ON THE CURVATURES OF TUBULAR SURFACE WITH BISHOP FRAME

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ABSTRACT. A canal surface is the envelope of a moving sphere with varying radius, defined by the trajectory $C(t)$ (spine curve) of its center and a radius function $r(t)$ and it is parametrized through Frenet frame of the spine curve $C(t)$. If the radius function $r(t) = r$ is a constant, then the canal surface is called a tube or tubular surface. In this work, we investigate tubular surface with Bishop frame in place of Frenet frame and afterwards give some characterizations about special curves lying on this surface

1. INTRODUCTION

Canal surfaces are useful for representing long thin objects, e.g., pipes, poles, ropes, 3D fonts or intestines of body. Canal surfaces are also frequently used in solid and surface modelling for CAD/CAM. Representative examples are natural quadrics, torus, tubular surfaces and Dupin cyclides.

Maekawa *et.al.* [6] researched necessary and sufficient conditions for the regularity of pipe (tubular) surfaces. More recently, Xu *et.al.* [8] studied these conditions for canal surfaces and examined principle geometric properties of these surfaces like computing the area and Gaussian curvature.

Gross [3] gave the concept of generalized tubes (briefly GT) and classified them in two types as ZGT and CGT. Here, ZGT refers to the spine curve (the axis) that has torsion-free and CGT refers to tube that has circular cross sections. He investigated the properties of GT and showed that parameter curves of a generalized tube are also lines of curvature if and only if the spine curve has torsion free (planar).

Bishop [1] displayed that there exists orthonormal frames which he called relatively paralled adapted frames other than the Frenet frame and compared features of them with the Frenet frame.

This paper is organized as follows. We introduce canal and tubular surfaces in section 2. Section 3 gives us information concerning the curvatures of tubular surface with the Frenet frame. In section 4, we define tubular surface with respect to the Bishop frame. Subsequently, we compute the curvatures of this new tubular surface and give some characterizations regarding special curves lying on it.

2. Preliminaries

Initially, we parametrize a canal surface via characteristic circles of it. Later, we define a tube as a special case of the canal surface. A canal surface is defined as the envelope of a family of one parameter spheres. Alternatively, a canal surface is the envelope of a moving sphere with varying radius, defined by the trajectory $C(t)$ of its center and a radius function $r(t)$. This moving sphere $S(t)$ touches the canal surface at a characteristic circle $K(t)$. If the radius function $r(t) = r$ is a constant, then the canal surface is called a tube or pipe surface.

Since the canal surface $K(s, \theta)$ is the envelope of a family of one parameter spheres with the center $C(t)$ and radius function $r(t)$, a surface point $p = K(t, \theta) \in \mathbb{E}^3$ satisfies the following equations.
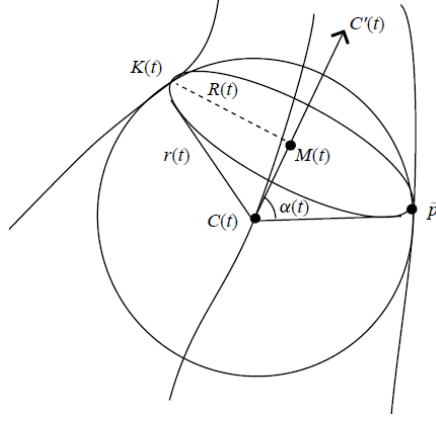

Figure 1.[4] A circle $K(t)$ on the sphere $S(t)$

$$\left\| \overrightarrow{p} - C(t) \right\| = r(t)$$

$$(\overrightarrow{p} - C(t)) \cdot C^{'}(t) + r(t)r^{'}(t) = 0. \tag{2.1}$$

Now, we decompose the canal surface into a family of characteristic circles. Let $M(t)$ be center of characteristic circles $K(t)$. For a point $p = K(t, \theta)$, the vector $\overrightarrow{C(t)M(t)}$ is the orthogonal projection of $\overrightarrow{C(t)p}$ onto the tangent $C^{'}(t)$ as obtained below.

$$\overrightarrow{C(t)M(t)} = \frac{\overrightarrow{C(t)p} \cdot C^{'}(t)}{C^{'}(t) \cdot C^{'}(t)} C^{'}(t)$$

$$M(t) - C(t) = \frac{(p - C(t) \cdot C^{'}(t)}{C^{'}(t) \cdot C^{'}(t)} C^{'}(t).$$

