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ON SOME NEW DOUBLE SEQUENCE SPACES OF INVARIANT MEANS DEFINED BY ORLICZ FUNCTIONS

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Abstract. The sequence space BV_σ was introduced and studied by Mursaleen[14]. In this paper we extend BV_σ to ${}_2BV_\sigma(p,r,s)$ and study some properties and inclusion relations on this space.

1. Introduction

Let l_∞, and c denote the Banach spaces of bounded and convergent sequences $x = (x_i)$, with complex terms, respectively, normed by $\|x\|_\infty = \sup_i |x_i|$, where $i \in \mathbb{N}$. Let σ be an injection of the set of positive integers $\mathbb{N}$ into itself having no finite orbits that is to say, if and only if, for all $i = 0, j = 0, \sigma^j(i) \neq i$ and T be the operator defined on l_∞ by $(T(x_i)_{i=1}^\infty) = (x_{\sigma(i)})_{i=1}^\infty$.

A continuous linear functional ϕ on l_∞ is said to be an invariant mean or σ-mean if and only if

(1) $\phi(x) \geq 0$, when the sequence $x = (x_i)$ has $x_i \geq 0$ for all i,
(2) $\phi(e) = 1$, where $e = \{1, 1, 1, ...\}$ and
(3) $\phi(x_{\sigma(i)}) = \phi(x)$ for all $x \in l_\infty$.

If $x = (x_i)$ write $Tx = (Tx_i) = (x_{\sigma(i)})$. It can be shown that

$$V_\sigma = \left\{ x = (x_i) : \sum_{m=1}^{\infty} t_{m,i}(x) = L \text{ uniformly in i}, L = \sigma - \lim x \right\} \tag{1}$$

where $m \geq o, i > 0$.

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$$t_{m,i}(x) = \frac{x_i + x_{\sigma(i)} + ... + x_{\sigma^m(i)}}{m+1}$$
$$\text{and } t_{-1,i} = 0 \tag{2}$$

. Where $\sigma^m(i)$ denote the mth iterate of $\sigma(i)$ at i. In the case σ is the translation mapping, $\sigma(i) = i + 1$ is often called a Banach limit and V_σ, the set of bounded sequences of all whose invariant means are equal, is the set of almost convergent sequence. Subsequently invariant means have been studied by Ahmad and Mursaleen[1], Mursaleen[12,13], Raimi[15] and many others.

The concept of paranorm is closely related to linear metric spaces. It is generalization of that of absolute value. Let X be a linear space. A *Paranorm* is a function $g : X \to \mathbb{R}$ which satisfies the following axioms: for any $x, y, x_0 \in X$, $\lambda, \lambda_0 \in \mathbb{C}$,

(i) $g(\theta) = 0$;
(ii) $g(x) = g(-x)$;
(iii) $g(x + y) \leq g(x) + g(y)$
(iv) the scalar multiplication is continuous, that is $\lambda \to \lambda_0$, $x \to x_0$ imply $\lambda x \to \lambda_0 x_0$.

Any function g which satisfies all the condition (i)-(iv) together with the condition

(v) $g(x) = 0$ if only if $x = \theta$,

is called a *Total Paranorm* on X and the pair (X, g) is called *Total paranormed space*. It is well known that the metric of any linear metric space is given by some total paranorm (cf.[18],Theorm 10.42,p183])

An *Orlicz Function* is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$. If convexity of M is replaced by $M(x + y) \leq M(x) + M(y)$ then it is called Modulus function.

An Orlicz function M satisfies the $\Delta_2 -$ *condition* ($M \in \Delta_2$ for short) if there exist constant $k \geq 2$ and $u_0 > 0$ such that

$$M(2u) \leq KM(u)$$

whenever $|u| \leq u_0$.

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An Orlicz function M can always be represented in the integral form $M(x) = \int_0^x q(t)dt$, where q known as the kernel of M, is right differentiable for $t \geq 0, q(t) > 0$ for $t > 0, q$is non-decreasing and $q(t) \to \infty$ as $t \to \infty$.

