell-1}(n-1)^\ell}$	$(\varepsilon)^\ell(n-1)^\ell g$
para-Sasakian	$\dfrac{1}{(-1)^{\ell-1}(n-1)^\ell}$	$(-1)^\ell(n-1)^\ell g$
(ε)-*para-Sasakian*	$\dfrac{1}{(-\varepsilon)^{\ell-1}(n-1)^\ell}$	$(-\varepsilon)^\ell(n-1)^\ell g$

Corollary 5.31. *Let M be an n-dimensional $(R, \mathcal{W}_3, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_3$-semisymmetric, then M is an Einstein manifold with scalar curvature $kn(n-1)$ and* $L_S = \dfrac{1}{n-1}$. *Consequently, we have the following:*

M	$L_S =$	$S =$
$N(k)$-*contact metric*	$\dfrac{1}{n-1}$	$k(n-1)g$
Sasakian	$\dfrac{1}{n-1}$	$(n-1)g$
Kenmotsu	$\dfrac{1}{n-1}$	$-(n-1)g$
(ε)-*Sasakian*	$\dfrac{1}{n-1}$	$\varepsilon(n-1)g$
para-Sasakian	$\dfrac{1}{n-1}$	$-(n-1)g$
(ε)-*para-Sasakian*	$\dfrac{1}{n-1}$	$-\varepsilon(n-1)g$

Corollary 5.32. *Let M be an n-dimensional $(R, \mathcal{W}_4, S^\ell)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_4$-semisymmetric, then*

$$L_{S^\ell} S^{\ell+1} = k(n-1)L_{S^\ell} S^\ell + kS - k^2(n-1)g.$$

Consequently, we have the following:

M	$L_{S^\ell} S^{\ell+1} =$
$N(k)$-*contact metric*	$k(n-1)L_{S^\ell}S^\ell + kS - k^2(n-1)g$
Sasakian	$(n-1)L_{S^\ell}S^\ell + S - (n-1)g$
Kenmotsu	$-(n-1)L_{S^\ell}S^\ell - S - (n-1)g$
(ε)-*Sasakian*	$\varepsilon(n-1)L_{S^\ell}S^\ell + \varepsilon S - (n-1)g$
para-Sasakian	$-(n-1)L_{S^\ell}S^\ell - S - (n-1)g$
(ε)-*para-Sasakian*	$-\varepsilon(n-1)L_{S^\ell}S^\ell - \varepsilon S - (n-1)g$

Corollary 5.33. *Let M be an n-dimensional $(R, \mathcal{W}_4, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_4$-semisymmetric, then*

$$L_S S^2 = k\left((n-1)L_S + 1\right) S - k^2(n-1)g.$$

Consequently, we have the following:

M	$L_S S^2 =$
$N(k)$-contact metric	$k\left((n-1)L_S+1\right)S - k^2(n-1)g$
Sasakian	$\left((n-1)L_S+1\right)S-(n-1)g$
Kenmotsu	$-\left((n-1)L_S+1\right)S-(n-1)g$
(ε)-Sasakian	$\varepsilon\left((n-1)L_S+1\right)S-(n-1)g$
para-Sasakian	$-\left((n-1)L_S+1\right)S-(n-1)g$
(ε)-para-Sasakian	$-\varepsilon\left((n-1)L_S+1\right)S-(n-1)g$

Corollary 5.34. *Let M be an n-dimensional $(R,\mathcal{W}_5,S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_5$-semisymmetric, then either*

$$S = k(n-1)g$$

or $L_{S^\ell} = \dfrac{1}{k^{\ell-1}(n-1)^\ell}$. *Consequently, we have the following:*

M	$L_{S^\ell} =$	$S =$
$N(k)$-contact metric	$\frac{1}{k^{\ell-1}(n-1)^\ell}$	$k(n-1)g$
Sasakian	$\frac{1}{(n-1)^\ell}$	$(n-1)g$
Kenmotsu	$\frac{1}{(-1)^{\ell-1}(n-1)^\ell}$	$-(n-1)g$
(ε)-Sasakian	$\frac{1}{(-1)^{\ell-1}(n-1)^\ell}$	$\varepsilon(n-1)g$
para-Sasakian	$\frac{1}{(-1)^{\ell-1}(n-1)^\ell}$	$-(n-1)g$
(ε)-para-Sasakian	$\frac{1}{(-\varepsilon)^{\ell-1}(n-1)^\ell}$	$-\varepsilon(n-1)g$

Corollary 5.35. *Let M be an n-dimensional $(R,\mathcal{W}_5,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_5$-semisymmetric, then M is an Einstein manifold with scalar curvature $kn(n-1)$ and $L_S = \dfrac{1}{n-1}$. Consequently, we have the following:*

M	$L_S =$	$S =$
$N(k)$-contact metric	$\frac{1}{n-1}$	$k(n-1)g$
Sasakian	$\frac{1}{n-1}$	$(n-1)g$
Kenmotsu	$\frac{1}{n-1}$	$-(n-1)g$
(ε)-Sasakian	$\frac{1}{n-1}$	$\varepsilon(n-1)g$
para-Sasakian	$\frac{1}{n-1}$	$-(n-1)g$
(ε)-para-Sasakian	$\frac{1}{n-1}$	$-\varepsilon(n-1)g$

Corollary 5.36. *Let M be an n-dimensional $(R,\mathcal{W}_6,S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_6$-semisymmetric, then*

$$(n-1)L_{S^\ell}S^\ell = (1-k^{\ell-1}(n-1)^\ell L_{S^\ell})S + (2k^\ell(n-1)^{\ell+1}L_{S^\ell} - k(n-1))g.$$

Consequently, we have the following:

M	$(n-1)L_{S^\ell}S^\ell =$
$N(k)$-contact metric	$(1-k^{\ell-1}(n-1)^\ell L_{S^\ell})S + (2k^\ell(n-1)^{\ell+1}L_{S^\ell} - k(n-1))g$
Sasakian	$(1-k^{\ell-1}(n-1)^\ell L_{S^\ell})S + (2k^\ell(n-1)^{\ell+1}L_{S^\ell} - k(n-1))g$
Kenmotsu	$(1-k^{\ell-1}(n-1)^\ell L_{S^\ell})S + (2k^\ell(n-1)^{\ell+1}L_{S^\ell} - k(n-1))g$
(ε)-Sasakian	$(1-k^{\ell-1}(n-1)^\ell L_{S^\ell})S + (2k^\ell(n-1)^{\ell+1}L_{S^\ell} - k(n-1))g$
para-Sasakian	$(1-k^{\ell-1}(n-1)^\ell L_{S^\ell})S + (2k^\ell(n-1)^{\ell+1}L_{S^\ell} - k(n-1))g$
(ε)-para-Sasakian	$(1-k^{\ell-1}(n-1)^\ell L_{S^\ell})S + (2k^\ell(n-1)^{\ell+1}L_{S^\ell} - k(n-1))g$

Corollary 5.37. *Let M be an n-dimensional $(R, \mathcal{W}_6, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_6$-semisymmetric, then M is an Einstein manifold with scalar curvature $kn(n-1)$ and $L_S = \frac{1}{2(n-1)}$. Consequently, we have the following:*

M	$L_S =$	$S =$
$N(k)$-contact metric	$\frac{1}{2(n-1)}$	$k(n-1)g$
Sasakian	$\frac{1}{2(n-1)}$	$(n-1)g$
Kenmotsu	$\frac{1}{2(n-1)}$	$-(n-1)g$
(ε)-Sasakian	$\frac{1}{2(n-1)}$	$\varepsilon(n-1)g$
para-Sasakian	$\frac{1}{2(n-1)}$	$-(n-1)g$
(ε)-para-Sasakian	$\frac{1}{2(n-1)}$	$-\varepsilon(n-1)g$

Corollary 5.38. *Let M be an n-dimensional $(R, \mathcal{W}_7, S^\ell)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_7$-semisymmetric, then*

$$(n-1)L_{S^\ell}S^\ell = (1 - k^{\ell-1}(n-1)^\ell L_{S^\ell})S + (2k^\ell(n-1)^{\ell+1}L_{S^\ell} - k(n-1))g.$$

Consequently, we have the following:

M	$(n-1)L_{S^\ell}S^\ell =$
$N(k)$-contact metric	$(1 - k^{\ell-1}(n-1)^\ell L_{S^\ell})S + (2k^\ell(n-1)^{\ell+1}L_{S^\ell} - k(n-1))g$
Sasakian	$(1 - (n-1)^\ell L_{S^\ell})S + (2(n-1)^{\ell+1}L_{S^\ell} - (n-1))g$
Kenmotsu	$(1 - (-1)^{\ell-1}(n-1)^\ell L_{S^\ell})S + (2(-1)^\ell(n-1)^{\ell+1}L_{S^\ell} + (n-1))g$
(ε)-Sasakian	$(1 - (\varepsilon)^{\ell-1}(n-1)^\ell L_{S^\ell})S + (2(\varepsilon)^\ell(n-1)^{\ell+1}L_{S^\ell} - \varepsilon(n-1))g$
para-Sasakian	$(1 - (-1)^{\ell-1}(n-1)^\ell L_{S^\ell})S + (2(-1)^\ell(n-1)^{\ell+1}L_{S^\ell} + (n-1))g$
(ε)-para-Sasakian	$(1 - (-\varepsilon)^{\ell-1}(n-1)^\ell L_{S^\ell})S + (2(-\varepsilon)^\ell(n-1)^{\ell+1}L_{S^\ell} + \varepsilon(n-1))g$