By Eq (2.1), because $(p - C(t)) \cdot C^{'}(t) = -r(t)r^{'}(t)$ we get the center $M(t)$ and the radius function $R(t)$ of characteristic circles as

$$M(t) = C(t) + r(t) \cos \alpha(t) \frac{C^{'}(t)}{\|C^{'}(t)\|}; \quad \cos \alpha(t) = -\frac{r^{'}(t)}{\|C^{'}(t)\|}$$

$$R(t) = r(t) \sin \alpha(t) = \mp r(t) \frac{\sqrt{\|C^{'}(t)\|^2 - r^{'}(t)^2}}{\|C^{'}(t)\|},$$

where $\alpha(t)$ is the angle between $\overrightarrow{C(t)p}$ and $C'(t)$. Thus, the canal surface is parametrized as follows.

$$K(t,\theta) = M(t) + R(t)\left(\cos\theta N(t) + \sin\theta B(t)\right)$$

$$K(t,\theta) = C(t) - r(t)r'(t)\frac{C'(t)}{\|C'(t)\|^2} \mp r(t)\frac{\sqrt{\|C'(t)\|^2 - r'(t)^2}}{\|C'(t)\|}\left(\cos\theta N + \sin\theta B\right)$$

$$\tag{2.2}$$

where $N(t)$ and $B(t)$ are the principal normal and binormal to $C(t)$, respectively. Alternatively, $N(t)$ and $B(t)$ are the basis vectors of the plane containing characteristic circle. If the spine curve $C(t)$ has an arclenght parametrization ($\|C'(t)\| = 1$), then the canal surface is reparametrized as

$$K(s,\theta) = C(s) - r(s)r'(s)T(s) \mp r(s)\sqrt{1 - r'(s)^2}\left(\cos\theta N(s) + \sin\theta B(s)\right). \tag{2.3}$$

In the event $r(t) = r$ is a constant, the canal surface is called a tube or pipe surface and it turns into the form

$$L(s,\theta) = C(s) + r\left(\cos\theta N(s) + \sin\theta B(s)\right),\ 0 \le \theta < 2\pi. \tag{2.4}$$

Let $T(s)$ be tangent to $C(s)$ and let $N_1(s)$ be arbitrary orthogonal unit vector to $T(s)$. If $N_2(s)$ is orthogonal to both $T(s)$ and $N_1(s)$, then $N_2(s) = T(s) \times N_1(s)$. This means that $\{T(s), N_1(s), N_2(s)\}$ is an orthonormal frame. The frame is called Bishop frame (relatively parallel adapted frame accordance with Frenet frame). If we rotate the Bishop frame by the angle ϕ around the tangent vector T, we obtain the Frenet frame as below.

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}$$

The derivative formulas for Frenet frame are given by

$$\begin{aligned} T'(s) &= \kappa(s)N(s) \\ N'(s) &= -\kappa(s)T(s) + \tau(s)B(s) \\ B'(s) &= -\tau(s)N(s), \end{aligned} \tag{2.5}$$

where κ and τ are the curvature and the torsion of the spine curve $C(s)$, respectively. Let $k_1(s)$ and $k_2(s)$ be Bishop parameters (normal development). The derivative formulas which correspond to Bishop frame and Bishop parameters are as follows.

$$\begin{aligned} T'(s) &= k_1(s)N_1(s) + k_2(s)N_2(s) \\ N_1'(s) &= -k_1(s)T(s) \\ N_2'(s) &= -k_2(s)T(s) \\ k_1 &= \kappa\cos\phi \\ k_2 &= \kappa\sin\phi \\ \tau &= \phi'. \end{aligned} \tag{2.6}$$

In next sections, first we will give the curvatures of the tube $L(s,\theta)$. Afterwards, by taking $N_1(s)$ and $N_2(s)$ instead of $N(s)$ and $B(s)$ we will compute the curvatures of this new tubular surface

$$P(s,\theta) = C(s) + r\left(\cos\theta N_1(s) + \sin\theta N_2(s)\right),\ 0 \le \theta < 2\pi \tag{2.7}$$

and obtain some characterizations as regards special curves lying on $P(s, \theta)$.

