International Electronic Journal of Geometry
Volume 5 No. 2 pp. 19–26 (2012) ©IEJG

ON SUBMANIFOLDS OF AN ALMOST PSEUDO CONTACT METRIC MANIFOLD

MEHMET ATÇEKEN

(Communicated by Levent KULA)

Abstract. In [3], a new structure is defined in an even dimensional on a differentiable Riemannian manifold which is called almost pseudo contact structure. In this paper, we introduce the fundamental properties of submanifolds of an almost pseudo contact metric manifold and research the induced structures on submanifold. They have been categorized.

1. Introduction

The notion of an almost pseudo contact structure has been defined by P. Debnath and A. Konar [3]. They defined an almost pseudo contact metric structure on an even dimensional Riemannian manifold and given an example for such structures.

They have also shown that such structures are not unique on a manifold. Authors given the necessary and sufficient conditions for an even dimensional Riemannian admits an almost pseudo contact metric structure. Such structures are a generalization of contact metric structures. Since almost pseudo contact metric manifolds are quite a new type manifold, the geometry of their submanifolds an open problem.

So, in this paper, we have devoted ourselves to this problem. In this paper, a submanifold of an almost pseudo contact metric manifold is introduced by means of an isometric immersion. These properties are protected under isometric immersion have been researched. I have achieved some results used in differential geometry.

2. Preliminaries

In [3], P. Debnath and A. Konar introduced an almost pseudo contact structure on even dimensional manifold as follows.

Let $M_{2n}(n \geq 2)$ be an even dimensional manifold, ϕ be a tensor field of type $(1,1)$ on M and ξ and $\xi^{'}$ be two linearly independent vector fields and η and $\eta^{'}$ be

2000 *Mathematics Subject Classification.* 53C15, 53C25.
Key words and phrases. Almost Hermite structure, Almost contact metric structure, almost pseudo contact metric structure and invariant submanifold.

two non-zero 1-forms. If the systems $(\phi, \xi, \xi^{'}, \eta, \eta^{'})$ satisfies the conditions

$$\phi\xi = \phi\xi^{'} = 0, \tag{2.1}$$

and

$$\phi^2 X = -X + \eta(X)\xi + \eta^{'}(X)\xi^{'}, \quad rank(\phi) = 2n-2, \tag{2.2}$$

for any $X \in \Gamma(TM)$, then $(\phi, \xi, \xi^{'}, \eta, \eta^{'})$ is said to be an almost pseudo contact structure on M and such a manifold is called an almost pseudo contact manifold. Such manifolds have relations $\eta \circ \phi = \eta^{'} \circ \phi = 0$, $\eta(\xi) = \eta^{'}(\xi^{'})$=1 and $\eta(\xi^{'}) = \eta^{'}(\xi) = 0$.

Furthermore, every almost pseudo contact manifold M_{2n} admits a Riemannian metric tensor g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) - \eta^{'}(X)\eta^{'}(Y) \tag{2.3}$$

and

$$\eta(X) = g(X, \xi), \quad \eta^{'}(X) = g(X, \xi^{'}), \tag{2.4}$$

for any $X, Y \in \Gamma(TM)$. A Riemannian manifold which accepts almost pseudo contact structure is called the almost pseudo-contact metric manifold[3]. We note that the structure tensor ϕ of the almost pseudo contact structure is skew-symmetric, that is,

$$g(\phi X, Y) + g(X, \phi Y) = 0,$$

for any $X, Y \in \Gamma(TM)$.

3. Submanifolds of an almost pseudo contact metric manifold

Now, let $\bar{M}$ be an isometrically immersed submanifold of an almost pseudo contact metric manifold M. We define an isometric immersed by $i : \bar{M} \longrightarrow M$ and denote by B the differential of i. The induced Riemannian metric tensor $\bar{g}$ on $\bar{M}$ by g satisfies $\bar{g}(X, Y) = g(BX, BY)$ for any $X, Y \in \Gamma(T\bar{M})$.