Corollary 5.39. *Let M be an n-dimensional $(R, \mathcal{W}_7, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_7$-semisymmetric, then M is an Einstein manifold with scalar curvature $kn(n-1)$ and $L_S = \frac{1}{2(n-1)}$. Consequently, we have the following:*

M	$L_S =$	$S =$
$N(k)$-contact metric	$\frac{1}{2(n-1)}$	$k(n-1)g$
Sasakian	$\frac{1}{2(n-1)}$	$(n-1)g$
Kenmotsu	$\frac{1}{2(n-1)}$	$-(n-1)g$
(ε)-Sasakian	$\frac{1}{2(n-1)}$	$\varepsilon(n-1)g$
para-Sasakian	$\frac{1}{2(n-1)}$	$-(n-1)g$
(ε)-para-Sasakian	$\frac{1}{2(n-1)}$	$-\varepsilon(n-1)g$

Corollary 5.40. *Let M be an n-dimensional $(R, \mathcal{W}_8, S^\ell)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_8$-semisymmetric, then*

$$S^\ell = k^\ell(n-1)^\ell g$$

and $L_{S^\ell} = \dfrac{1}{k^{\ell-1}(n-1)^\ell}$. *Consequently, we have the following:*

M	$L_{S^\ell} =$	$S^\ell =$
$N(k)$-*contact metric*	$\frac{1}{k^{\ell-1}(n-1)^\ell}$	$k^\ell(n-1)^\ell g$
Sasakian	$\frac{1}{(n-1)^\ell}$	$(n-1)^\ell g$
Kenmotsu	$\frac{1}{(-1)^{\ell-1}(n-1)^\ell}$	$(-1)^\ell(n-1)^\ell g$
(ε)-*Sasakian*	$\frac{1}{(\varepsilon)^{\ell-1}(n-1)^\ell}$	$(\varepsilon)^\ell(n-1)^\ell g$
para-Sasakian	$\frac{1}{(-1)^{\ell-1}(n-1)^\ell}$	$(-1)^\ell(n-1)^\ell g$
(ε)-*para-Sasakian*	$\frac{1}{(-\varepsilon)^{\ell-1}(n-1)^\ell}$	$(-\varepsilon)^\ell(n-1)^\ell g$

Corollary 5.41. *Let M be an n-dimensional $(R,\mathcal{W}_8,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_8$-semisymmetric, then M is an Einstein manifold with scalar curvature $kn(n-1)$ and $L_S = \dfrac{1}{n-1}$. Consequently, we have the following:*

M	$L_S =$	$S =$
$N(k)$-*contact metric*	$\frac{1}{n-1}$	$k(n-1)g$
Sasakian	$\frac{1}{n-1}$	$(n-1)g$
Kenmotsu	$\frac{1}{n-1}$	$-(n-1)g$
(ε)-*Sasakian*	$\frac{1}{n-1}$	$\varepsilon(n-1)g$
para-Sasakian	$\frac{1}{n-1}$	$-(n-1)g$
(ε)-*para-Sasakian*	$\frac{1}{n-1}$	$-\varepsilon(n-1)g$

Corollary 5.42. *Let M be an n-dimensional $(R,\mathcal{W}_9,S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_9$-semisymmetric, then*

$$S^\ell = k^\ell(n-1)^\ell g$$

and $L_{S^\ell} = \dfrac{1}{k^{\ell-1}(n-1)^\ell}$. *Consequently, we have the following:*

M	$L_{S^\ell} =$	$S^\ell =$
$N(k)$-*contact metric*	$\frac{1}{k^{\ell-1}(n-1)^\ell}$	$k^\ell(n-1)^\ell g$
Sasakian	$\frac{1}{(n-1)^\ell}$	$(n-1)^\ell g$
Kenmotsu	$\frac{1}{(-1)^{\ell-1}(n-1)^\ell}$	$(-1)^\ell(n-1)^\ell g$
(ε)-*Sasakian*	$\frac{1}{(\varepsilon)^{\ell-1}(n-1)^\ell}$	$(\varepsilon)^\ell(n-1)^\ell g$
para-Sasakian	$\frac{1}{(-1)^{\ell-1}(n-1)^\ell}$	$(-1)^\ell(n-1)^\ell g$
(ε)-*para-Sasakian*	$\frac{1}{(-\varepsilon)^{\ell-1}(n-1)^\ell}$	$(-\varepsilon)^\ell(n-1)^\ell g$

Corollary 5.43. *Let M be an n-dimensional $(R,\mathcal{W}_9,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is not $\mathcal{W}_9$-semisymmetric, then M is an Einstein manifold*

with scalar curvature $kn(n-1)$ *and* $L_S = \dfrac{1}{n-1}$. *Consequently, we have the following:*

M	$L_S =$	$S =$
$N(k)$-contact metric	$\frac{1}{n-1}$	$k(n-1)g$
Sasakian	$\frac{1}{n-1}$	$(n-1)g$
Kenmotsu	$\frac{1}{n-1}$	$-(n-1)g$
(ε)-Sasakian	$\frac{1}{n-1}$	$\varepsilon(n-1)g$
para-Sasakian	$\frac{1}{n-1}$	$-(n-1)g$
(ε)-para-Sasakian	$\frac{1}{n-1}$	$-\varepsilon(n-1)g$

6. $(\mathcal{T}_a, S_{\mathcal{T}_b})$-PSEUDOSYMMETRY

In this section, we determine the results for an n-dimensional $(N(k),\xi)$-semi-Riemannian manifold satisfy $\mathcal{T}_a \cdot S_{\mathcal{T}_b} = LQ(g, S_{\mathcal{T}_b})$.

Definition 6.1. A semi-Riemannian manifold is said to be $(\mathcal{T}_a, S_{\mathcal{T}_b})$-pseudosymmetric if

$$\mathcal{T}_a \cdot S_{\mathcal{T}_b} = LQ(g, S_{\mathcal{T}_b}), \tag{6.1}$$

where L is some smooth function defined on M. In particular, it is said to be $(R \cdot S_{\mathcal{T}_a})$-pseudosymmetric if it satisfies

$$R \cdot S_{\mathcal{T}_a} = LQ(g, S_{\mathcal{T}_a}), \tag{6.2}$$

holds on the set $\mathcal{U} = \left\{ x \in M : \left(S_{\mathcal{T}_a} - \dfrac{tr(S_{\mathcal{T}_a})}{n} g \right)_x \neq 0 \right\}$, where L is some function defined on $\mathcal{U}$.

Remark 6.1. If in (6.2), $S_{\mathcal{T}_a}$ is replaced by S then it is said to be Ricci-pseudosymmetric.

Theorem 6.1. *Let M be an n-dimensional $(\mathcal{T}_a, S_{\mathcal{T}_b})$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$\begin{aligned}
& \varepsilon a_5(b_0 + nb_1 + b_2 + b_3 + b_5 + b_6)S^2(Y,U) \\
& + \{\varepsilon(b_0 + nb_1 + b_2 + b_3 + b_5 + b_6)\times \\
& (-ka_0 + k(n-1)a_1 + k(n-1)a_2 - a_7 r) \\
& \quad + \varepsilon(a_1 + a_5)(b_4 r + (n-1)b_7 r)\} S(Y,U) \\
& + \{\varepsilon k(n-1)(a_2 + a_4)(b_4 r + (n-1)b_7 r) \\
& \quad + \varepsilon k(n-1)(b_0 + nb_1 + b_2 + b_3 + b_5 + b_6) \times \\
& (ka_0 + k(n-1)a_4 + a_7 r)\} g(Y,U) \\
& + k(n-1)(a_1 + a_2 + 2a_3 + a_4 + a_5 + 2a_6) \times \\
& \{b_4 r + (n-1)b_7 r \\
& \quad + k(n-1)(b_0 + nb_1 + b_2 + b_3 + b_5 + b_6)\} \eta(Y)\eta(U) \\
= \; & L(b_0 + nb_1 + b_2 + b_3 + b_5 + b_6)(\varepsilon k(n-1)g(Y,U) - \varepsilon S(Y,U)).
\end{aligned}$$

In particular, if M is an n-dimensional $(\mathcal{T}_a, S_{\mathcal{T}_a})$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold, then

$$\begin{aligned} & \varepsilon a_5(a_0+na_1+a_2+a_3+a_5+a_6)S^2(Y,U) \\ & + \{\varepsilon(a_0+na_1+a_2+a_3+a_5+a_6)\times \\ & (-ka_0+k(n-1)a_1+k(n-1)a_2-a_7r) \\ & \quad + \varepsilon(a_1+a_5)(a_4r+(n-1)a_7r)\}\, S(Y,U) \\ & + \{\varepsilon k(n-1)(a_2+a_4)(a_4r+(n-1)a_7r) \\ & \quad + \varepsilon k(n-1)(a_0+na_1+a_2+a_3+a_5+a_6)\times \\ & (ka_0+k(n-1)a_4+a_7r)\}\, g(Y,U) \\ & + k(n-1)(a_1+a_2+2a_3+a_4+a_5+2a_6) \\ & \{a_4r+(n-1)a_7r \\ & \quad + \varepsilon k(n-1)(a_0+na_1+a_2+a_3+a_5+a_6)\}\,\eta(Y)\eta(U) \\ = \; & L(a_0+na_1+a_2+a_3+a_5+a_6)(\varepsilon k(n-1)g(Y,U)-\varepsilon S(Y,U)). \end{aligned}$$