3. The curvatures of tubular surfaces with respect to the Frenet frame

For the tubular surface $L(s, \theta)$, the surface normal vector U and the coefficients of the first and second fundamental form are given by

$$U = \frac{L_s \times L_\theta}{\|L_s \times L_\theta\|} = -\cos\theta N - \sin\theta B$$

$$
\begin{aligned}
L_\theta &= r(-\sin\theta N + \cos\theta B) \\
L_s &= (1 - r\kappa\cos\theta)T + \tau L_\theta \\
L_{\theta\theta} &= -r(\cos\theta N + \sin\theta B) \\
L_{ss} &= \left(-r\kappa'\cos\theta + r\kappa\tau\sin\theta\right)T + \left[\kappa - r(\kappa^2 + \tau^2)\cos\theta - r\tau'\sin\theta\right] N \\
&\quad + \left(-r\tau^2\sin\theta + r\tau'\cos\theta\right) B \\
E &= L_s \cdot L_s = (1 - r\kappa\cos\theta)^2 + r^2\tau^2 \\
F &= L_s \cdot L_\theta = r^2\tau \\
G &= L_\theta \cdot L_\theta = r^2 \\
e &= U \cdot L_{ss} = -\kappa\cos\theta(1 - r\kappa\cos\theta) + r\tau^2 \\
f &= U \cdot L_{s\theta} = r\tau \\
g &= U \cdot L_{\theta\theta} = r
\end{aligned}
\tag{3.1}
$$

$$\|L_s \times L_\theta\|^2 = EG - F^2 = r^2(1 - r\kappa\cos\theta)^2. \tag{3.2}$$

Theorem 3.1. $L(s, \theta)$ *is a regular tube if and only if* $1 - r\kappa\cos\theta \neq 0$.

Proof. For a regular surface, $EG - F^2 \neq 0$. By Eq (3.2), we have

$$EG - F^2 = r^2(1 - r\kappa\cos\theta)^2.$$

Since $EG - F^2 \neq 0$ and $r > 0$, $L(s, \theta)$ is a regular tube if and only if

$$1 - r\kappa\cos\theta \neq 0.$$

$\square$

Thus, the Gaussian and mean curvature for a regular tube $L(s, \theta)$ are computed as

$$
\begin{aligned}
K &= \frac{eg - f^2}{EG - F^2} = \frac{-\kappa\cos\theta}{r(1 - r\kappa\cos\theta)} \\
H &= \frac{eG - 2fF + gE}{2(EG - F^2)} = \frac{1}{2}\left[\frac{1}{r} + Kr\right].
\end{aligned}
\tag{3.3}
$$

Theorem 3.2. *If the Gaussian curvature K is zero, then $L(s,\theta)$ is generated by a moving sphere with the radius $r = 1$.*

Proof. When $K = 0$, from Eq (3.3) $\cos\theta = 0$ and so the normal of $L(s,\theta)$ becomes

$$U = -\cos\theta N - \sin\theta B = \pm B.$$

Again, when $\cos\theta = 0$ it follows that

$$
\begin{aligned}
L(s,\theta) - C(s) &= r\left(\cos\theta N(s) + \sin\theta B(s)\right) \\
U &= \pm rB \\
\pm B &= \pm rB.
\end{aligned}
$$

From the last equation we must have $r = 1$. $\qquad\square$

Theorem 3.3. *Let $L(s,\theta)$ be a regular tube. In that case, we have the following.*
(1) The $s-$parameter curves of $L(s,\theta)$ are also asymptotic curves if and only if

$$\frac{\tau^2}{\kappa} = \frac{1}{r}\cos\theta(1 - r\kappa\cos\theta).$$

(2) The $\theta-$parameter curves of $L(s,\theta)$ cannot also be asymptotic curves.

Proof. (1) A curve α lying on a surface is an asymptotic curve if and only if the acceleration vector α'' is tangent to the surface that is $U \cdot \alpha'' = 0$. Then, for the $s-$parameter curves we have

$$U \cdot L_{ss} = -\kappa\cos\theta(1 - r\kappa\cos\theta) + r\tau^2 = 0. \tag{3.4}$$

From this, we get $\dfrac{\tau^2}{\kappa} = \dfrac{1}{r}\cos\theta(1 - r\kappa\cos\theta)$ for $s-$parameter curves.