We denote the tangent and normal spaces of $\bar{M}$ at point $p \in \bar{M}$ by $T_{\bar{M}}(p)$ and $T^{\perp}_{\bar{M}}(p)$, respectively, and let $\{N_1, N_2, ..., N_s\}$ be an orthonormal basis of the normal space $T^{\perp}_{\bar{M}}(p)$, where s=codim$(\bar{M})$.

For any $X \in \Gamma(TM)$, ϕX and ϕN_i can be written in the following way;

$$\phi BX = B\psi X + \sum_{i=1}^{s} \upsilon_i(X)N_i \tag{3.1}$$

and

$$\phi N_i = BU_i + \sum_{j=1}^{s} \lambda_{ij}N_j, \quad 1 \leq i \leq s, \tag{3.2}$$

where ψ, υ_i, U_i denote the induced $(1-1)$-tensor field, 1-forms, vector fields on $\bar{M}$, respectively, and λ_{ij} are arbitrary functions on M. By using (3.1) and (3.2), we can easily to see that

$$\upsilon_k(X) = g(\phi BX, N_k) = -g(BX, \phi N_k) = -g(BX, BU_k) = -\bar{g}(X, U_k) \tag{3.3}$$

and

$$\lambda_{ik} = g(\phi N_i, N_k) = -g(N_i, \phi N_k) = -\lambda_{ki}, \quad 1 \leq i, k \leq s.$$

On the other hand, the vector fields ξ and $\xi^{'}$ on almost pseudo contact metric manifold M can be written as follows

$$\xi = BV + \sum_{i=1}^{s} \alpha_i N_i \ \ \textit{and} \ \ \xi^{'} = BV^{'} + \sum_{i=1}^{s} \alpha_i^{'} N_i, \tag{3.4}$$

respectively, where V, $V^{'}$ and α_i, $\alpha_i^{'}$ are vector fields and functions on $\bar{M}$ and M, respectively. By direct a calculations, we mean that

$$\alpha_k = g(\xi, N_k) = \eta(N_k)$$

and

$$\alpha_k^{'} = g(\xi^{'}, N_k) = \eta^{'}(N_k), \quad 1 \leq k \leq s.$$

Here we note that the induced (1-1)-tensor field ψ is also skew symmetric because ϕ is skew-symmetric, V and $V^{'}$ are linearly independent vector field.

Next, we will prove three lemmas for later used.

Lemma 3.1. *Let $\bar{M}$ be an isometrically immersed submanifold of an almost pseudo contact metric manifold M. Then the following assertions are true;*

$$\sum_{i=1}^{s} \upsilon_i \oplus U_i = -\psi^2 - I + \bar{\eta} \oplus V + \bar{\eta}^{'} \oplus V^{'} \tag{3.5}$$

$$\sum_{i=1}^{s} \upsilon_i \lambda_{ik} = \alpha_k V + \alpha_k^{'} V^{'} + \psi U_k \tag{3.6}$$

$$\sum_{j=1}^{s} \lambda_{ij} \lambda_{jp} = -\delta_{ip} + \alpha_i \alpha_p + \alpha_i^{'} \alpha_p^{'} - \upsilon_p(U_i), \tag{3.7}$$

where $\bar{\eta}$ and $\bar{\eta}^{'}$ denote the induced 1-forms on $\bar{M}$ by η and $\eta^{'}$, respectively.

Proof. Taking into account that the equations (2.2), (3.1) and (3.2), we have

$$\phi^2 BX = B\psi^2 X + \sum_{j=1}^{s} \upsilon_j(\psi X) N_j + \sum_{i=1}^{s} \upsilon_i(X) \phi N_i$$

from which, we see that

$$\begin{aligned} -BX &+ \bar{\eta}(X) BV + \bar{\eta}(X) \sum_{i=1}^{s} \alpha_i N_i + \bar{\eta}^{'}(X) BV^{'} + \bar{\eta}^{'}(X) \sum_{i=1}^{s} \alpha_i^{'} N_i = \\ B\psi^2 X &+ \sum_{j=1}^{s} \upsilon_j(\psi X) N_j + \sum_{i=1}^{s} \upsilon_i(X) BU_i + \sum_{i=1}^{s} \upsilon_i(X) \sum_{p=1}^{s} \lambda_{ip} N_p, \end{aligned} \tag{3.8}$$