Proof. Let M be an n-dimensional $(\mathcal{T}_a, S_{\mathcal{T}_b})$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then

$$\mathcal{T}_a(X,Y)\cdot S_{\mathcal{T}_b}(U,V) = LQ(g,S_{\mathcal{T}_b})(U,V;X,Y). \tag{6.3}$$

Taking $X=\xi=V$ in (6.3), we have

$$\mathcal{T}_a(\xi,Y)\cdot S_{\mathcal{T}_b}(U,\xi) = LQ(g,S_{\mathcal{T}_b})(U,\xi;\xi,Y),$$

which gives

$$\begin{aligned} & S_{\mathcal{T}_b}(\mathcal{T}_a(\xi,Y)U,\xi)+S_{\mathcal{T}_b}(U,\mathcal{T}_a(\xi,Y)\xi) \\ = \; & L\left(S_{\mathcal{T}_b}((\xi\wedge Y)U,\xi)+S_{\mathcal{T}_b}(U,(\xi\wedge Y)\xi)\right). \end{aligned} \tag{6.4}$$

Using (3.1), (3.23), (3.30), (3.31), (3.35) and (3.36) in (6.4), we get the result. □

Theorem 6.2. *Let M be an n-dimensional $(\mathcal{T}_a, S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$\begin{aligned} & \varepsilon a_5\, S^2(Y,U) - E\, S(Y,U) - Fg(Y,U) - G\eta(Y)\eta(U) \\ = \; & L(\varepsilon k(n-1)g(Y,U)-\varepsilon S(Y,U)), \end{aligned}$$

where

$$E = \varepsilon\,(ka_0+a_7r-k(n-1)a_1-k(n-1)a_2),$$

$$F = -\,\varepsilon k(n-1)(ka_0+k(n-1)a_4+a_7r),$$

$$G = -\,k^2(n-1)^2(a_1+a_2+2a_3+a_4+a_5+2a_6).$$

In view of Theorem 6.2, we have the following

Corollary 6.1. *Let M be an n-dimensional Ricci-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$L=$	$S=$
$N(k)$-*contact metric*	k	$k(n-1)g$
Sasakian	1	$(n-1)g$
Kenmotsu	-1	$-(n-1)g$
(ε)-*Sasakian*	ε	$\varepsilon(n-1)g$
para-Sasakian	-1	$-(n-1)g$
(ε)-*para-Sasakian*	$-\varepsilon$	$-\varepsilon(n-1)g$

Corollary 6.2. *Let M be an n-dimensional $(\mathcal{C}_*, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$S^2 =$
$N(k)$-contact metric	$-\left(\left(k - \frac{r}{n(n-1)}\right)\frac{a_0}{a_1} - \frac{2r}{n} - \frac{L}{a_1}\right)S$ $+k(n-1)\left(\left(k - \frac{r}{n(n-1)}\right)\frac{a_0}{a_1} + k(n-1) - \frac{2r}{n} - \frac{L}{a_1}\right)g$
Sasakian	$-\left(\left(1 - \frac{r}{n(n-1)}\right)\frac{a_0}{a_1} - \frac{2r}{n} - \frac{L}{a_1}\right)S$ $+(n-1)\left(\left(1 - \frac{r}{n(n-1)}\right)\frac{a_0}{a_1} + (n-1) - \frac{2r}{n} - \frac{L}{a_1}\right)g$
Kenmotsu	$\left(\left(1 + \frac{r}{n(n-1)}\right)\frac{a_0}{a_1} + \frac{2r}{n} + \frac{L}{a_1}\right)S$ $+(n-1)\left(\left(1 + \frac{r}{n(n-1)}\right)\frac{a_0}{a_1} + (n-1) + \frac{2r}{n} + \frac{L}{a_1}\right)g$
(ε)-Sasakian	$-\varepsilon\left(\left(1 - \frac{\varepsilon r}{n(n-1)}\right)\frac{a_0}{a_1} - \frac{2\varepsilon r}{n} - \frac{\varepsilon L}{a_1}\right)S$ $+\varepsilon(n-1)\left(\left(\varepsilon - \frac{r}{n(n-1)}\right)\frac{a_0}{a_1} + \varepsilon(n-1) - \frac{2r}{n} - \frac{L}{a_1}\right)g$
para-Sasakian	$\left(\left(1 + \frac{r}{n(n-1)}\right)\frac{a_0}{a_1} + \frac{2r}{n} + \frac{L}{a_1}\right)S$ $+(n-1)\left(\left(1 + \frac{r}{n(n-1)}\right)\frac{a_0}{a_1} + (n-1) + \frac{2r}{n} + \frac{L}{a_1}\right)g$
(ε)-para-Sasakian	$\varepsilon\left(\left(1 + \frac{\varepsilon r}{n(n-1)}\right)\frac{a_0}{a_1} + \frac{2\varepsilon r}{n} + \frac{\varepsilon L}{a_1}\right)S$ $+\varepsilon(n-1)\left(\left(\varepsilon + \frac{r}{n(n-1)}\right)\frac{a_0}{a_1} + \varepsilon(n-1) + \frac{2r}{n} + \frac{L}{a_1}\right)g$

Corollary 6.3. *Let M be an n-dimensional $(\mathcal{C}, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$S^2 =$
$N(k)$-contact metric	$\left(\frac{r}{n-1} + (k-L)(n-2)\right)S - k(r - (L(n-2)+k)(n-1))g$
Sasakian	$\left(\frac{r}{n-1} + (1-L)(n-2)\right)S - (r - (L(n-2)+1)(n-1))g$
Kenmotsu	$\left(\frac{r}{n-1} - (1+L)(n-2)\right)S + (r - (L(n-2)-1)(n-1))g$
(ε)-Sasakian	$\left(\frac{r}{n-1} + (\varepsilon-L)(n-2)\right)S - \varepsilon(r - (L(n-2)+\varepsilon)(n-1))g$
para-Sasakian	$\left(\frac{r}{n-1} - (1+L)(n-2)\right)S + 1(r - (L(n-2)-1)(n-1))g$
(ε)-para-Sasakian	$\left(\frac{r}{n-1} - (\varepsilon+L)(n-2)\right)S + \varepsilon(r - (L(n-2)-\varepsilon)(n-1))g$

Corollary 6.4. *Let M be an n-dimensional $(\mathcal{L}, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$S^2 =$
$N(k)$-contact metric	$(n-2)(k-L)S + k(n-1)(k+(n-2)L)g$
Sasakian	$(n-2)(1-L)S + (n-1)(1+(n-2)L)g$
Kenmotsu	$-(n-2)(1+L)S - (n-1)(-1+(n-2)L)g$
(ε)-Sasakian	$(n-2)(\varepsilon-L)S + \varepsilon(n-1)(\varepsilon+(n-2)L)g$
para-Sasakian	$-(n-2)(1+L)S - (n-1)(-1+(n-2)L)g$
(ε)-para-Sasakian	$-(n-2)(\varepsilon+L)S - \varepsilon(n-1)(-\varepsilon+(n-2)L)g$

Corollary 6.5. *Let M be an n-dimensional $(\mathcal{V}, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$L =$	$S =$
$N(k)$-contact metric	$k - \frac{r}{n(n-1)}$	$k(n-1)g$
Sasakian	$1 - \frac{r}{n(n-1)}$	$(n-1)g$
Kenmotsu	$-1 - \frac{r}{n(n-1)}$	$-(n-1)g$
(ε)-Sasakian	$\varepsilon - \frac{r}{n(n-1)}$	$\varepsilon(n-1)g$
para-Sasakian	$-1 - \frac{r}{n(n-1)}$	$-(n-1)g$
(ε)-para-Sasakian	$-\varepsilon - \frac{r}{n(n-1)}$	$-\varepsilon(n-1)g$

Corollary 6.6. *Let M be an n-dimensional $(\mathcal{P}_*, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$L =$	$S =$
$N(k)$-contact metric	$\left(k - \frac{r}{n(n-1)}\right) a_0 - \frac{r}{n} a_1$	$k(n-1)g$
Sasakian	$\left(1 - \frac{r}{n(n-1)}\right) a_0 - \frac{r}{n} a_1$	$(n-1)g$
Kenmotsu	$\left(-1 - \frac{r}{n(n-1)}\right) a_0 - \frac{r}{n} a_1$	$-(n-1)g$
(ε)-Sasakian	$\left(\varepsilon - \frac{r}{n(n-1)}\right) a_0 - \frac{r}{n} a_1$	$\varepsilon(n-1)g$
para-Sasakian	$\left(-1 - \frac{r}{n(n-1)}\right) a_0 - \frac{r}{n} a_1$	$-(n-1)g$
(ε)-para-Sasakian	$\left(-\varepsilon - \frac{r}{n(n-1)}\right) a_0 - \frac{r}{n} a_1$	$-\varepsilon(n-1)g$

Corollary 6.7. *Let M be an n-dimensional $(\mathcal{P}, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$L =$	$S =$
$N(k)$-contact metric	k	$k(n-1)g$
Sasakian	1	$(n-1)g$
Kenmotsu	-1	$-(n-1)g$
(ε)-Sasakian	ε	$\varepsilon(n-1)g$
para-Sasakian	-1	$-(n-1)g$
(ε)-para-Sasakian	$-\varepsilon$	$-\varepsilon(n-1)g$