(2) On account of the fact that $U \cdot L_{\theta\theta} = r \neq 0$, $\theta-$parameter curves cannot also be asymptotic curves.

Here, the equation $\dfrac{\tau^2}{\kappa} = \dfrac{1}{r}\cos\theta(1 - r\kappa\cos\theta)$ is satisfied for a circular helix $C(s)$.
In the case of general helix, we get the curvature of spine curve $C(s)$ as

$$\kappa = \frac{\cos\theta}{r(\tan^2\beta + \cos^2\theta)},$$

where β is the angle between tangent line T and the fixed direction of the general helix. $\dfrac{\tau}{\kappa} = \tan\beta$ is a constant for a general helix. Hence, if we substitute this in the equation $\dfrac{\tau^2}{\kappa} = \dfrac{1}{r}\cos\theta(1 - r\kappa\cos\theta)$, it gathers that

$$r\left(\tan^2\beta + \cos^2\theta\right)\kappa = \cos\theta.$$

In this situation, we obtain the curvature as $\kappa(s) = \dfrac{\cos\theta}{r(\tan^2\beta + \cos^2\theta)}$. Because θ and β are constants, it follows that $\kappa(s)$ is a constant. Therefore,

$$\tau = \kappa\tan\beta = \frac{\tan\beta\cos\theta}{r(\tan^2\beta + \cos^2\theta)}$$

is also a constant. We see that the general helix becomes a circular helix and finally the equation is satisfied for a circular helix. $\qquad\square$

Theorem 3.4. *Let $L(s,\theta)$ be a regular tube.*
(1) The $\theta-$parameter curves of $L(s,\theta)$ are also geodesics.
(2) The $s-$parameter curves are also geodesics of $L(s,\theta)$ if and only if the curvatures of $C(s)$ satisfy the equation

$$r\cos^2\theta\kappa^2 - 2\cos\theta\kappa + r\tau^2 = c,$$

where c is a constant.

Proof. A curve α lying on a surface is a geodesic curve if and only if the acceleration vector α'' is normal to the surface. This means that α'' and the surface normal U are linearly dependent namely $U\times\alpha'' = 0$. In this case, for the $s-$ and $\theta-$parameter curves we conclude

$$
\begin{aligned}
U \times L_{\theta\theta} &= r\sin\theta\cos\theta T - r\sin\theta\cos\theta T = 0 \\
U \times L_{ss} &= [\kappa\sin\theta(1 - r\kappa\cos\theta) - r\tau']\,T + [r\kappa'\sin\theta\cos\theta - r\kappa\tau\sin^2\theta]N \quad (3.5)\\
&\quad + [-r\kappa'\cos^2\theta + r\kappa\tau\sin\theta\cos\theta]B.
\end{aligned}
$$

(1) As immediately seen above, $\theta-$parameter curves of $L(s,\theta)$ are also geodesics.
(2) Since $\{T,N,B\}$ is an orthonormal basis, $U \times L_{ss} = 0$ if and only if

$$
\begin{aligned}
\kappa\sin\theta(1 - r\kappa\cos\theta) - r\tau' &= 0 \\
r\sin\theta[\kappa'\cos\theta - \kappa\tau\sin\theta] &= 0 \quad\quad (3.6)\\
r\cos\theta[\kappa'\cos\theta - \kappa\tau\sin\theta] &= 0
\end{aligned}
$$

By the last two equations we have $\kappa'\cos\theta - \kappa\tau\sin\theta = 0$. If this equation is solved with the first equation of (3.6) it concludes that

$$\cos\theta\kappa' - r\cos^2\theta\kappa\kappa' - r\tau\tau' = 0.$$

Because θ is a constant, if we take integral of the above differential equation we obtain

$$r\cos^2\theta\kappa^2 - 2\cos\theta\kappa + r\tau^2 = c.$$

It is clear that this equation is satisfied for a circular helix or a circle $C(s)$.　　□