Note that an Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1,$$

since M is convex and $M(0) = 0$.

W.Orlicz used the idea of Orlicz function to construct the space (L^M). Lindesstrauss and Tzafriri [9] used the idea of Orlicz sequence space;

$$l_M := \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is Banach space with the norm the norm

$$\|x\|_M = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.
Orlicz functons have been studied by V.A.Khan[3,5,6,7,8] and many others.

Throughout a double sequence is denoted by $x = (x_{ij})$.A double sequence is a double infinite array of elements $x_{ij} \in \mathbb{R}$ for all $i, j \in \mathbb{N}$. Let ${}_2l_\infty$ and ${}_2c$ denote the Banach spaces of bounded and convergent double sequence $x = (x_{i,j})$ respectively. Doube sequence spaces have been studied by Moricz and Rhoads[11], E.Savas and R.F.Patterson[16], V.A.Khan[4] and many others.

Let σ be an injection having no finite orbits and T be the operator defined on ${}_2l_\infty$ by

$$T((x_{i,j})_{i,j=1}^{\infty}) = (x_{\sigma(i,j)})_{i,j}^{\infty}$$

The idea of σ-convergence for double sequences has recently been introduced in [2] and further studied by Mursaleen and Mohiuddine [12].
For double sequences,

$${}_2V_\sigma = \left\{ x = (x_{i,j}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} t_{mnpq}(x) = L \text{ uniformly in } p, q, L = \sigma - \lim x \right\} \text{see[16]} \tag{3}$$

$$t_{mnpq}(x) = \frac{1}{(m+1)(n+1)} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{\sigma^i(p),\sigma^j(q)}, \ p, q = 0, 1, 2... \tag{4}$$

$$\begin{aligned} t_{0,0,p,q}(x) &= x_{pq}, t_{-1,0,p,q}(x) = x_{p-1,q}(x), t_{0,-1,p,q}(x) \\ &= x_{p,q-1}, t_{-1,-1,p,q}(x) = x_{p-1,q-1}, \end{aligned}$$

and $x_{\sigma^i(p),\sigma^j(q)} = 0$ for all i or j or both negative.

A double sequence space E is said to be *solid* if $(\alpha_{i,j} x_{i,j}) \in E$, whenever $(x_{i,j}) \in E$, for all double sequences $(\alpha_{i,j})$ of scalars with $|\alpha_{i,j}| \leq 1$, for all $i, j \in \mathbb{N}$.

Let

$$K = \{(n_i, k_j) : i, j \in \mathbb{N}; n_1 < n_2 < n_3 < .... \text{ and } k_1 < k_2 < k_3 < ...\} \subseteq N \otimes N$$

and E be a double sequence space. A *K-step space* of E is a sequence space

$$\lambda_K^E = \{(\alpha_{i,j} x_{i,j}) : (x_{i,j}) \in E\}.$$

A *canonical pre-image* of a sequence $(x_{n_i,k_j}) \in E$ is a sequence $(b_{n,k}) \in E$ defined as follows:

$$b_{nk} = \begin{cases} a_{nk} & \text{if } (n,k) \in K, \\ 0 & \text{otherwise} . \end{cases}$$

A *canonical pre-image of step space* λ_K^E is a set of canonical pre-images of all elements in λ_K^E.

A double sequence space E is said to be *monotone* if it contains the canonical pre-images of all its step spaces.

A double sequence space E is said to be symmetric if $(x_{i,j}) \in E$ implies $(x_{\pi(i),\pi(j)}) \in E$, where π is a permutation of $\mathbb{N}$.

2. Main Results

Lemma 1 A sequence space E is solid implies E is monotone.