Corollary 6.8. *Let M be an n-dimensional $(\mathcal{M}, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$S^2 =$
$N(k)$-contact metric	$2(n-1)(k-L)S - k(n-1)^2(k-2L)g$
Sasakian	$2(n-1)(1-L)S - (n-1)^2(1-2L)g$
Kenmotsu	$-2(n-1)(1+L)S - (n-1)^2(1+2L)g$
(ε)-Sasakian	$2(n-1)(\varepsilon-L)S - \varepsilon(n-1)^2(\varepsilon-2L)g$
para-Sasakian	$-2(n-1)(1+L)S - (n-1)^2(1+2L)g$
(ε)-para-Sasakian	$-2(n-1)(\varepsilon+L)S - \varepsilon(n-1)^2(\varepsilon+2L)g$

Corollary 6.9. *Let M be an n-dimensional $(\mathcal{W}_0, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$S^2 =$
$N(k)$-contact metric	$(n-1)(2k-L)S + k(n-1)^2 (L-k)\, g$
Sasakian	$(n-1)(2-L)S + (n-1)^2 (L-1)\, g$
Kenmotsu	$-(n-1)(2+L)S - (n-1)^2 (L+1)\, g$
(ε)-Sasakian	$(n-1)(2\varepsilon - L)S + \varepsilon(n-1)^2 (L-\varepsilon)\, g$
para-Sasakian	$-(n-1)(2+L)S - (n-1)^2 (L+1)\, g$
(ε)-para-Sasakian	$-(n-1)(2\varepsilon + L)S - \varepsilon(n-1)^2 (L+\varepsilon)\, g$

Corollary 6.10. *Let M be an n-dimensional $(\mathcal{W}_0^*, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$S^2 =$
$N(k)$-contact metric	$L(n-1)S + k(n-1)^2(k-L)g$
Sasakian	$L(n-1)S + (n-1)^2(1-L)g$
Kenmotsu	$L(n-1)S + (n-1)^2(1+L)g$
(ε)-Sasakian	$L(n-1)S + \varepsilon(n-1)^2(\varepsilon-L)g$
para-Sasakian	$L(n-1)S + (n-1)^2(1+L)g$
(ε)-para-Sasakian	$L(n-1)S + \varepsilon(n-1)^2(\varepsilon+L)g$

Corollary 6.11. *Let M be an n-dimensional $(\mathcal{W}_1, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$L =$	$S =$
$N(k)$-contact metric	k	$k(n-1)g$
Sasakian	1	$(n-1)g$
Kenmotsu	-1	$-(n-1)g$
(ε)-Sasakian	ε	$\varepsilon(n-1)g$
para-Sasakian	-1	$-(n-1)g$
(ε)-para-Sasakian	$-\varepsilon$	$-\varepsilon(n-1)g$

Corollary 6.12. *Let M be an n-dimensional $(\mathcal{W}_1^*, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$L =$	$S =$
$N(k)$-contact metric	k	$k(n-1)g$
Sasakian	1	$(n-1)g$
Kenmotsu	-1	$-(n-1)g$
(ε)-Sasakian	ε	$\varepsilon(n-1)g$
para-Sasakian	-1	$-(n-1)g$
(ε)-para-Sasakian	$-\varepsilon$	$-\varepsilon(n-1)g$

Corollary 6.13. *Let M be an n-dimensional $(\mathcal{W}_2, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$S^2 =$
$N(k)$-contact metric	$(n-1)(k-L)S + k(n-1)^2 Lg$
Sasakian	$(n-1)(1-L)S + (n-1)^2 Lg$
Kenmotsu	$-(n-1)(1+L)S - (n-1)^2 Lg$
(ε)-Sasakian	$(n-1)(\varepsilon-L)S + \varepsilon(n-1)^2 Lg$
para-Sasakian	$-(n-1)(1+L)S - (n-1)^2 Lg$
(ε)-para-Sasakian	$-(n-1)(\varepsilon+L)S - \varepsilon(n-1)^2 Lg$

Corollary 6.14. *Let M be an n-dimensional $(\mathcal{W}_3,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$L=$	$S=$
$N(k)$-contact metric	$2k$	$k(n-1)g$
Sasakian	2	$(n-1)g$
Kenmotsu	-2	$-(n-1)g$
(ε)-Sasakian	2ε	$\varepsilon(n-1)g$
para-Sasakian	-2	$-(n-1)g$
(ε)-para-Sasakian	-2ε	$-\varepsilon(n-1)g$

Corollary 6.15. *Let M be an n-dimensional $(\mathcal{W}_4,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$S^2=$
$N(k)$-contact metric	$(n-1)(k-L)S+k(n-1)^2(L-k)g+\varepsilon k^2(n-1)^2\eta\otimes\eta$
Sasakian	$(n-1)(1-L)S+(n-1)^2(L-1)g+\varepsilon(n-1)^2\eta\otimes\eta$
Kenmotsu	$-(n-1)(1+L)S-(n-1)^2(L+1)g+\varepsilon(n-1)^2\eta\otimes\eta$
(ε)-Sasakian	$(n-1)(\varepsilon-L)S+\varepsilon(n-1)^2(L-\varepsilon)g+\varepsilon(n-1)^2\eta\otimes\eta$
para-Sasakian	$-(n-1)(1+L)S-(n-1)^2(L+1)g+\varepsilon(n-1)^2\eta\otimes\eta$
(ε)-para-Sasakian	$-(n-1)(\varepsilon+L)S-\varepsilon(n-1)^2(L+\varepsilon)g+\varepsilon(n-1)^2\eta\otimes\eta$

Corollary 6.16. *Let M be an n-dimensional $(\mathcal{W}_5,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$S^2=$
$N(k)$-contact metric	$(n-1)(2k-L)S+k(n-1)^2(L-k)g$
Sasakian	$(n-1)(2-L)S+(n-1)^2(L-1)g$
Kenmotsu	$-(n-1)(2+L)S-(n-1)^2(L+1)g$
(ε)-Sasakian	$(n-1)(2\varepsilon-L)S+\varepsilon(n-1)^2(L-\varepsilon)g$
para-Sasakian	$-(n-1)(2+L)S-(n-1)^2(L+1)g$
(ε)-para-Sasakian	$-(n-1)(2\varepsilon+L)S+k(n-1)^2(L+\varepsilon)g$

Corollary 6.17. *Let M be an n-dimensional $(\mathcal{W}_6,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	***Result***
$N(k)$-contact metric	$(2k-L)S+k(n-1)(L-k)g+k^2(n-1)\eta\otimes\eta=0$
Sasakian	$(2-L)S+(n-1)(L-1)g+(n-1)\eta\otimes\eta=0$
Kenmotsu	$-(2+L)S-(n-1)(L+1)g+(n-1)\eta\otimes\eta=0$
(ε)-Sasakian	$(2\varepsilon-L)S+\varepsilon(n-1)(L-\varepsilon)g+\varepsilon(n-1)\eta\otimes\eta=0$
para-Sasakian	$-(2+L)S-(n-1)(L+1)g+(n-1)\eta\otimes\eta=0$
(ε)-para-Sasakian	$-(2\varepsilon+L)S-\varepsilon(n-1)(L+\varepsilon)g+\varepsilon(n-1)\eta\otimes\eta=0$

Corollary 6.18. *Let M be an n-dimensional $(\mathcal{W}_7,S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$L=$	$S=$
$N(k)$-contact metric	$2k$	$k(n-1)g$
Sasakian	2	$(n-1)g$
Kenmotsu	-2	$-(n-1)g$
(ε)-Sasakian	2ε	$\varepsilon(n-1)g$
para-Sasakian	-2	$-(n-1)g$
(ε)-para-Sasakian	-2ε	$-\varepsilon(n-1)g$

Corollary 6.19. *Let M be an n-dimensional $(\mathcal{W}_8, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table:*

M	***Result***
$N(k)$-contact metric	$(2k - L)\,S + k(n-1)\,(L-k)\,g - k^2(n-1)\eta\otimes\eta = 0$
Sasakian	$(2 - L)\,S + (n-1)\,(L-1)\,g - (n-1)\eta\otimes\eta = 0$
Kenmotsu	$-\,(2+L)\,S - (n-1)\,(L+1)\,g - (n-1)\eta\otimes\eta = 0$
(ε)-Sasakian	$(2\varepsilon - L)\,S + \varepsilon(n-1)\,(L-\varepsilon)\,g - \varepsilon(n-1)\eta\otimes\eta = 0$
para-Sasakian	$-\,(2+L)\,S - (n-1)\,(L+1)\,g - (n-1)\eta\otimes\eta = 0$
(ε)-para-Sasakian	$-\,(2\varepsilon + L)\,S - \varepsilon(n-1)\,(L+\varepsilon)\,g - \varepsilon(n-1)\eta\otimes\eta = 0$

Corollary 6.20. *Let M be an n-dimensional $(\mathcal{W}_9, S)$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table:*