Definition 3.5. A generalized tube around the spine curve $\Gamma(s)$ is defined as

$$X(s,\theta) = \Gamma(s) + u(\theta)\left(\cos\theta N(s) + \sin\theta B(s)\right),\ 0 \leq \theta < 2\pi \quad\quad (3.7)$$

where u is twice differentiable, $u(\theta) > 0$ and $u(0) = u(2\pi)$ [3].
Let U be the normal vector field of the generalized tube $X(s,\theta)$. In that case,

$$
\begin{aligned}
X_\theta &= (1 - \kappa u\cos\theta)\,T - u\tau\sin\theta N + u\tau\cos\theta B \\
X_s &= (u'\cos\theta - u\sin\theta)N + (u'\sin\theta + u\cos\theta)B \quad\quad (3.8)\\
U &= X_\theta \times X_s = uu'\tau T + (1 - \kappa u\cos\theta)\begin{bmatrix}\left(u'\sin\theta + u\cos\theta\right)N \\ + \left(u\sin\theta - u'\cos\theta\right)B\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
F &= X_s \cdot X_\theta = u^2\tau \quad\quad (3.9)\\
f &= U \cdot X_{s\theta} = \frac{1}{\|U\|}\tau\kappa u^2(u'\sin\theta + u\cos\theta).
\end{aligned}
$$

Theorem 3.6 (Line of Curvature). *The directions of the parameter curves at a non-umbilical point on a patch are in the direction of the principal directions (line of curvature) if and only if $F = f = 0$ at the point, where F and f are the respective first and second fundamental coefficients* [3].

Proof. Weingarten equations are given by

$$
\begin{aligned}
-S(x_u) &= U_u = \frac{fF - eG}{EG - F^2}x_u + \frac{eF - fE}{EG - F^2}x_v \qquad (3.10) \\
-S(x_v) &= U_v = \frac{gF - fG}{EG - F^2}x_u + \frac{fF - gE}{EG - F^2}x_v.
\end{aligned}
$$

$\square$

($\Longrightarrow$) Assume that the directions of the parameter curves at a non-umbilical point on a patch are in the direction of the principal directions. In this case, from the Weingarten equations and definition of line of curvature we have

$$
\begin{aligned}
S(x_u) &= -\frac{fF - eG}{EG - F^2}x_u \\
S(x_v) &= -\frac{fF - gE}{EG - F^2}x_v
\end{aligned}
$$

This means that $\dfrac{eF - fE}{EG - F^2} = \dfrac{gF - fG}{EG - F^2} = 0$ in Eq (3.10). By the last two equations we obtain

$$
\begin{aligned}
eF - fE &= 0 \\
gF - fG &= 0.
\end{aligned}
$$

From this, $F = f = 0$.

($\Longleftarrow$) Let $F = f = 0$ at a non-umbilical point on a patch. By the Weingarten equations it follows that

$$
\begin{aligned}
S(x_u) &= \frac{e}{E}x_u \\
S(x_v) &= \frac{g}{G}x_v.
\end{aligned}
$$

Then, $u-$ and $v-$parameter curves are lines of curvature concurrently.

We view that the parameter curves of a tube or generalized tube are also lines of curvature if and only if the spine curve is planar. For the tube $L(s, \theta)$, we have

$$
\begin{aligned}
F &= L_s \cdot L_\theta = r^2\tau \\
f &= r\tau.
\end{aligned}
$$

Also, for the generalized tube $X(s, \theta)$ we have

$$
\begin{aligned}
F &= u^2\tau \\
f &= \frac{1}{\|U\|}\tau\kappa u^2(u^{'}\sin\theta + u\cos\theta).
\end{aligned}
$$

Truthfully, in both two cases, $F = f = 0$ if and only if the torsion τ of the spine curve is zero, i.e., the spine curve is planar.

4. The curvatures of tubular surfaces with respect to the Bishop frame

From this time, we will compute the curvatures of tubular surfaces with Bishop frame and then give some characterizations relative to it. Let $P(s, \theta)$ be a tubular surface with Bishop frame. By applying Eq (2.6), the first and second derivatives of $P(s, \theta)$ with respect to s and θ are obtained as

$$
\begin{aligned}
P_s &= (1 - rk_1 \cos\theta - rk_2 \sin\theta)T \\
P_\theta &= r(-\sin\theta N_1 + \cos\theta N_2) \\
P_{ss} &= -r\left(k_1' \cos\theta + k_2' \sin\theta\right)T + k_1(1 - rk_1 \cos\theta - rk_2 \sin\theta)N_1 \qquad (4.1)\\
&\quad + k_2(1 - rk_1 \cos\theta - rk_2 \sin\theta)N_2 \\
P_{s\theta} &= (rk_1 \sin\theta - rk_2 \cos\theta)T \\
P_{\theta\theta} &= -r(\cos\theta N_1 + \sin\theta N_2).
\end{aligned}
$$