Mursaleen[14] defined the sequence space

$$BV_\sigma = \{x \in l_\infty : \sum_m |\phi_{m,i}(x)| < \infty, \quad \text{uniformly in } i\}, \tag{5}$$

$$\text{where } \phi_{m,i}(x) = t_{m,i}(x) - t_{m-1,i}(x)$$

assuming that $t_{m,i}(x) = 0$ for $m = -1$
A straightforward calculation shows that

$$\phi_{m,n}(x) = \begin{cases} \frac{1}{m(m+1)} \sum_{n=1}^{m} n[x_\sigma^n(i) - x_\sigma^{n-1}(i)] \ (m \geq 1) \\ x_i \ (m = 0). \end{cases} \tag{6}$$

We define

$${}_2BV_\sigma = \{x \in {}_2l_\infty : \sum_{m,n} |\phi_{mnpq}(x)| < \infty, \ \text{ uniformly in } p \text{ and } q\}, \tag{7}$$

where

$$\phi_{mnpq}(x) = \begin{cases} \frac{1}{m(m+1)n(n+1)} \sum_{i=1}^{m} \sum_{j=1}^{n} ij[x_{\sigma^i(p),\sigma^j(q)} - x_{\sigma^{i-1}(p),\sigma^j(q)} \\ -x_{\sigma^i(p),\sigma^{j-1}(q)} + x_{\sigma^{i-1}(p),\sigma^{j-1}(q)}] \ (m, n \geq 1) \\ x_{ij} \ m \text{ or } n \text{ or both zero }. \end{cases} \quad (see[12]) \tag{8}$$

Let M be an Orlicz function, $p = (p_i)$ be any sequence of strictly positive real numbers and $r \geq 0$. V.A.Khan[5] defined the following sequence space:

$$BV_\sigma(M, p, r) = \left\{ x = (x_i) : \sum_{m=1}^{\infty} \frac{1}{m^r} \left[M\left(\frac{|\phi_{m,i}(x)|}{\rho} \right) \right]^{p_i} < \infty, \right.$$

$$\left. \text{uniformly in } i \text{ and for some } \rho > 0 \right\}.$$

Let $p = (p_{ij})$ be any double sequence of strictly positive real numbers and $r, s \geq 0$. We define the following double sequence spaces as:

$${}_2BV_\sigma(M, p, r, s) = \left\{ x = (x_{ij}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} \left[M\left(\frac{|\phi_{mnpq}(x)|}{\rho} \right) \right]^{p_{ij}} < \infty, \right.$$

$$\left. \text{uniformly in } p, q \text{ and for some } \rho > 0 \right\}.$$

For $M(x) = x$, we get

$${}_2BV_\sigma(p, r, s) = \left\{ x = (x_{ij}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} |\phi_{mnpq}(x)|^{p_{ij}} < \infty, \text{ uniformly in } p, q \right\}.$$

For $p_{i,j} = 1$ for all i, j we get

$${}_2BV_\sigma(M, r, s) = \left\{ x = (x_{ij}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} \left[M\left(\frac{|\phi_{mnpq}(x)|}{\rho} \right) \right] < \infty, \right.$$

$$\left. \text{uniformly in } p, q \text{ and for some } \rho > 0 \right\}.$$

For $r, s = 0$, we get

$$ {}_2BV_\sigma(M,p) = \Bigg\{ x = (x_{ij}) : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[M\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right]^{p_{ij}} < \infty, $$

$$ \text{uniformly in } p, q \text{ and for some } \rho > 0 \Bigg\}. $$

For $M(x) = x$ and $r, s = 0$, we get

$$ {}_2BV_\sigma(p) = \Bigg\{ x = (x_{ij}) : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\phi_{mnpq}(x)|^{p_{ij}} < \infty, \text{ uniformly in } p, q \Bigg\}. $$

For $p_{i,j} = 1$ for all i, j and $r, s = 0$, we get

$$ {}_2BV_\sigma(M) = \Bigg\{ x = (x_{ij}) : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[M\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right] < \infty, \text{ uniformly in } p, q $$

$$ \text{and for some } \rho > 0 \Bigg\}. $$

For $M(x) = x, p_{i,j} = 1$ and $r, s = 0$,we get

$$ {}_2BV_\sigma = \Bigg\{ x = (x_{ij}) : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\phi_{mnpq}(x)| < \infty, \text{ uniformly in } p, q \Bigg\}. $$

Theorem 1 The sequence space ${}_2BV_\sigma(M, p, r, s)$ is a linear space over the field $\mathbb{C}$ of complex numbers.