M	***Result***
$N(k)$-contact metric	$(L-k)\,S - k(n-1)Lg + k^2(n-1)\eta\otimes\eta = 0$
Sasakian	$(L-1)\,S - (n-1)Lg + (n-1)\eta\otimes\eta = 0$
Kenmotsu	$(L+1)\,S + (n-1)Lg + (n-1)\eta\otimes\eta = 0$
(ε)-Sasakian	$(L-\varepsilon)\,S - \varepsilon(n-1)Lg + \varepsilon(n-1)\eta\otimes\eta = 0$
para-Sasakian	$(L+1)\,S + (n-1)Lg + (n-1)\eta\otimes\eta = 0$
(ε)-para-Sasakian	$(L+\varepsilon)\,S + \varepsilon(n-1)Lg + \varepsilon(n-1)\eta\otimes\eta = 0$

Theorem 6.3. *Let M be an n-dimensional $(R, S_{\mathcal{T}_a})$-pseudosymmetric $(N(k), \xi)$-semi-Riemannian manifold such that*

$$a_0 + na_1 + a_2 + a_3 + a_5 + a_6 \neq 0.$$

Then M is either Einstein manifold, that is,

$$S = k(n-1)g$$

or $L = k$ holds on M. Consequently, we have the following table:

Manifold	***Condition***	$S =$	$L =$
$N(k)$-contact metric	$R\cdot S_{\mathcal{T}_a} = LQ(g, S_{\mathcal{T}_a})$	$k(n-1)g$	k
Sasakian	$R\cdot S_{\mathcal{T}_a} = LQ(g, S_{\mathcal{T}_a})$	$(n-1)g$	1
Kenmotsu	$R\cdot S_{\mathcal{T}_a} = LQ(g, S_{\mathcal{T}_a})$	$-\,(n-1)g$	-1
(ε)-Sasakian	$R\cdot S_{\mathcal{T}_a} = LQ(g, S_{\mathcal{T}_a})$	$\varepsilon(n-1)g$	ε
para-Sasakian	$R\cdot S_{\mathcal{T}_a} = LQ(g, S_{\mathcal{T}_a})$	$-\,(n-1)g$	-1
(ε)-para-Sasakian	$R\cdot S_{\mathcal{T}_a} = LQ(g, S_{\mathcal{T}_a})$	$-\,\varepsilon(n-1)g$	$-\,\varepsilon$

Remark 6.2. The conclusions of Theorem 6.3 remain true if $S_{\mathcal{T}_a}$ is replaced by S.

Corollary 6.21. *(*[19], [14]*) If an n-dimensional Kenmotsu manifold M is Ricci-pseudosymmetric then either M is an Einstein manifold with the scalar curvature $r = n(1-n)$ or $L = -1$ holds on M.*

7. $(\mathcal{T}_a, S_{\mathcal{T}_b}, S^{\ell})$-PSEUDOSYMMETRY

In this section, we determine the result for an n-dimensional $(N(k), \xi)$-semi-Riemannian manifold satisfy $\mathcal{T}_a \cdot S_{\mathcal{T}_b} = LQ(S^{\ell}, S_{\mathcal{T}_b})$.

Definition 7.1. A semi-Riemannian manifold M is called $(\mathcal{T}_a, S_{\mathcal{T}_b}, S^{\ell})$-pseudosymmetric if

$$\mathcal{T}_a \cdot S_{\mathcal{T}_b} = LQ(S^{\ell}, S_{\mathcal{T}_b}),$$

where L is some smooth function defined on M. In particular, M is said to be $(R, S_{\mathcal{T}_a}, S^{\ell})$-pseudosymmetric if

$$R \cdot S_{\mathcal{T}_a} = LQ(S^{\ell}, S_{\mathcal{T}_a}).$$

Theorem 7.1. *Let M be an n-dimensional $(\mathcal{T}_a, S_{\mathcal{T}_b}, S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$\begin{aligned}
& \varepsilon a_5(b_0+nb_1+b_2+b_3+b_5+b_6)S^2(Y,U) \\
& + \{\varepsilon(b_0+nb_1+b_2+b_3+b_5+b_6)\times \\
& (-ka_0+k(n-1)a_1+k(n-1)a_2-a_7r) \\
& \quad + \varepsilon(a_1+a_5)(b_4r+(n-1)b_7r)\}\, S(Y,U) \\
& + \{\varepsilon k(n-1)(a_2+a_4)(b_4r+(n-1)b_7r) \\
& \quad + \varepsilon k(n-1)(b_0+nb_1+b_2+b_3+b_5+b_6)\times \\
& (ka_0+k(n-1)a_4+a_7r)\}\, g(Y,U) \\
& + k(n-1)(a_1+a_2+2a_3+a_4+a_5+2a_6)\times \\
& \{b_4r+(n-1)b_7r \\
& \quad + k(n-1)(b_0+nb_1+b_2+b_3+b_5+b_6)\}\, \eta(Y)\eta(U) \\
= \; & L\varepsilon((b_0+nb_1+b_2+b_3+b_5+b_6)\times \\
& (k(n-1)S^\ell(Y,U)-k^\ell(n-1)^\ell S(Y,U)) \\
& +(b_4+(n-1)b_7)r(k^\ell(n-1)^\ell g(Y,U)-S^\ell(Y,U))).
\end{aligned}$$

In particular, if M be an n-dimensional $(\mathcal{T}_a, S_{\mathcal{T}_a}, S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then

$$\begin{aligned}
& \varepsilon a_5(a_0+na_1+a_2+a_3+a_5+a_6)S^2(Y,U) \\
& + \{\varepsilon(a_0+na_1+a_2+a_3+a_5+a_6)\times \\
& (-ka_0+k(n-1)a_1+k(n-1)a_2-a_7r) \\
& \quad + \varepsilon(a_1+a_5)(a_4r+(n-1)a_7r)\}\, S(Y,U) \\
& + \{\varepsilon k(n-1)(a_2+a_4)(a_4r+(n-1)a_7r) \\
& \quad + \varepsilon k(n-1)(a_0+na_1+a_2+a_3+a_5+a_6)\times \\
& (ka_0+k(n-1)a_4+a_7r)\}\, g(Y,U) \\
& + k(n-1)(a_1+a_2+2a_3+a_4+a_5+2a_6) \\
& \{a_4r+(n-1)a_7r \\
& \quad + \varepsilon k(n-1)(a_0+na_1+a_2+a_3+a_5+a_6)\}\, \eta(Y)\eta(U) \\
= \; & L\varepsilon((a_0+na_1+a_2+a_3+a_5+a_6)\times \\
& (k(n-1)S^\ell(Y,U)-k^\ell(n-1)^\ell S(Y,U)) \\
& +(a_4+(n-1)a_7)r(k^\ell(n-1)^\ell g(Y,U)-S^\ell(Y,U))).
\end{aligned}$$

Proof. Let M be an n-dimensional $(\mathcal{T}_a, S_{\mathcal{T}_b}, S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then

$$\mathcal{T}_a(X,Y)\cdot S_{\mathcal{T}_b}(U,V) = LQ(S^\ell, S_{\mathcal{T}_b})(U,V;X,Y). \tag{7.1}$$

Taking $X=\xi=V$ in (7.1), we have

$$\mathcal{T}_a(\xi,Y)\cdot S_{\mathcal{T}_b}(U,\xi) = LQ(S^\ell, S_{\mathcal{T}_b})(U,\xi;\xi,Y),$$

which gives

$$\begin{aligned}
& S_{\mathcal{T}_b}(\mathcal{T}_a(\xi,Y)U,\xi)+S_{\mathcal{T}_b}(U,\mathcal{T}_a(\xi,Y)\xi) \\
= \; & L\left(S_{\mathcal{T}_b}((\xi\wedge_{S^\ell}Y)U,\xi)+S_{\mathcal{T}_b}(U,(\xi\wedge_{S^\ell}Y)\xi)\right).
\end{aligned} \tag{7.2}$$

Using (3.1), (3.28), (3.30), (3.31), (3.35) and (3.36) in (7.2), we get the result. □

For $\ell = 1$, we have the following result.

Corollary 7.1. *Let M be an n-dimensional $(\mathcal{T}_a, S_{\mathcal{T}_b}, S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$\begin{aligned}
& \varepsilon a_5(b_0+nb_1+b_2+b_3+b_5+b_6)S^2(Y,U) \\
& + \ \{\varepsilon(b_0+nb_1+b_2+b_3+b_5+b_6)\times \\
& (-ka_0+k(n-1)a_1+k(n-1)a_2-a_7r) \\
& \qquad + \varepsilon(a_1+a_5)(b_4r+(n-1)b_7r)\}\,S(Y,U) \\
& + \ \{\varepsilon k(n-1)(a_2+a_4)(b_4r+(n-1)b_7r) \\
& \qquad + \varepsilon k(n-1)(b_0+nb_1+b_2+b_3+b_5+b_6)\times \\
& (ka_0+k(n-1)a_4+a_7r)\}\,g(Y,U) \\
& + k(n-1)(a_1+a_2+2a_3+a_4+a_5+2a_6)\times \\
& \{b_4r+(n-1)b_7r \\
& \qquad + k(n-1)(b_0+nb_1+b_2+b_3+b_5+b_6)\}\,\eta(Y)\eta(U) \\
= \ & L\varepsilon(b_4+(n-1)b_7)r(k(n-1)g(Y,U)-S(Y,U)).
\end{aligned}$$