for any $X \in \Gamma(T\bar{M})$. From the tangent components of (3.8) and consider i is an immersion, we conclude that

$$-X + \bar{\eta}(X) V + \bar{\eta}^{'}(X) V^{'} = \psi^2 X + \sum_{i=1}^{s} \upsilon_i(X) U_i,$$

that is,

$$\psi^2 = -I + \bar{\eta} \oplus V + \bar{\eta}^{'} \oplus V^{'} - \sum_{i=1}^{s} \upsilon_i \oplus U_i,$$

and normal components of (3.8), we arrive at

$$\bar{\eta}(X)\alpha_k + \bar{\eta}^{'}(X)\alpha_k^{'} = \upsilon_k(\psi X) + \sum_{i=1}^{s} \upsilon_i(X)\lambda_{ik},$$

which implies that

$$\alpha_k V + \alpha_k^{'} V^{'} = -\psi U_k + \sum_{i=1}^{s} \upsilon_i \lambda_{ik},$$

which gives us to (3.6).

To prove (3.7), it is enought to take account(2.2) and (3.2). Thus we have

$$\begin{aligned} \phi^2 N_i &= \phi\{BU_i + \sum_{j=1}^{s} \lambda_{ij} N_j\} \\ -N_i &+ \eta(N_i)\xi + \eta^{'}(N_i)\xi^{'} = \phi BU_i + \sum_{j=1}^{s} \lambda_{ij}\phi N_j \\ -N_i &+ B\alpha_i V + B\alpha_i^{'} V^{'} + \alpha_i \sum_{\ell=1}^{s} \alpha_\ell N_\ell + \alpha_i^{'} \sum_{k=1}^{s} \alpha_k^{'} N_k = B\psi U_i \\ &+ \sum_{j=1}^{s} \upsilon_j(U_i) N_j + \sum_{j=1}^{s} \lambda_{ij}\{BU_j + \sum_{k=1}^{s} \lambda_{jk} N_k\}. \end{aligned}$$

Taking inner product both sides of this equation by N_p, $1 \leq p \leq s$, we have

$$-\delta_{ip} + \alpha_i\alpha_p + \alpha_i^{'}\alpha_p^{'} = \upsilon_p(U_i) + \sum_{j=1}^{s} \lambda_{ij}\lambda_{jp}, \quad 1 \leq i, p \leq s.$$

So the proof is completes. □

Lemma 3.2. *Let $\bar{M}$ be an isometrically immersed submanifold of an almost pseudo contact metric manifold M. Then the following assertions are true.*

$$\psi V + \sum_{i=1}^{s} \alpha_i U_i = 0, \quad \psi V^{'} + \sum_{i=1}^{s} \alpha_i^{'} U_i = 0, \tag{3.9}$$

$$\upsilon_p(V) + \sum_{i=1}^{s} \alpha_i \lambda_{ip} = 0, \quad \upsilon_p(V^{'}) + \sum_{i=1}^{s} \alpha_i^{'} \lambda_{ip} = 0, \tag{3.10}$$

$$\bar{\eta}(V) = 1 - \sum_{i=1}^{s} \alpha_i^2, \qquad \bar{\eta}^{'}(V^{'}) = 1 - \sum_{i=1}^{s} \alpha_i^{'2} \tag{3.11}$$

and

$$g(\psi X, \psi Y) = g(X,Y) - \bar{\eta}(X)\bar{\eta}(Y) - \bar{\eta}^{'}(X)\bar{\eta}^{'}(Y) - \sum_{i=1}^{s} \upsilon_i(X)\upsilon_i(Y), \tag{3.12}$$

for any $X, Y \in \Gamma(TM)$.