Proof Let $x = (x_{i,j})$ and $y = (y_{i,j}) \in {}_2BV_\sigma(M, p, r, s)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^r n^s}\left[M\left(\frac{|\phi_{mnpq}(x)|}{\rho_1}\right)\right]^{p_{ij}} < \infty $$

and

$$ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^r n^s}\left[M\left(\frac{|\phi_{mnpq}(y)|}{\rho_2}\right)\right]^{p_{ij}} < \infty $$

uniformly in p and q and $r, s \geq 0$
Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non decreasing and convex we have,

$$ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^r n^s}\left[M\left(\frac{|\alpha\phi_{mnpq}(x) + \beta\phi_{mnpq}(y)|}{\rho_3}\right)\right]^{p_{ij}} < \infty $$

$$ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^r n^s}\left[M\left(\frac{|\alpha\phi_{mnpq}(x)|}{\rho_3} + \frac{|\beta\phi_{mnpq}(y)|}{\rho_3}\right)\right]^{p_{ij}} < \infty $$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^r n^s}\frac{1}{2}\left[M\left(\frac{\phi_{mnpq}(x)}{\rho_1}\right)+M\left(\frac{\phi_{mnpq}(y)}{\rho_2}\right)\right]<\infty$$

uniformly in p and q and $r, s \geq 0$.
This proves that ${}_2BV_\sigma(M,p,r,s)$ is a linear space over the field $\mathbb{C}$ of complex numbers.

Theorem 2 For any Orlicz function M and a bounded sequence $p = (p_{i,j})$ of strictly positive real numbers, ${}_2BV_\sigma(M,p,r,s)$ is a paranormed space with paranorm

$$g((x_{ij})) = \sup_i |x_{i,1}| + \sup_j |x_{1,j}| + \inf\left\{\rho^{\frac{p_{ij}}{H}} : \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[M\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right]^{p_{ij}}\right)^{\frac{1}{H}} \leq 1 \right.$$

$$\left. \text{uniformly in } p \text{ and } q\right\}$$

where $H = \max(1, \sup_{i,j} p_{i,j})$.

Proof Clearly $g(0) = 0, g(-(x_{ij})) = g((x_{i,j}))$.
Using Theorem[1], for $\alpha = \beta = 1$, we get

$$g(x+y) \leq g(x) + g(y).$$

For continuity of scalar multiplication let $\eta \neq 0$ be any complex number. Then by definition we have

$$g(\eta(x_{ij})) = \sup_i |\eta x_{i,1}| + \sup_j |\eta x_{1,j}| + \inf\left\{\rho^{\frac{p_{ij}}{H}} : \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[M\left(\frac{|\phi_{mnpq}(\eta x)|}{\rho}\right)\right]^{p_{ij}}\right)^{\frac{1}{H}} \leq 1 \right.$$

$$\left. \text{uniformly in } p \text{ and } q\right\}$$

$$= \sup_i |\eta||x_{i,1}| + \sup_j |\eta||x_{1,j}| + \inf\left\{(|\eta| r)^{\frac{p_{ij}}{H}} : \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[M\left(\frac{|\phi_{mnpq}(x)|}{r}\right)\right]^{p_{ij}}\right)^{\frac{1}{H}} \leq 1 \right.$$

$$\left. \text{uniformly in } p \text{ and } q\right\}$$

where $\frac{1}{r} = \frac{|\eta|}{\rho} = \max(1, |\eta|^H g((x_{i,j}))$
and therefore $g(\eta(x_{ij}))$ converges to zero when $g((x_{ij}))$ converges to zero in ${}_2BV_\sigma(M,p,r,s)$.
Now let x be fixed element in ${}_2BV_\sigma(M,p,r,s)$. There exist $\rho > 0$ such that