In particular, if M be an n-dimensional $(\mathcal{T}_a, S_{\mathcal{T}_a}, S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then

$$\begin{aligned}
& \varepsilon a_5(a_0+na_1+a_2+a_3+a_5+a_6)S^2(Y,U) \\
& + \ \{\varepsilon(a_0+na_1+a_2+a_3+a_5+a_6)\times \\
& (-ka_0+k(n-1)a_1+k(n-1)a_2-a_7r) \\
& \qquad + \varepsilon(a_1+a_5)(a_4r+(n-1)a_7r)\}\,S(Y,U) \\
& + \ \{\varepsilon k(n-1)(a_2+a_4)(a_4r+(n-1)a_7r) \\
& \qquad + \varepsilon k(n-1)(a_0+na_1+a_2+a_3+a_5+a_6)\times \\
& (ka_0+k(n-1)a_4+a_7r)\}\,g(Y,U) \\
& + k(n-1)(a_1+a_2+2a_3+a_4+a_5+2a_6) \\
& \{a_4r+(n-1)a_7r \\
& \qquad + \varepsilon k(n-1)(a_0+na_1+a_2+a_3+a_5+a_6)\}\,\eta(Y)\eta(U) \\
= \ & L\varepsilon(a_4+(n-1)a_7)r(k(n-1)g(Y,U)-S(Y,U)).
\end{aligned}$$

Theorem 7.2. *Let M be an n-dimensional $\left(\mathcal{T}_a, S, S^\ell\right)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$\begin{aligned}
& \varepsilon a_5\,S^2(Y,U)-E\,S(Y,U)-Fg(Y,U)-G\eta(Y)\eta\,(U) \\
= \ & \varepsilon L(k(n-1)S^\ell(Y,U)-k^\ell(n-1)^\ell S(Y,U)),
\end{aligned}$$

where

$$E=\varepsilon\,(ka_0+a_7r-k(n-1)a_1-k(n-1)a_2),$$

$$F=-\,\varepsilon k(n-1)(ka_0+k(n-1)a_4+a_7r),$$

$$G=-\,k^2(n-1)^2(a_1+a_2+2a_3+a_4+a_5+2a_6).$$

In view of Theorem 7.2, we have the following

Corollary 7.2. *Let M be an n-dimensional $\left(R,S,S^{\ell}\right)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^{\ell}=$
$N(k)$-*contact metric*	$\left(Lk^{\ell-1}(n-1)^{\ell-1}-\frac{1}{n-1}\right)S+kg$
Sasakian	$\left(L(n-1)^{\ell-1}-\frac{1}{n-1}\right)S+g$
Kenmotsu	$\left(L(-1)^{\ell-1}(n-1)^{\ell-1}-\frac{1}{n-1}\right)S-g$
(ε)-*Sasakian*	$\left(L(\varepsilon)^{\ell-1}(n-1)^{\ell-1}-\frac{1}{n-1}\right)S+\varepsilon g$
para-Sasakian	$\left(L(-1)^{\ell-1}(n-1)^{\ell-1}-\frac{1}{n-1}\right)S-g$
(ε)-*para-Sasakian*	$\left(L(-\varepsilon)^{\ell-1}(n-1)^{\ell-1}-\frac{1}{n-1}\right)S-\varepsilon g$

Corollary 7.3. *Let M be an n-dimensional $\left(\mathcal{C}_*,S,S^{\ell}\right)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$L(n-1)S^{\ell}=$
$N(k)$-*contact metric*	$-\frac{a_1}{k}S^2-\left(\left(1-\frac{r}{kn(n-1)}\right)a_0-\frac{2r}{kn}a_1-k^{\ell-1}(n-1)^{\ell}L\right)S$ $+(n-1)\left(\left(k-\frac{r}{n(n-1)}\right)a_0+\left(k(n-1)-\frac{2r}{n}\right)a_1\right)g$
Sasakian	$-a_1S^2-\left(\left(1-\frac{r}{n(n-1)}\right)a_0-\frac{2r}{n}a_1-(n-1)^{\ell}L\right)S$ $+(n-1)\left(\left(1-\frac{r}{n(n-1)}\right)a_0+\left((n-1)-\frac{2r}{n}\right)a_1\right)g$
Kenmotsu	$a_1S^2-\left(\left(1+\frac{r}{n(n-1)}\right)a_0+\frac{2r}{n}a_1-(-1)^{\ell-1}(n-1)^{\ell}L\right)S$ $-(n-1)\left(\left(1+\frac{r}{n(n-1)}\right)a_0+\left((n-1)+\frac{2r}{n}\right)a_1\right)g$
(ε)-*Sasakian*	$-\varepsilon a_1S^2-\left(\left(1-\frac{\varepsilon r}{n(n-1)}\right)a_0-\frac{2\varepsilon r}{n}a_1-(\varepsilon)^{\ell-1}(n-1)^{\ell}L\right)S$ $+(n-1)\left(\left(\varepsilon-\frac{r}{n(n-1)}\right)a_0+\left(\varepsilon(n-1)-\frac{2r}{n}\right)a_1\right)g$
para-Sasakian	$a_1S^2-\left(\left(1+\frac{r}{n(n-1)}\right)a_0+\frac{2r}{n}a_1-(-1)^{\ell-1}(n-1)^{\ell}L\right)S$ $-(n-1)\left(\left(1+\frac{r}{n(n-1)}\right)a_0+\left((n-1)+\frac{2r}{n}\right)a_1\right)g$
(ε)-*para-Sasakian*	$\varepsilon a_1S^2-\left(\left(1+\frac{\varepsilon r}{n(n-1)}\right)a_0+\frac{2\varepsilon r}{n}a_1-(-\varepsilon)^{\ell-1}(n-1)^{\ell}L\right)S$ $-(n-1)\left(\left(\varepsilon+\frac{r}{n(n-1)}\right)a_0+\left(\varepsilon(n-1)+\frac{2r}{n}\right)a_1\right)g$

Corollary 7.4. *Let M be an n-dimensional $\left(\mathcal{C},S,S^{\ell}\right)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^{\ell} =$
$N(k)$-*contact metric*	$-\left(\frac{r}{k(n-1)^2(n-2)}+\frac{1}{n-1}-k^{\ell-1}(n-1)^{\ell-1}L\right)S$ $+\left(k-\frac{n-1}{n-2}+\frac{r}{(n-1)(n-2)}\right)g+\frac{1}{k(n-1)(n-2)}S^2$
Sasakian	$-\left(\frac{r}{(n-1)^2(n-2)}+\frac{1}{n-1}-(n-1)^{\ell-1}L\right)S$ $+\left(1-\frac{n-1}{n-2}+\frac{r}{(n-1)(n-2)}\right)g+\frac{1}{(n-1)(n-2)}S^2$
Kenmotsu	$-\left(-\frac{r}{(n-1)^2(n-2)}+\frac{1}{n-1}-(-1)^{\ell-1}(n-1)^{\ell-1}L\right)S$ $+\left(-1-\frac{n-1}{n-2}+\frac{r}{(n-1)(n-2)}\right)g-\frac{1}{(n-1)(n-2)}S^2$
(ε)-*Sasakian*	$-\left(\frac{\varepsilon r}{(n-1)^2(n-2)}+\frac{1}{n-1}-(\varepsilon)^{\ell-1}(n-1)^{\ell-1}L\right)S$ $+\left(\varepsilon-\frac{n-1}{n-2}+\frac{r}{(n-1)(n-2)}\right)g+\frac{\varepsilon}{(n-1)(n-2)}S^2$
para-Sasakian	$-\left(-\frac{r}{(n-1)^2(n-2)}+\frac{1}{n-1}-(-1)^{\ell-1}(n-1)^{\ell-1}L\right)S$ $+\left(-1-\frac{n-1}{n-2}+\frac{r}{(n-1)(n-2)}\right)g-\frac{1}{(n-1)(n-2)}S^2$
(ε)-*para-Sasakian*	$-\left(-\frac{\varepsilon r}{(n-1)^2(n-2)}+\frac{1}{n-1}-(-\varepsilon)^{\ell-1}(n-1)^{\ell-1}L\right)S$ $+\left(-\varepsilon-\frac{n-1}{n-2}+\frac{r}{(n-1)(n-2)}\right)g-\frac{\varepsilon}{(n-1)(n-2)}S^2$

Corollary 7.5. *Let M be an n-dimensional $\left(\mathcal{L},S,S^{\ell}\right)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^{\ell} =$
$N(k)$-*contact metric*	$\frac{1}{k(n-1)(n-2)}S^2+\left(k^{\ell-1}(n-1)^{\ell-1}L-\frac{1}{n-1}\right)S-\frac{k}{n-2}g$
Sasakian	$\frac{1}{(n-1)(n-2)}S^2+\left((n-1)^{\ell-1}L-\frac{1}{n-1}\right)S-\frac{1}{n-2}g$
Kenmotsu	$-\frac{1}{(n-1)(n-2)}S^2+\left((-1)^{\ell-1}(n-1)^{\ell-1}L-\frac{1}{n-1}\right)S+\frac{1}{n-2}g$
(ε)-*Sasakian*	$\frac{\varepsilon}{(n-1)(n-2)}S^2+\left((\varepsilon)^{\ell-1}(n-1)^{\ell-1}L-\frac{1}{n-1}\right)S-\frac{\varepsilon}{n-2}g$
para-Sasakian	$-\frac{1}{(n-1)(n-2)}S^2+\left((-1)^{\ell-1}(n-1)^{\ell-1}L-\frac{1}{n-1}\right)S+\frac{1}{n-2}g$
(ε)-*para-Sasakian*	$-\frac{\varepsilon}{(n-1)(n-2)}S^2+\left((-\varepsilon)^{\ell-1}(n-1)^{\ell-1}L-\frac{1}{n-1}\right)S+\frac{\varepsilon}{n-2}g$