Proof. To prove (3.9) and (3.10), taking into account (2.1) and (3.4), we have

$$\begin{aligned}\phi\xi &= \phi BV + \sum_{i=1}^{s}\alpha_i\phi N_i = B\psi V + \sum_{j=1}^{s}\upsilon_j(V)N_j + \sum_{i=1}^{s}\alpha_i\{BU_i + \sum_{j=1}^{s}\lambda_{ij}N_j\}\\ 0 &= B\psi V + \sum_{j=1}^{s}\upsilon_j(V)N_j + \sum_{i=1}^{s}\alpha_i BU_i + \sum_{i,j=1}^{s}\alpha_i\lambda_{ij}N_j.\end{aligned}$$

Comparing the tangent and normal components of this equation, respectively, we have

$$\psi V + \sum_{i=1}^{s}\alpha_i U_i = 0 \quad and \quad \upsilon_p(V) + \sum_{i=1}^{s}\alpha_i\lambda_{ip} = 0, \quad 1 \leq p \leq s.$$

which gives us the left sides of (3.9) and (3.10), respectively. From $\phi\xi^{'} = 0$, we can find the right sides of (3.9) and (3.10). On the other hand, we have

$$\begin{aligned}g(\xi,\xi) &= g(BV + \sum_{i=1}^{s}\alpha_i N_i, BV + \sum_{j=1}^{s}\alpha_j N_j)\\ 1 &= \bar{\eta}(V) + \sum_{i=1}^{s}\alpha_i^2,\end{aligned}$$

and $g(\xi^{'},\xi^{'}) = 1$ is also equivalent to $\bar{\eta}^{'}(V^{'}) = 1 - \sum_{i=1}^{s}\alpha_i^{'2}$. So (3.11) is completed.

In order to prove (3.12), considering (2.3), (3.1) and (3.3), we have

$$\begin{aligned}g(\psi X,\psi Y) &= \bar{g}(B\psi X, B\psi Y) = \bar{g}(\phi BX - \sum_{i=1}^{s}\upsilon_i(X)N_i, \phi BY - \sum_{j=1}^{s}\upsilon_j(Y)N_j)\\ &= \bar{g}(\phi BX,\phi BY) + \bar{g}(BX, \sum_{j=1}^{s}\upsilon_j(X)\phi N_j) + \bar{g}(BY, \sum_{i=1}^{s}\upsilon_i(Y)\phi N_i)\\ &+ \sum_{i=1}^{s}\upsilon_i(X)\upsilon_i(Y)\\ &= \bar{g}(BX,BY) - \eta(BX)\eta(BY) - \eta^{'}(BX)\eta^{'}(BY) - \sum_{i=1}^{s}\upsilon_i(X)g(X,U_i)\\ &- \sum_{i=1}^{s}\upsilon_i(Y)g(Y,U_i) + \sum_{i=1}^{s}\upsilon_i(X)\upsilon_i(Y)\\ &= g(X,Y) - \bar{\eta}(X)\bar{\eta}(Y) - \bar{\eta}^{'}(X)\bar{\eta}^{'}(Y) - \sum_{i=1}^{s}\upsilon_i(X)\upsilon_i(Y).\end{aligned}$$

Hence the lemma is proved completely. □

Next, let $\bar{M}$ be an isometrically immersed submanifold of an almost pseudo contact metric manifold M. If $\phi(B(T_{\bar{M}}(p))) \subseteq T_{\bar{M}}(p)$ for each point $p \in \bar{M}$, then $\bar{M}$ is said to be an invariant submanifold of M[2]. In this case (3.1) and (3.2) equations reduce to, respectively,

(3.13) $$\phi BX = B\psi X, \quad for \;\; all \quad X \in \Gamma(T\bar{M}),$$

and

$$\phi N_i = \sum_{j=1}^{s} \lambda_{ij} N_j, \quad 1 \leq i \leq s. \tag{3.14}$$