$$g((x_{ij})) = \sup_i |x_{i,1}| + \sup_j |x_{1,j}| + \inf\left\{\rho^{\frac{p_{ij}}{H}} : \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[M\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right]^{p_{ij}}\right)^{\frac{1}{H}} \leq 1 \right.$$

$$\left. \text{uniformly in } p \text{ and } q\right\}$$

Now

$$\begin{aligned} g(\eta(x_{ij})) = & \sup_i |\eta x_{i,1}| + \sup_j |\eta x_{1,j}| \\ & + \inf\left\{\rho^{\frac{p_{ij}}{H}} : \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^r n^s}\left[M\left(\frac{|\phi_{mnpq}(\eta x)|}{\rho}\right)\right]^{p_{ij}}\right)^{\frac{1}{H}} \leq 1 \right. \\ & \left. \text{uniformly in } p \text{ and } q\right\} \to 0 \quad \text{as } \eta \to 0. \end{aligned}$$

This copmletes the proof.

Theorem 3 Suppose that $0 < p_{ij} \leq q_{ij} < \infty$ for each $m \in \mathbb{N}$ and $r, s \geq 0$. Then

(i) ${}_2BV_\sigma(M,p) \subseteq {}_2BV_\sigma(M,q)$.
(ii) ${}_2BV_\sigma(M) \subseteq {}_2BV_\sigma(M,r,s)$.

Proof(i) Suppose $x \in {}_2BV_\sigma(M,p)$. This implies that

$$\left[M\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right]^{p_{ij}} \leq 1$$

for sufficiently large values m, n say $m \geq m_0,\ n \geq n_0$ for some fixed $m_0,\ n_0 \in \mathbb{N}$.
Since M is non decreasing, we have

$$\begin{aligned} \sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\left[M\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right]^{q_{ij}} & \leq \sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\left[M\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right]^{p_{ij}} \\ & < \infty. \end{aligned}$$

uniformy in p, q. Hence $x \in {}_2BV_\sigma(M,q)$.
The second proof is trivial.

The following result is a consequence of the above result.

Corollary 1 If $0 \leq p_{ij} \leq 1$ for each i and j, then ${}_2BV_\sigma(M,p) \subseteq {}_2BV_\sigma(M)$. If $0 \leq p_{ij} \leq 1$ for all i,j then ${}_2BV_\sigma(M) \subseteq {}_2BV_\sigma(M,p)$.

Theorem 4 The sequence space ${}_2BV_\sigma(M,p,r,s)$ is solid.

Proof Let $x \in {}_2BV_\sigma(M,p,r,s)$.
This implies $\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}\frac{1}{m^r n^s}\left[M\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right]^{p_{ij}} < \infty$.

Let (α_{ij}) be sequence of scalars such that $|\alpha_{ij}| \leq 1$ for all $i,j \in \mathbb{N}$. Then the result follows from the following inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^r n^s}\left[M\left(\frac{|\alpha_{ij}\phi_{mnpq}(x)|}{\rho}\right)\right]^{p_{ij}} \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^r n^s}\left[M\left(\frac{|\alpha_{ij}\phi_{mnpq}(x)|}{\rho}\right)\right]^{p_{ij}} < \infty.$$

Hence $\alpha x \in {}_2BV_\sigma(M,p,r,s)$, for all sequences of scalars (α_{ij}) with $|\alpha_{ij}| \leq 1$ for all $i,j \in \mathbb{N}$ whenever $x \in {}_2BV_\sigma(M,p,r,s)$.

From Theorem[4] and Lemma we have:

Corollary 2 The sequence space ${}_2BV_\sigma(M,p,r,s)$ is monotone.