Corollary 7.6. *Let M be an n-dimensional $\left(\mathcal{V},S,S^{\ell}\right)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^{\ell} =$
$N(k)$-*contact metric*	$-\left(\frac{1}{n-1}-\frac{r}{kn(n-1)^2}-k^{\ell-1}(n-1)^{\ell-1}\right)S$ $+\left(k-\frac{r}{n(n-1)}\right)g$

M	$LS^{\ell} =$
Sasakian	$-\left(\frac{1}{n-1} - \frac{r}{n(n-1)^2} - (n-1)^{\ell-1}\right) S + \left(1 - \frac{r}{n(n-1)}\right) g$
Kenmotsu	$-\left(\frac{1}{n-1} + \frac{r}{n(n-1)^2} - (-1)^{\ell-1}(n-1)^{\ell-1}\right) S - \left(1 + \frac{r}{n(n-1)}\right) g$
(ε)-*Sasakian*	$-\left(\frac{1}{n-1} - \frac{\varepsilon r}{n(n-1)^2} - (\varepsilon)^{\ell-1}(n-1)^{\ell-1}\right) S + \left(\varepsilon - \frac{r}{n(n-1)}\right) g$
para-Sasakian	$-\left(\frac{1}{n-1} + \frac{r}{n(n-1)^2} - (-1)^{\ell-1}(n-1)^{\ell-1}\right) S - \left(1 + \frac{r}{n(n-1)}\right) g$
(ε)-*para-Sasakian*	$-\left(\frac{1}{n-1} + \frac{\varepsilon r}{n(n-1)^2} - (-\varepsilon)^{\ell-1}(n-1)^{\ell-1}\right) S - \left(\varepsilon + \frac{r}{n(n-1)}\right) g$

Corollary 7.7. *Let M be an n-dimensional $\left(\mathcal{P}_*, S, S^{\ell}\right)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^{\ell}=$
$N(k)$-*contact metric*	$-\left(\left(\frac{1}{n-1} - \frac{r}{kn(n-1)^2}\right) a_0 - \frac{r}{kn(n-1)} a_1 - k^{\ell-1}(n-1)^{\ell-1}\right) S + \left(\left(k - \frac{r}{n(n-1)}\right) a_0 - \frac{r}{n} a_1\right) g$
Sasakian	$-\left(\left(\frac{1}{n-1} - \frac{r}{n(n-1)^2}\right) a_0 - \frac{r}{n(n-1)} a_1 - (n-1)^{\ell-1}\right) S + \left(\left(1 - \frac{r}{n(n-1)}\right) a_0 - \frac{r}{n} a_1\right)$
Kenmotsu	$-\left(\left(\frac{1}{n-1} + \frac{r}{n(n-1)^2}\right) a_0 + \frac{r}{n(n-1)} a_1 - (-1)^{\ell-1}(n-1)^{\ell-1}\right) S + \left(\left(-1 - \frac{r}{n(n-1)}\right) a_0 - \frac{r}{n} a_1\right)$
(ε)-*Sasakian*	$-\left(\left(\frac{1}{n-1} - \frac{\varepsilon r}{n(n-1)^2}\right) a_0 - \frac{\varepsilon r}{n(n-1)} a_1 - (\varepsilon)^{\ell-1}(n-1)^{\ell-1}\right) S + \left(\left(\varepsilon - \frac{r}{n(n-1)}\right) a_0 - \frac{r}{n} a_1\right)$
para-Sasakian	$-\left(\left(\frac{1}{n-1} + \frac{r}{n(n-1)^2}\right) a_0 + \frac{r}{n(n-1)} a_1 - (-1)^{\ell-1}(n-1)^{\ell-1}\right) S + \left(\left(-1 - \frac{r}{n(n-1)}\right) a_0 - \frac{r}{n} a_1\right)$
(ε)-*para-Sasakian*	$-\left(\left(\frac{1}{n-1} + \frac{\varepsilon r}{n(n-1)^2}\right) a_0 + \frac{\varepsilon r}{n(n-1)} a_1 - (-\varepsilon)^{\ell-1}(n-1)^{\ell-1}\right) S + \left(\left(-\varepsilon - \frac{r}{n(n-1)}\right) a_0 - \frac{r}{n} a_1\right)$

Corollary 7.8. *Let M be an n-dimensional $\left(\mathcal{P},S,S^{\ell}\right)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^{\ell}=$
$N(k)$-contact metric	$\left(k^{\ell-1}(n-1)^{\ell-1}L-\frac{1}{n-1}\right)S+kg$
Sasakian	$\left((n-1)^{\ell-1}L-\frac{1}{n-1}\right)S+g$
Kenmotsu	$\left((-1)^{\ell-1}(n-1)^{\ell-1}L-\frac{1}{n-1}\right)S-g$
(ε)-Sasakian	$\left((\varepsilon)^{\ell-1}(n-1)^{\ell-1}L-\frac{1}{n-1}\right)S+\varepsilon g$
para-Sasakian	$\left((-1)^{\ell-1}(n-1)^{\ell-1}L-\frac{1}{n-1}\right)S-g$
(ε)-para-Sasakian	$\left((-\varepsilon)^{\ell-1}(n-1)^{\ell-1}L-\frac{1}{n-1}\right)S-\varepsilon g$

Corollary 7.9. *Let M be an n-dimensional $\left(\mathcal{M},S,S^{\ell}\right)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^{\ell}=$
$N(k)$-contact metric	$\frac{1}{2k(n-1)^2}S^2+\left(Lk^{\ell-1}(n-1)^{\ell-1}-\frac{1}{n-1}\right)S+\frac{k}{2}g$
Sasakian	$\frac{1}{2(n-1)^2}S^2+\left(L(n-1)^{\ell-1}-\frac{1}{n-1}\right)S+\frac{1}{2}g$
Kenmotsu	$-\frac{1}{2(n-1)^2}S^2+\left(L(-1)^{\ell-1}(n-1)^{\ell-1}-\frac{1}{n-1}\right)S-\frac{1}{2}g$
(ε)-Sasakian	$\frac{\varepsilon}{2(n-1)^2}S^2+\left(L(\varepsilon)^{\ell-1}(n-1)^{\ell-1}-\frac{1}{n-1}\right)S+\frac{\varepsilon}{2}g$
para-Sasakian	$-\frac{1}{2(n-1)^2}S^2+\left(L(-1)^{\ell-1}(n-1)^{\ell-1}-\frac{1}{n-1}\right)S-\frac{1}{2}g$
(ε)-para-Sasakian	$-\frac{\varepsilon}{2(n-1)^2}S^2+\left(L(-\varepsilon)^{\ell-1}(n-1)^{\ell-1}-\frac{1}{n-1}\right)S-\frac{\varepsilon}{2}g$

Corollary 7.10. *Let M be an n-dimensional $\left(\mathcal{W}_0,S,S^{\ell}\right)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^{\ell}=$
$N(k)$-contact metric	$\frac{1}{k(n-1)^2}S^2+\left(k^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S+kg$
Sasakian	$\frac{1}{(n-1)^2}S^2+\left((n-1)^{\ell-1}L-\frac{2}{n-1}\right)S+g$
Kenmotsu	$-\frac{1}{(n-1)^2}S^2+\left((-1)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S-g$
(ε)-Sasakian	$\frac{\varepsilon}{(n-1)^2}S^2+\left((\varepsilon)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S+\varepsilon g$
para-Sasakian	$-\frac{1}{(n-1)^2}S^2+\left((-1)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S-g$
(ε)-para-Sasakian	$-\frac{1}{(n-1)^2}S^2+\left((-\varepsilon)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S-\varepsilon g$

Corollary 7.11. *Let M be an n-dimensional $(\mathcal{W}_0^*, S, S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^\ell =$
$N(k)$-contact metric	$-\frac{1}{k(n-1)^2}S^2 + k^{\ell-1}(n-1)^{\ell-1}LS + kg$
Sasakian	$-\frac{1}{(n-1)^2}S^2 + (n-1)^{\ell-1}LS + g$
Kenmotsu	$\frac{1}{(n-1)^2}S^2 + (-1)^{\ell-1}(n-1)^{\ell-1}LS - g$
(ε)-Sasakian	$-\frac{\varepsilon}{(n-1)^2}S^2 + (\varepsilon)^{\ell-1}(n-1)^{\ell-1}LS + \varepsilon g$
para-Sasakian	$\frac{1}{(n-1)^2}S^2 + (-1)^{\ell-1}(n-1)^{\ell-1}LS - g$
(ε)-para-Sasakian	$\frac{1}{(n-1)^2}S^2 + (-\varepsilon)^{\ell-1}(n-1)^{\ell-1}LS - \varepsilon g$

Corollary 7.12. *Let M be an n-dimensional $(\mathcal{W}_1, S, S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^\ell =$
$N(k)$-contact metric	$\left(k^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S + kg$
Sasakian	$\left((n-1)^{\ell-1}L - \frac{1}{n-1}\right)S + g$
Kenmotsu	$\left((-1)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S - g$
(ε)-Sasakian	$\left((\varepsilon)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S + \varepsilon g$
para-Sasakian	$\left((-1)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S - g$
(ε)-para-Sasakian	$\left((-\varepsilon)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S - \varepsilon g$