Since $T \times N_1 = N_2$ and $T \times N_2 = -N_1$, the cross product of P_s and P_θ is that

$$
P_s \times P_\theta = -r(1 - rk_1 \cos\theta - rk_2 \sin\theta)\left[\cos\theta N_1 + \sin\theta N_2\right]. \qquad (4.2)
$$

For this reason, the normal vector field U and the coefficients of the first and second fundamental form are computed as

$$
U = \frac{P_s \times P_\theta}{\|P_s \times P_\theta\|} = -\cos\theta N_1 - \sin\theta N_2
$$

$$
\begin{aligned}
E &= P_s \cdot P_s = (1 - rk_1 \cos\theta - rk_2 \sin\theta)^2 \\
F &= P_s \cdot P_\theta = 0 \\
G &= P_\theta \cdot P_\theta = r^2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.3)\\
e &= U \cdot P_{ss} = -(k_1 \cos\theta + k_2 \sin\theta)(1 - rk_1 \cos\theta - rk_2 \sin\theta) \\
f &= U \cdot P_{s\theta} = 0 \\
g &= U \cdot P_{\theta\theta} = r.
\end{aligned}
$$

Theorem 4.1. $P(s, \theta)$ *is a regular tube if and only if* $k_1 \cos\theta + k_2 \sin\theta \neq \dfrac{1}{r}$.

Proof. By using Eq (4.2) we attain $EG - F^2 = r^2(1 - rk_1 \cos\theta - rk_2 \sin\theta)^2$. When $1 - rk_1 \cos\theta - rk_2 \sin\theta \neq 0$, $EG - F^2 \neq 0$. As a result, $P(s, \theta)$ is a regular tube if and only if $k_1 \cos\theta + k_2 \sin\theta \neq \dfrac{1}{r}$. □

In this case, the Gaussian curvature K and mean curvature H for the regular tube $P(s, \theta)$ are obtained as

$$
\begin{aligned}
K &= \frac{k_1 \cos\theta + k_2 \sin\theta}{r(rk_1 \cos\theta + rk_2 \sin\theta - 1)} \qquad\qquad\qquad (4.4)\\
H &= rK - \frac{K}{2(k_1 \cos\theta + k_2 \sin\theta)}.
\end{aligned}
$$

Theorem 4.2. *The Gaussian curvature K of the regular tube $P(s, \theta)$ is zero if and only if the spine curve $C(s)$ is planar for the $s-$parameter curves.*

Proof. Let K be zero. Then,

$$K = \frac{k_1 \cos \theta + k_2 \sin \theta}{r \left(r k_1 \cos \theta + r k_2 \sin \theta - 1 \right)} = 0.$$

Hence, we conclude that $k_1 \cos \theta + k_2 \sin \theta = 0$. Because of the fact that θ is a constant for the $s-$parameter curves, the normal development (k_1, k_2) lies on a line through the origin. According to [1], this means that the spine curve $C(s)$ is a plane curve. The sufficiency part of proof is obvious. □

Theorem 4.3. *The parameter curves of tubular surface $P(s, \theta)$ are also lines of curvature.*

Proof. For the tubular surface $P(s, \theta)$, by Eq (4.3) we have $F = f = 0$. Therefore, from Theorem of line of curvature the parameter curves of $P(s, \theta)$ are also lines of curvature. □

Theorem 4.4. *Let $P(s, \theta)$ be a regular tube. Then, we have the following.*
(1) The $s-$parameter curves of $P(s, \theta)$ are concurrently asymptotic curves if and only if the spine curve $C(s)$ is planar.
(2) The $\theta-$parameter curves of $P(s, \theta)$ cannot concurrently be asymptotic curves.