Theorem 5 Let M_1, M_2 be Orlicz functions satisfying Δ_2-condition and r, r_1, r_2, $s, s_1, s_2 \geq 0$. Then we have

(i) if $r,s > 1$ then ${}_2BV_\sigma(M,p,r,s) \subseteq {}_2BV_\sigma(M \circ M_1,p,r,s)$,
(ii) ${}_2BV_\sigma(M_1,p,r,s) \cap {}_2BV_\sigma(M_2,p,r) \subseteq {}_2BV_\sigma(M_1+M_2,p,r,s)$,
(iii) if $r_1 \leq r_2$ and $s_1 \leq s_2$ then ${}_2BV_\sigma(M,p,r_1,s_1) \subseteq {}_2BV_\sigma(M,p,r_2,s_2)$.

Proof(i) Since M is continuous at 0 from right, for $\epsilon > 0$, there exists $0 < \delta < 1$ such that $0 \leq c \leq \delta$ implies $M(c) < \epsilon$. If we define

$$I_1 = \left\{m \in \mathbb{N} : M_1\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right) \leq \delta \text{ for some } \rho > 0\right\}.$$

$$I_2 = \left\{m \in \mathbb{N} : M_1\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right) > \delta \text{ for some } \rho > 0\right\}.$$

then, when $M_1\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right) > \delta$ we get

$$M\left(M_1\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right) \leq \left\{2\frac{M(1)}{\delta}\right\}M_1\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)$$

Hence for $x \in {}_2BV_\sigma(M,p,r,s)$ and $r,s>1$

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^r n^s}\left[M\circ M_1\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right]^{p_{ij}}\\
&=\sum_{m\in I_1}\sum_{n\in I_1}\frac{1}{m^r n^s}\left[M\circ M_1\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right]^{p_{ij}}\\
&+\sum_{m\in I_2}\sum_{n\in I_2}\frac{1}{m^r n^s}\left[M\circ M_1\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right]^{p_{ij}}\\
&\le \sum_{m\in I_1}\sum_{n\in I_1}\frac{1}{m^r n^s}[\epsilon]^{p_{ij}}+\sum_{m\in I_2}\sum_{n\in I_2}\frac{1}{m^r n^s}\left[\left\{2\frac{M(1)}{\delta}\right\}M_1\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right]^{p_{ij}}\\
&\le \max(\epsilon^h,\epsilon^H)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^r n^s}+\max\left(\left\{2\frac{M(1)}{\delta}\right\}^h\left\{2\frac{M(1)}{\delta}\right\}^H\right)
\end{aligned}$$

(where $0<h=\inf p_{ij}\le p_{ij}\le H=\sup_{i,j} p_{ij}<\infty$.)

(ii) The proof follows from the following inequality

$$\begin{aligned}
\frac{1}{m^r n^s}\left[(M_1+M_2)\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right]^{p_{ij}} &\le \frac{C}{m^r n^s}\left[M_1\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right]^{p_{ij}}\\
&+\frac{C}{m^r n^s}\left[M_2\left(\frac{|\phi_{mnpq}(x)|}{\rho}\right)\right]^{p_{ij}}
\end{aligned}$$

.

(iii) The proof is trivial.

Corollary 3 Let M be an Orlicz function satisfying Δ_2-condition. Then we have.

(i) if $r,s>1$ then ${}_2BV_\sigma(p,r,s)\subseteq {}_2BV_\sigma(M,p,r,s)$,
(ii) ${}_2BV_\sigma(M,p)\subseteq {}_2BV_\sigma(M,p,r,s)$,
(iii) ${}_2BV_\sigma(M)\subseteq {}_2BV_\sigma(M,r,s)$.

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ÖZET: BV_σ dizi uzayı, Mursaleen tarafından tanımlanmış ve incelenmiştir. Bu makalede ise BV_σ uzayı, ${}_2BV_\sigma(p,r,s)$ uzayına genişletilmiş ve bazı özellikleri ile içerme bağıntıları incelenmiştir.

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