Lemma 3.3. *Let $\bar{M}$ be an invariant submanifold of an almost pseudo contact metric manifold M. The following assertions are true;*

$$\psi^2 = -I + \bar{\eta} \oplus V + \bar{\eta}^{'} \oplus V^{'}, \quad \alpha_i V + \alpha_i^{'} V^{'} = 0, \tag{3.15}$$

$$\sum_{j=1}^{s} \lambda_{ij}\lambda_{jp} = -\delta_{ip} + \alpha_i\alpha_p + \alpha_i^{'}\alpha_p^{'}, \quad \psi U_p + \sum_{i=1}^{s} U_i \lambda_{ip} = 0, \tag{3.16}$$

$$\psi V = \psi V^{'} = 0, \quad \sum_{i=1}^{s} \alpha_i \lambda_{ij} = \sum_{i=1}^{s} \alpha_i^{'} \lambda_{ij} = 0, \tag{3.17}$$

and

$$g(\psi X, \psi Y) = g(X, Y) - \bar{\eta}(X)\bar{\eta}(Y) - \bar{\eta}^{'}(X)\bar{\eta}^{'}(Y), \tag{3.18}$$

for any $X, Y \in \Gamma(T\bar{M})$.

Proof. Since $\bar{M}$ is an invariant submanifold of M, we have $\upsilon_i(X) = 0$, $1 \leq i \leq s$, for any $X \in \Gamma(T\bar{M})$. So the equation (3.5) reduce to the left side of (3.15).

To prove (3.16), considering (2.2) and (3.14), we have

$$\begin{aligned} \phi^2 N_i = \sum_{j=1}^{s} \lambda_{ij}\phi N_j &= \sum_{j=1}^{s} \lambda_{ij} \sum_{k=1}^{s} \lambda_{jk} N_k \\ -N_i + \alpha_i BV + \alpha_i \sum_{t=1}^{s} \alpha_t N_t + \alpha_i^{'} BV^{'} + \alpha_i^{'} \sum_{t=1}^{s} \alpha_t^{'} N_t &= \sum_{j,k=1}^{s} \lambda_{ij}\lambda_{jk} N_k, \end{aligned}$$

which is equivalent to

$$\alpha_i V + \alpha_i^{'} V^{'} = 0 \;\; and \;\; \sum_{j=1}^{s} \lambda_{ij}\lambda_{jp} = -\delta_{ip} + \alpha_i\alpha_p + \alpha_i^{'}\alpha_p^{'}, \quad 1 \leq i, p \leq s.$$

On the other hand, $\phi\xi = \phi\xi^{'} = 0$ imply that (3.17). Making use of $\bar{M}$ being invariant submanifold and (3.12), we get (3.18). □

In this rest of this paper, we will prove the main theorems.

Theorem 3.1. *Let $\bar{M}$ be an invariant submanifold of an almost pseudo contact metric manifold M. The one of the following cases occur.*
1.) If ξ and $\xi^{'}$ are normal to $\bar{M}$, then induced structure $(\psi, \bar{g})$ on $\bar{M}$ is an almost Hermite structure.
2.) If ξ is tangent to $\bar{M}$ and $\xi^{'}$ is normal to $\bar{M}$, then induced structure $(\psi, V, \bar{\eta}, \bar{g})$ on $\bar{M}$ is an almost contact metric structure.
3.) If ξ and $\xi^{'}$ are tangent to $\bar{M}$, then induced structure $(\psi, V, V^{'}, \bar{\eta}, \bar{\eta}^{'}, \bar{g})$ is an almost pseudo contact metric structure.

Proof. 1.) If ξ and $\xi^{'}$ are normal to $\bar{M}$, then the induced vector fields $V = V^{'} = 0$(that is, $\sum_{i=1}^{s} \alpha_i^2 = \sum_{i=1}^{s} \alpha_i^{'2} = 1$). From (3.15) and (3.18), we have $\psi^2 = -I$ and $\bar{g}(X, Y) = \bar{g}(\psi X, \psi Y)$ for any $X, Y \in \Gamma(T\bar{M})$, that is, $(\psi, \bar{g})$ is an almost Hermite structure on $\bar{M}$.

2.) If ξ tangent to $\bar{M}$ and $\xi^{'}$ is normal to $\bar{M}$, then from (3.4), we obtain $V^{'} = 0$ and $\bar{\eta}^{'}(X) = 0$ for any $X \in \Gamma(T\bar{M})$. Also by using (3.15) and (3.18), we conclude that

$$\psi^2 = -I + \bar{\eta} \oplus V \quad and \quad g(\psi X, \psi Y) = g(X,Y) - \bar{\eta}(X)\bar{\eta}(Y),$$

which proves our assertion.