Proof. A curve α lying on a surface is an asymptotic curve if and only if $U \cdot \alpha'' = 0$. For the $s-$ and $\theta-$parameter curves, it follows that

$$
\begin{aligned}
U \cdot P_{ss} &= -\left(k_1 \cos \theta + k_2 \sin \theta \right)\left(1 - r k_1 \cos \theta - r k_2 \sin \theta \right) \qquad (4.5) \\
U \cdot P_{\theta\theta} &= r \neq 0.
\end{aligned}
$$

(1) Because $P(s, \theta)$ is a regular tube, $k_1 \cos \theta + k_2 \sin \theta \neq \dfrac{1}{r}$. Then

$$U \cdot P_{ss} = 0 \iff k_1 \cos \theta + k_2 \sin \theta = 0.$$

From this, it follows that the spine curve $C(s)$ is planar.
(2) Since $U \cdot P_{\theta\theta} \neq 0$, the $\theta-$parameter curves of the regular tube $P(s, \theta)$ cannot concurrently be asymptotic curves. □

Theorem 4.5. *Let $P(s, \theta)$ be a regular tube. Then, we have the following.*
(1) The $\theta-$parameter curves of $P(s, \theta)$ are also geodesics.
(2) The $s-$parameter curves of $P(s, \theta)$ cannot also be geodesic curves.

Proof. A curve lying on a surface is a geodesic curve if and only if the acceleration vector α'' is normal to the surface, i.e., $U \times \alpha'' = 0$. In this case, for the $s-$ and $\theta-$parameter curves, we obtain

$$
\begin{aligned}
U \times P_{ss} &= \left(k_1 \sin \theta - k_2 \cos \theta \right) T + r \sin \theta \left(k_1' \cos \theta + k_2' \sin \theta \right) N_1 \\
&\quad - r \cos \theta \left(k_1' \cos \theta + k_2' \sin \theta \right) N_2 \qquad (4.6) \\
U \times P_{\theta\theta} &= r \sin \theta \cos \theta T - r \sin \theta \cos \theta T = 0.
\end{aligned}
$$

(1) Seeing that, $U \times P_{\theta\theta} = 0$, $\theta-$parameter curves are also geodesics of $P(s, \theta)$.

(2) $U \times P_{ss} = 0$ if and only if

$$
\begin{aligned}
k_1 \sin\theta - k_2 \cos\theta &= 0 \\
r \sin\theta \left(k_1' \cos\theta + k_2' \sin\theta \right) &= 0 \\
r \cos\theta \left(k_1' \cos\theta + k_2' \sin\theta \right) &= 0.
\end{aligned} \tag{4.7}
$$

We know that $\sin\theta$ and $\cos\theta$ cannot be zero at the same time. Now that $r > 0$, the system of equations above is held if and only if

$$
\begin{aligned}
k_1 \sin\theta - k_2 \cos\theta &= 0 \\
k_1' \cos\theta + k_2' \sin\theta &= 0.
\end{aligned}
$$

Since $k_1 \sin\theta - k_2 \cos\theta = 0$, we get $\dfrac{k_1}{k_2} = \cot\theta$. Besides, we have $\dfrac{k_1'}{k_2'} = -\tan\theta$. By the last two equations, $\cot\theta + \tan\theta = 0$. From this, it follows that $\dfrac{1}{\sin\theta\cos\theta} \neq 0$. This is a contradiction, i.e., the system of equations does not have a solution. Then, the $s-$parameter curves of $P(s, \theta)$ cannot also be geodesic curves. $\qquad\square$

5. Conclusions

In this paper, we defined a tube with respect to the Bishop frame. Later, we computed the curvatures of this tube and examined special curves on it. Surprisingly, we viewed that $\theta-$parameter curves of $P(s, \theta)$ are both lines of curvature and geodesics in other words $\theta-$parameter curves are planar. Furthermore, while a $s-$parameter curve of $L(s, \theta)$ can also be a geodesic none of the $s-$parameter curves of $P(s, \theta)$ can concurrently be a geodesic.

ÖZET: Kanal yüzeyi, merkezlerinin yörüngesi $C(t)$ eğrisi (spine eğrisi) ve yarıçap fonksiyonu $r(t)$ olan hareketli bir kürenin zarfı olarak tanımlanır ve spine eğrisinin Frenet çatısı yardımı ile parametrize edilir. Eğer yarıçap fonksiyonu $r(t) = r$ olacak şekilde bir sabit ise, kanal yüzeyine bir tüp adı verilir. Bu çalışmada tüp yüzeyini Frenet çatısı yerine Bishop çatısı ile birlikte araştıracağız ve daha sonra bu yüzey üzerinde yatan özel eğrilerle ilgili bazı karakterizasyonlar vereceğiz.

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