Corollary 7.13. *Let M be an n-dimensional $(\mathcal{W}_1^*, S, S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^\ell =$
$N(k)$-contact metric	$\left(k^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S + kg$
Sasakian	$\left((n-1)^{\ell-1}L - \frac{1}{n-1}\right)S + g$
Kenmotsu	$\left((-1)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S - g$
(ε)-Sasakian	$\left((\varepsilon)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S + \varepsilon g$
para-Sasakian	$\left((-1)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S - g$
(ε)-para-Sasakian	$\left((-\varepsilon)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S - \varepsilon g$

Corollary 7.14. *Let M be an n-dimensional $(\mathcal{W}_2, S, S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^\ell =$
$N(k)$-contact metric	$\frac{1}{k(n-1)^2}S^2 + \left(k^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S$
Sasakian	$\frac{1}{(n-1)^2}S^2 + \left((n-1)^{\ell-1}L - \frac{1}{n-1}\right)S$
Kenmotsu	$-\frac{1}{(n-1)^2}S^2 + \left((-1)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S$
(ε)-Sasakian	$\frac{\varepsilon}{(n-1)^2}S^2 + \left((\varepsilon)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S$
para-Sasakian	$-\frac{1}{(n-1)^2}S^2 + \left((-1)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S$
(ε)-para-Sasakian	$-\frac{1}{(n-1)^2}S^2 + \left((-\varepsilon)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S$

Corollary 7.15. *Let M be an n-dimensional $(\mathcal{W}_3, S, S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^\ell =$
$N(k)$-contact metric	$\left(k^{\ell-1}(n-1)^{\ell-1}L - \frac{2}{n-1}\right)S + 2kg$
Sasakian	$\left((n-1)^{\ell-1}L - \frac{2}{n-1}\right)S + 2g$
Kenmotsu	$\left((-1)^{\ell-1}(n-1)^{\ell-1}L - \frac{2}{n-1}\right)S - 2g$
(ε)-Sasakian	$\left((\varepsilon)^{\ell-1}(n-1)^{\ell-1}L - \frac{2}{n-1}\right)S + 2\varepsilon g$
para-Sasakian	$\left((-1)^{\ell-1}(n-1)^{\ell-1}L - \frac{2}{n-1}\right)S - 2g$
(ε)-para-Sasakian	$\left((-\varepsilon)^{\ell-1}(n-1)^{\ell-1}L - \frac{2}{n-1}\right)S - 2\varepsilon g$

Corollary 7.16. *Let M be an n-dimensional $(\mathcal{W}_4, S, S^\ell)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^\ell =$
$N(k)$-contact metric	$\frac{1}{k(n-1)^2}S^2 + \left(k^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S + kg - \varepsilon k\eta\otimes\eta$
Sasakian	$\frac{1}{(n-1)^2}S^2 + \left((n-1)^{\ell-1}L - \frac{1}{n-1}\right)S + g - \eta\otimes\eta$
Kenmotsu	$-\frac{1}{(n-1)^2}S^2 + \left((-1)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S - g + \eta\otimes\eta$
(ε)-Sasakian	$\frac{\varepsilon}{(n-1)^2}S^2 + \left((\varepsilon)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S + \varepsilon g - \eta\otimes\eta$
para-Sasakian	$-\frac{1}{(n-1)^2}S^2 + \left((-1)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S - g + \eta\otimes\eta$
(ε)-para-Sasakian	$-\frac{\varepsilon}{(n-1)^2}S^2 + \left((-\varepsilon)^{\ell-1}(n-1)^{\ell-1}L - \frac{1}{n-1}\right)S - \varepsilon g + \eta\otimes\eta$

Corollary 7.17. *Let M be an n-dimensional $\left(\mathcal{W}_5, S, S^\ell\right)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^\ell =$
$N(k)$-contact metric	$\frac{1}{k(n-1)^2}S^2+\left(k^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S+kg$
Sasakian	$\frac{1}{(n-1)^2}S^2+\left((n-1)^{\ell-1}L-\frac{2}{n-1}\right)S+g$
Kenmotsu	$-\frac{1}{(n-1)^2}S^2+\left((-1)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S-g$
(ε)-Sasakian	$\frac{\varepsilon}{(n-1)^2}S^2+\left((\varepsilon)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S+\varepsilon g$
para-Sasakian	$-\frac{1}{(n-1)^2}S^2+\left((-1)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S-g$
(ε)-para-Sasakian	$-\frac{\varepsilon}{(n-1)^2}S^2+\left((-\varepsilon)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S-\varepsilon g$

Corollary 7.18. *Let M be an n-dimensional $\left(\mathcal{W}_6, S, S^\ell\right)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^\ell =$
$N(k)$-contact metric	$\left(k^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S+kg+k\eta\otimes\eta$
Sasakian	$\left((n-1)^{\ell-1}L-\frac{2}{n-1}\right)S+g+\eta\otimes\eta$
Kenmotsu	$\left((-1)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S-g-\eta\otimes\eta$
(ε)-Sasakian	$\left((\varepsilon)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S+\varepsilon g+\varepsilon\eta\otimes\eta$
para-Sasakian	$\left((-1)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S-g-\eta\otimes\eta$
(ε)-para-Sasakian	$\left((-\varepsilon)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S-\varepsilon g-\varepsilon\eta\otimes\eta$

Corollary 7.19. *Let M be an n-dimensional $\left(\mathcal{W}_7, S, S^\ell\right)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^\ell =$
$N(k)$-contact metric	$\left(k^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S+2kg$
Sasakian	$\left((n-1)^{\ell-1}L-\frac{2}{n-1}\right)S+2g$
Kenmotsu	$\left((-1)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S-2g$
(ε)-Sasakian	$\left((\varepsilon)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S+2\varepsilon g$
para-Sasakian	$\left((-1)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S-2g$
(ε)-para-Sasakian	$\left((-\varepsilon)^{\ell-1}(n-1)^{\ell-1}L-\frac{2}{n-1}\right)S-2\varepsilon g$

Corollary 7.20. *Let M be an n-dimensional $\left(\mathcal{W}_8, S, S^{\ell}\right)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^{\ell} =$
$N(k)$-*contact metric*	$\left(Lk^{\ell-1}(n-1)^{\ell-1} - \dfrac{2}{n-1}\right) S + kg + k\eta\otimes\eta$
Sasakian	$\left(L(n-1)^{\ell-1} - \dfrac{2}{n-1}\right) S + kg + \eta\otimes\eta$
Kenmotsu	$\left(L(-1)^{\ell-1}(n-1)^{\ell-1} - \dfrac{2}{n-1}\right) S + kg - \eta\otimes\eta$
(ε)-*Sasakian*	$\left(L(\varepsilon)^{\ell-1}(n-1)^{\ell-1} - \dfrac{2}{n-1}\right) S + kg + \eta\otimes\eta$
para-Sasakian	$\left(L(-1)^{\ell-1}(n-1)^{\ell-1} - \dfrac{2}{n-1}\right) S + kg - \eta\otimes\eta$
(ε)-*para-Sasakian*	$\left(L(-\varepsilon)^{\ell-1}(n-1)^{\ell-1} - \dfrac{2}{n-1}\right) S + kg - \eta\otimes\eta$

Corollary 7.21. *Let M be an n-dimensional $\left(\mathcal{W}_9, S, S^{\ell}\right)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table:*

M	$LS^{\ell} =$
$N(k)$-*contact metric*	$\left(k^{\ell-1}(n-1)^{\ell-1}L - \dfrac{1}{n-1}\right) S + k\eta\otimes\eta$
Sasakian	$\left((n-1)^{\ell-1}L - \dfrac{1}{n-1}\right) S + \eta\otimes\eta$
Kenmotsu	$\left((-1)^{\ell-1}(n-1)^{\ell-1}L - \dfrac{1}{n-1}\right) S - \eta\otimes\eta$
(ε)-*Sasakian*	$\left((\varepsilon)^{\ell-1}(n-1)^{\ell-1}L - \dfrac{1}{n-1}\right) S + \eta\otimes\eta$
para-Sasakian	$\left((-1)^{\ell-1}(n-1)^{\ell-1}L - \dfrac{1}{n-1}\right) S - \eta\otimes\eta$
(ε)-*para-Sasakian*	$\left((-\varepsilon)^{\ell-1}(n-1)^{\ell-1}L - \dfrac{1}{n-1}\right) S - \eta\otimes\eta$

Remark 7.1. If in the Theorem 7.2, we take M be an n-dimensional $(\mathcal{T}_a, S, S)$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then the result is same as given in [31, Theorem 7.6].

Corollary 7.22. *Let M be an n-dimensional $(R, S_{\mathcal{T}_a}, S^{\ell})$-pseudosymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$\begin{aligned} & (a_0 + na_1 + a_2 + a_3 + a_5 + a_6)\times \\ & (Lk(n-1)S^{\ell} - Lk^{\ell}(n-1)^{\ell}S + kS - k^2(n-1)g) \\ = \;& Lr(a_4 + (n-1)a_7)(S^{\ell} - k^{\ell}(n-1)^{\ell}g). \end{aligned}$$

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