3.) If ξ and $\xi^{'}$ are tangent to $\bar{M}$, then from (3.15), (3.17) and (3.18) we conclude that the induced structure $(\psi, V, V^{'}, \bar{\eta}, \bar{\eta}^{'}, \bar{g})$ on $\bar{M}$ is an almost pseudo contact metric structure. Thus the proof is completes. □

As can be seen form Theorem 3.1, almost pseudo-contact metric structures are generalization of almost Hermite structures and almost contact metric structures.

Theorem 3.2. *Let $\bar{M}$ be an isometrically immersed submanifold of an almost pseudo contact metric manifold M. $\bar{M}$ is invariant submanifold of M is and only if the induced structure $(\psi, V, V^{'}, \bar{\eta}, \bar{\eta}^{'}, \bar{g})$ on $\bar{M}$ is an almost pseudo contact metric structure or the induced structure $(\psi, \bar{\eta}, \bar{g})$ on $\bar{M}$ is almost contact metric structure or the induced structure $(\psi, \bar{g})$ on $\bar{M}$ is an almost Hermite structure*

Proof. If $\bar{M}$ is invariant submanifold of M, then induced structures on $\bar{M}$ are categorized in Theorem 3.1.

Now, conversely, we first assume that the induced structure $(\psi, V, V^{'}, \bar{\eta}, \bar{\eta}^{'}, \bar{g})$ is an almost pseudo contact metric structure. Then from (3.5), we mean that $\upsilon_i(X) = 0$, $1 \leq i \leq s$, for any $X \in \Gamma(T\bar{M})$. This implies that $\bar{M}$ is an invariant submanifold.

Secondly,we suppose that the induced structure $(\psi, V, \bar{g})$ is a contact metric structure. Then from (3.5), we have

$$\bar{\eta}^{'}(X)V - \sum_{i=1}^{s} \upsilon_i(X)U_i = 0,$$

for any $X \in \Gamma(T\bar{M})$, from which, we get

$$\bar{\eta}^{'2}(X) + \sum_{i=1}^{s} \upsilon_i^2(X) = 0,$$

that is, $\bar{\eta}^{'}(X) = \upsilon_i(X) = 0$, $1 \leq i \leq s$. So $\bar{M}$ is invariant submanifold.

Finally, we assume that the induced structure $(\psi, \bar{g})$ is an almost Hermite structure. Then from (3.5), we have

$$\bar{\eta}(X)V + \bar{\eta}^{'}(X)V^{'} - \sum_{i=1}^{s} \upsilon_i(X)U_i = 0,$$

for any $X \in \Gamma(T\bar{M})$. Taking inner product by X both sides of this equality, we can easily to see that

$$\bar{\eta}^2(X) + \bar{\eta}^{'2}(X) + \sum_{i=1}^{s} \upsilon_i^2(X) = 0,$$

which is equivalent to

$$\bar{\eta}(X) = \bar{\eta}^{'}(X) = \upsilon_i(X) = 0, \quad 1 \leq i \leq s.$$

Since $\upsilon_i(X) = 0$, $1 \leq i \leq s$, $\bar{M}$ is invariant submanifold. □

References

[1] Atçeken, M., On Geometry of Submanifolds of (LCS)n-Manifolds, Hindawi Publishing Corporation International Journal of Mathematics and Mathematical Sciences., (2012), Article ID 304647, 11 pages doi:10.1155/2012/304647.
[2] Karadag, H. B. and Atçeken, M., Invariant Submanifolds of Sasakian Manifolds, Balkan Journal of Geometry and Its Applications., 12(2007), no.1, 68-75.
[3] Debnath, P. and Konar, A., Almost Pseudo Contact Structure, Commun. Korean Math. Soc., 26(2011), no.1, 125-133, DOI 10.4134/CKMS.2011.26.1.125.

Department of Mathematics, Faculty of Sciences and Arts, University of Gaziosmanpaşa, Tokat/TURKEY
E-mail address: mehmet.atceken@gop.edu.tr